An Overview of Generalized Basic Logic Algebras

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- Introduction and overview
- Description of congruences in residuated lattices
- Translation of positive universal sentences
- Discriminator varieties
- Decomposition of generalized BL-algebras
- Categorical equivalence of ℓ-groups and their negative cones
- Categorical equivalence of GMV-algebras and ℓ-group expansions

Introduction

Classical propositional logic $\equiv$ Boolean algebra

The *standard Boolean algebra* is $\{0, 1\}$, with operations

$x \wedge y = \min(x, y)$ $\quad$ $x \vee y = \max(x, y)$ $\quad$ $\neg x = 1 - x$

$x \rightarrow y = \max(1 - x, y)$ $\quad$ $1 =$ true $\quad$ $0 =$ false

The *residuation equivalences* relate conjunction and implication:

$$x \wedge y \leq z \iff x \leq y \rightarrow z \iff y \leq x \rightarrow z$$

They imply the following properties:

- commutativity of $\wedge$
- $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
- distributivity of $\wedge$ over $\vee$
- right-distributivity of $\rightarrow$ over $\wedge$

Fuzzy logic aims at finer control over degree of truth for statements.

Therefore use more truth values, such as the interval [0,1].

Don't want to assume conjunction is always $\min$.

So introduce a new symbol $\cdot$ called *fusion* for conjunction.

Want to retain some aspects of the residuation equivalences.

To avoid imposing commutativity on fusion, need two implications:

$$x \cdot y \leq z \quad \Longleftrightarrow \quad x \leq y \to z \quad \Longleftrightarrow \quad y \leq x \rightsquigarrow z$$

$\to$, $\rightsquigarrow$ are generalized divisions so a more suggestive notation is

$$x \cdot y \leq z \quad \Longleftrightarrow \quad x \leq z/y \quad \Longleftrightarrow \quad y \leq x\backslash z$$

Note: $\quad x \to y = y/x \quad$ and $\quad \backslash$ is another name for $\rightsquigarrow$

Mirror image principle: Any statement about residuated structures has an equivalent mirror image obtained by reading terms backwards

i.e. replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y \backslash x$.

Hence it suffices to state results in only one form.

A *residuated poset* $\langle P, \cdot, \backslash, /, \leq \rangle$ is a partially ordered set $\langle P, \leq \rangle$ with three binary operations that satisfy the residuation equivalences.

In many applications to logic there exists additional structure, such as a constant 1 to denote *true*, and suprema and infima for finite subsets of P.

Fusion is assumed to be at least associative, so we get Dilworth's *residuated lattices* = algebras of the form $\langle L, \vee, \wedge, \cdot, 1, \backslash, / \rangle$ such that

1. $\langle L, \vee, \wedge \rangle$ is a *lattice* ($\vee$, $\wedge$ are commutative, associative and mutually absorbtive)
2. $\langle L, \cdot, 1 \rangle$ is a *monoid* ($\cdot$ is associative, with identity element 1)
3. The *residuation equivalences*: for all $x, y, z \in L$
   $xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$

Since the first two properties are defined by identities, it is important to note that this can also be done for the third:

3. is e.g. equivalent to $x(x\backslash z \wedge y) \leq z$, $y \leq x\backslash(xy \vee z)$ and their mirror images.

Note: $x \cdot y = xy$ is performed first, then $\backslash, /$ and finally $\vee, \wedge$.

$s \leq t$ is an abbreviation for the equivalent identity $s = s \wedge t$.

Therefore residuated lattices form a *variety* (also called *equational class*), denoted by RL.

- L is **not** assumed to be bounded.
- 1 is **not** assumed to be the top element.
- $\cdot$ is **not** assumed to be commutative.

Such assumptions are handled by expanding the language with an additional constant 0, and/or adding further identities.

A *full Lambek algebra* or FL-*algebra* is a residuated lattice with a constant 0 (which can denote any element).

FL is the variety of all FL-algebras.

A *subvariety* of a variety is a subclass that is defined by identities.

The collection of all subvarieties of a variety $\mathcal{V}$ forms a complete *lattice of subvarieties* $\mathbf{L}(\mathcal{V})$

$\mathcal{V} \wedge \mathcal{W} = \mathcal{V} \cap \mathcal{W}$ and $\mathcal{V} \vee \mathcal{W} = \mathrm{Var}(\mathcal{V} \cup \mathcal{W})$

where $\mathrm{Var}(\mathcal{K})$ is the smallest variety that contains class $\mathcal{K}$.

This lattice is dual to a corresponding lattice of logics (more details later).

Important subvarieties of FL.

- FL_w [Ono]: FL-algebras that satisfy $0 \le x \le 1$.
- FL_e [Ono]: FL-algebras with *exchange*, i.e. $x \cdot y = y \cdot x$.

  In this case one usually writes $x \to y$ instead of $x\backslash y = y/x$.
- DFL $=$ *distributive* FL: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
- RFL $=$ *representable* FL: subdirect products of linearly ordered FL-algebras or equiv. [Blount Tsinakis] [J, Tsinakis 02] satisfy $1 \le u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$.
- psMTL $=$ *pseudo monoidal t-norm algebras, or weak-pseudo-BL algebras* [Flondor Georgescu Iorgulescu 01]: FL_w-algebras that satisfy prelinearity: $x\backslash y \vee y\backslash x = 1$ and $x/y \vee y/x = 1$.

- FL_{ew} [Kowalski Ono 01] $= \mathsf{FL}_e \cap \mathsf{FL}_w$.
- MTL $=$ *monoidal t-norm algebras* [Esteva Godo 01]: FL_{ew}-algebras that satisfy prelinearity.
- psBL $=$ *pseudo* BL [Flondor Georgescu Iorgulescu 01], [Di Nola Georgescu Iorgulescu 02]: psMTL-algebras that satisfy divisibility ($x \wedge y = x(x \backslash y) = (y/x)x$).
- BL $=$ *basic logic algebras* [Hajek 98]: MTL-algebras that satisfy divisibility.
- HA $=$ *Heyting algebras*: FL_w-algebras with $x \wedge y = xy$
- psMV $=$ *pseudo* MV [Georgescu Iorgulescu 01]: pseudo BL-algebras that satisfy $x \vee y = x/(y \backslash x) = (x/y) \backslash x$.

- MV $=$ *multi-valued logic algebras, or Łukasiewicz algebras* [Chang 59]: BL-algebras that satisfy $\neg\neg x = x$

- GA $=$ *Gödel logic algebras, or linear Heyting algebras* [Hajek 98]: BL-algebras that satisfy $x \cdot x = x$

- PA $=$ *product logic algebras* [Hajek 98]: BL-algebras that satisfy $\neg\neg x \leq (x \to xy) \to y(\neg\neg y)$.

- BA $=$ *Boolean algebras*: Heyting algebras that satisfy $\neg\neg x = x$

- BA$_n$ $=$ subdirect products of the linearly ordered $n+2$-element Heyting algebra

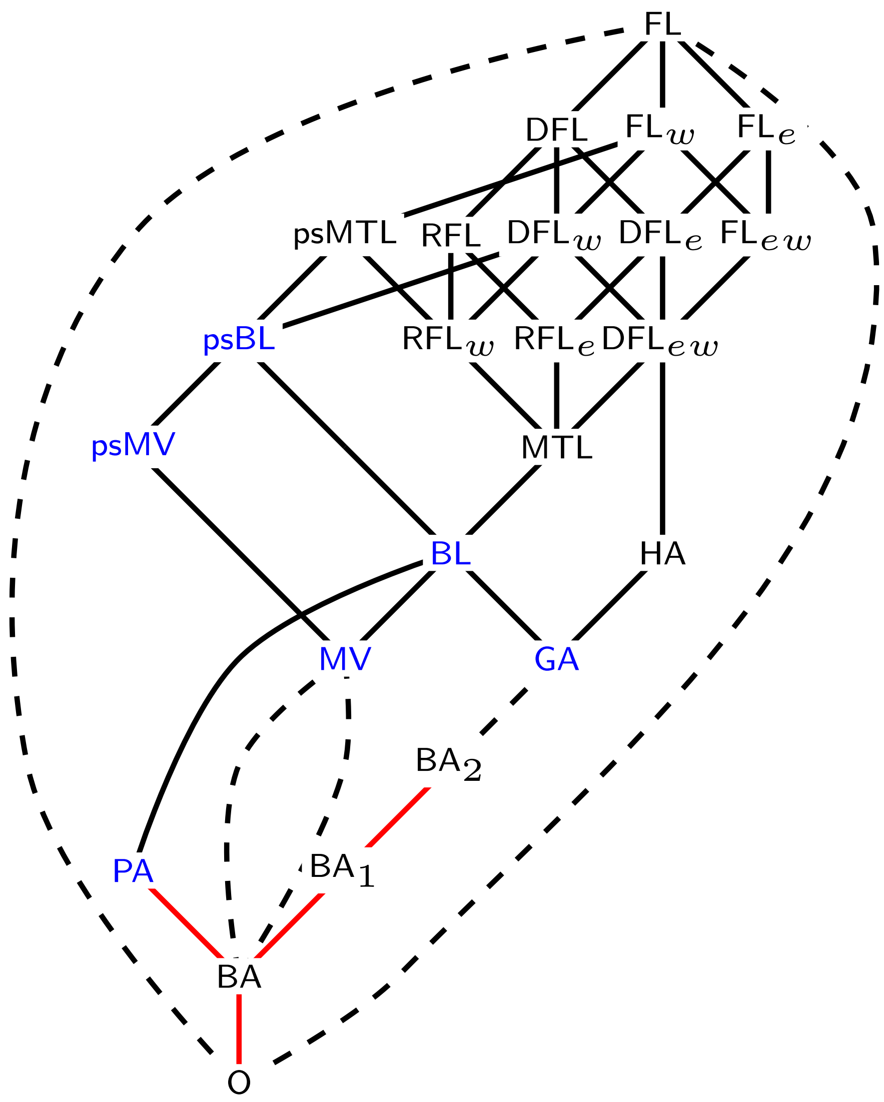

Figure 1: Some subvarieties of FL ordered by inclusion

Important subvarieties of RL.

- GBL $=$ *generalized* BL [J, Tsinakis 02]: Residuated lattices that satisfy $x \wedge y = x(x\backslash(x \wedge y)) = ((x \wedge y)/x)x$.
- GMV $=$ *generalized* MV [J, Tsinakis 02] [Galatos 03]: Residuated lattices such that $x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$.
- Fleas [Hajek 03]: Integral ($x \leq 1$) residuated lattices that satisfy prelinearity ($x\backslash y \vee y\backslash x = 1$ and $x/y \vee y/x = 1$).
- BH $=$ *basic hoops* [Agliano Ferreirim Montagna]: Commutative representable residuated lattices that satisfy divisibility: $x \wedge y = x(x\backslash y)$.

- $\mathsf{LG} =$ *lattice-ordered groups or ℓ-groups* [Birkhoff 67]:

  Residuated lattices that satisfy $1 = x(x\backslash 1)$.

- $\mathsf{NLG} =$ *normal-valued ℓ-groups*, defined by

  $(x \wedge 1)^2(y \wedge 1)^2 \leq (y \wedge 1)(x \wedge 1)$.

- $\mathsf{RLG} =$ *representable ℓ-groups*, defined by

  $1 \leq (1\backslash x)yx \vee 1\backslash y$.

- $\mathsf{CLG} =$ *commutative ℓ-groups*.

- $\mathcal{V}^- =$ *negative cones of members of $\mathcal{V}$* [J, Tsinakis 02]

  E.g. $\mathsf{CLG}^- =$ *negative cones of commutative ℓ-groups*, also

  defined as cancellative (basic) hoops.

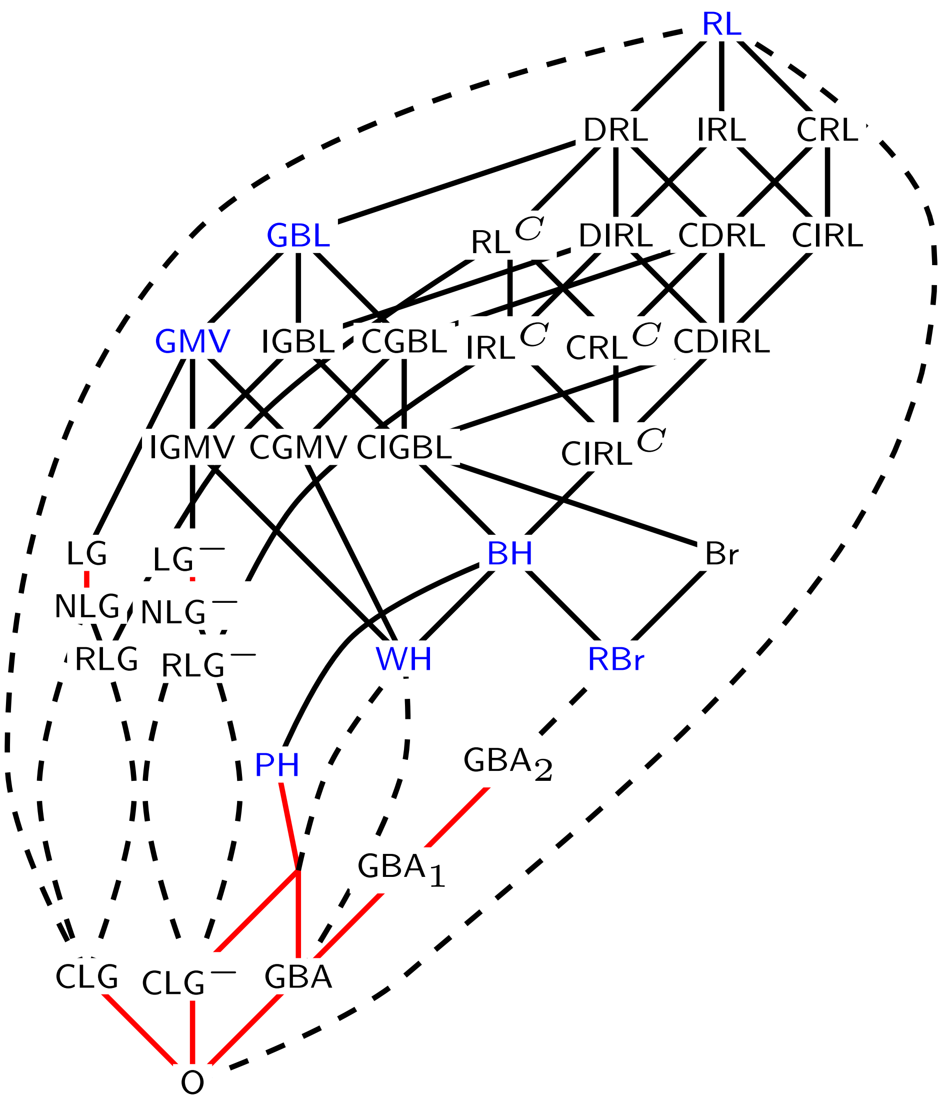

Figure 2: Some subvarieties of RL ordered by inclusion

Many further varieties can be obtained from these by combining some of the identities mentioned above.

The prefixes C, D, I, are used to denote the commutative, distributive and integral identities respectively.

There is a close correspondence between certain subvarieties of FL and RL.

In logic it is quite usual to have a constant 0 in the language to denote *falsity*.

From an algebraic perspective it is in some ways natural to consider the slightly less expressive signature without 0 since, for example, the variety of ℓ-groups is not a subvariety of FL_w.

FL	RL	Defining identities
FL_e	CRL	$xy = yx$
DFL	DRL	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
RFL	RL^C	$1 \leq u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$
		Below add $0 \leq x$ for subvarieties of FL
FL_w	IRL	$x \leq 1$
FL_{ew}	CIRL	$xy = yx$, $x \leq 1$
psMTL	Fleas	$x\backslash y \vee y\backslash x = 1 = x/y \vee y/x$

MTL	CIRL^C	$xy = yx$, $(x \to y) \vee (y \to x) = 1$
	GBL	$x \wedge y = x(x \backslash (x \wedge y)) = ((x \wedge y)/x)x$
psBL	IGBL	$x \wedge y = x(x \backslash y) = (y/x)x$
BL	BH	$xy = yx$, $x \wedge y = x(x \to y)$
		$(x \to y) \vee (y \to x) = 1$
	GMV	$x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$
psMV	IGMV	$x \vee y = x/(y \backslash x) = (x/y)\backslash x$
MV	WH	$xy = yx$, $x \vee y = (x \to y) \to y$
HA	Br	$x \wedge y = xy$
GA	RBr	$x \wedge y = xy$, $(x \to y) \vee (y \to x) = 1$
PA	PH	BL and $\neg\neg x \leq (x \to xy) \to y(\neg\neg y)$
BA	GBA	$x \wedge y = xy$, $x \vee y = (x \to y) \to y$

The notion of *triangular norm* has been studied extensively in the theory of probabilistic measures.

A *pseudo-t-norm* is an order-preserving monoid operation on the unit interval $[0, 1]$, with 1 as the identity.

A *t-norm* is a commutative pseudo-t-norm.

A t-norm is *continuous* if it is a continuous function from $[0, 1]^2$ to $[0, 1]$ in the standard topology of the unit interval.

FL	RL	Generated by
MTL	CIRL^{C}	all residuated t-norms [Jenei Montagna 02]
BL	BH	all continuous t-norms [Hajek 98] [CEGT00]
MV	WH	Łukasiewicz $xy = \max\{0, x + y - 1\}$ [Ch59]
GA	RBr	Gödel $xy = \min\{x, y\}$ [Hajek 98]
PA	PH	Product $xy =$ multiplication on $[0, 1]$ [Hajek 98]

Table 2: Some varieties generated by t-norms

The subvarieties of RL are obtained if we do not specify 0 as a constant of the algebra.

Basic logic was originally defined to formalize the logical properties of continuous t-norms.

It has turned out to be a remarkably well-behaved logic.

Studies of generalizations of BL aim to understand why this is the case.

They also allow applications outside of the domain of basic logic.

Continuous t-norms are very special semantic structures.

They mix together many nice properties, and it is not clear how they all contribute to the nice behavior of BL.

BL algebras are

- *integral*: $x \leq 1$
- *commutative*: $xy = yx$
- *prelinear*: $1 \leq (x\backslash y \vee y\backslash x) \wedge (x/y \vee y/x)$
- *representable*: subdirectly irreducible members are linear
- *divisible*: $x \leq y \implies \exists c, d(x = cy$ and $x = yd)$

A fundamental construction for residuated lattices is the following:

Let I be a linearly ordered set, and $\{A_i : i \in I\}$ a family of integral residuated lattices such that 1 is join-irreducible in each A_i, and $\bigcap_{i \in I} A_i = \{1\}$.

The *ordinal sum* $\bigoplus_{i \in I} A_i$ is defined on $\bigcup_{i \in I} A_i$ for $x \in A_i$ and $y \in A_j$ by

$$xy = \begin{cases} x \cdot^{A_i} y & \text{if } i = j \\ x & \text{if } i < j \\ y & \text{if } i > j \end{cases}$$

This is again an integral residuated lattice, with

$$x \leq y \quad \text{iff} \quad x \leq^{A_i} y \text{ or } i < j.$$

So the A_i's are stacked up according to the order of I, with the mutual 1 at the top.

- BL is closed under this (partial) operation.

The ordinal sum construction provides a straightforward proof [Agliano Montagna 03] of the following fundamental result of [Hajek 98] and [Cignoli Esteva Godo Torrens 00].

Theorem. $\mathsf{BL} = \mathrm{Var}\{$*continuous t-norms*$\}$.

Generalizations of basic logic are obtained by deleting some of the axioms of basic logic or relaxing restrictions on the semantics.

In the algebraic setting this corresponds to studying varieties that include BL.

Conversely, any subvariety of RL or FL defines a corresponding logic that is sound and complete with respect to the algebraic semantics of the subvariety.

A formula φ corresponds to the identity $\varphi \wedge 1 = 1$.

An identity $s = t$ corresponds to the formula $s \backslash t \wedge t \backslash s$.

(identify logical connectives and corresponding operation symbols)

Deductions in equational logic are equivalent to deductions in the corresponding logic.

Therefore many logical questions have algebraic counterparts and vice versa.

In the algebraic setting there are useful semantic concepts and methods.

E.g. subdirectly irreducible algebras, free algebras, homomorphic images, subalgebras, products,...

Further specific properties of BL and BH

Subdirectly irreducibles are linearly ordered.

Meet and join are definable by fusion and its residual.

BL and BH have uncountably many subvarieties.

They have the finite embedding property (every finite partial subalgebra of a member can be extended to a finite member) [Agliano Ferreirim Montagna].

Therefore the universal theory of BL and of BH is decidable.

[Montagna Pinna Tiezzi 03] give a Gentzen style decision procedure (with semantic rules) for the equational theory.

[Agliano Montagna 03] describe all varieties generated by finite ordinal sums of Wajsberg hoops.

Congruences in residuated lattices

We now consider the larger variety of residuated lattices.

To understand the structure of an algebra A, it is useful to map it homomorphically onto a smaller algebra.

This is done internally by examining *congruences* on A, i.e. equivalence relations θ on A such that

$$a_1 \theta b_1, \ldots, a_n \theta b_n \;\Rightarrow\; f(a_1, \ldots, a_n)\, \theta\, f(b_1, \ldots, b_n).$$

for all basic operations f of A.

The equivalence class of $a \in A$ is denoted by $[a]_\theta$.

A/θ is the set of all equivalence classes, and it is the universe of the *quotient algebra* A/θ.

The collection of all congruences of A is written $\mathrm{Con}(A)$.

It is a complete (in fact algebraic) lattice with intersection as meet.

In groups, and hence in ℓ-groups, any congruence is determined by its 1-congruence class, and

there is a one-one correspondence between 1-congruence classes and normal subgroups.

We want to find a similar characterization for congruences in residuated lattices.

Consider the term $d(x,y) = x\backslash y \wedge y\backslash x \wedge 1$.

Lemma. *Let L be a residuated lattice. For any congruence θ of L, we have*

$$a\,\theta\,b \text{ if and only if } d(a,b)\,\theta\,1.$$

Let $L^- = \{x \in L : x \leq 1\}$ denote the *negative part* of L.

Corollary. *Congruences in* RL *are determined by their 1-congruence classes. In fact, if θ and φ are congruences on a residuated lattice L, then $[1]_\theta \cap L^- = [1]_\varphi \cap L^-$ implies $\theta = \varphi$.*

Now we wish to characterize the 1-congruence classes.

A general framework for ideals in universal algebras was given by [Ursini 72] and elaborated by [Gumm and Ursini 84].

Definition A term $t(u_1, \ldots, u_m, x_1, \ldots, x_n)$ is called an *ideal term of* $\mathcal{K}$ if $\mathcal{K} \models t(u_1, \ldots, u_m, 1, \ldots, 1) = 1$.

We also write the term as $t_{u_1,\ldots,u_m}(x_1, \ldots, x_n)$ to indicate the distinction between the two types of variables.

Examples of ideal terms for RL are

- the *left conjugate* $\lambda_u(x) = (u\backslash xu) \wedge 1$ (since $1 \leq u\backslash u$)
- the *right conjugate* $\rho_u(x) = (ux/u) \wedge 1$
- $\kappa_u(x, y) = (u \vee x) \wedge y$ (since $u \vee 1 \geq 1$)
- $x \diamond y$ for $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$ (since $1 \diamond 1 = 1$).

A subset H of $A \in \mathcal{K}$ is *closed under an ideal term* t *of* $\mathcal{K}$ if for all $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_n \in H$ we have $t(a_1, \ldots, a_m, b_1, \ldots, b_n) \in H$.

Example. Any 1-congruence class is closed under all ideal terms, since if t is an ideal term and $\mathbf{a} \in A^m$, $b_1, \ldots, b_n \in [1]_\theta$ then

$$t(\mathbf{a}, b_1, \ldots, b_n) \, \theta \, t(\mathbf{a}, 1, \ldots, 1) = 1.$$

We will see that the 1-congruence classes of a residuated lattice are characterized as those subalgebras that are closed under the ideal terms λ, ρ and κ.

In analogy with groups, a subset S of a residuated lattice L is called *normal* if it is closed under λ and ρ.

S is called *convex* if for all $x \leq y \in S$, the elements $u \in L$ that satisfy $x \leq u \leq y$ are also in S.

Note: if S is closed under κ then it is a convex sublattice of L.

By the example above, every 1-congruence class is a convex normal subalgebra.

The converse requires some calculations in RL.

Note that $\cdot$ distributes over joins, hence it is order preserving.

$\backslash, /$ are order preserving in the numerator and order reversing in the denomintor.

Lemma. *Let H be a convex normal subalgebra of L, and define*

$$\theta_H = \{\langle a, b\rangle : d(a, b) \in H\}.$$

Then θ_H is a congruence of L and $H = [e]_{\theta_H}$.

The collection of all convex normal subalgebras of a residuated lattice L will be denoted by $\mathrm{CN}(L)$.

$\mathrm{CN}(L)$ is easily seen to be an algebraic lattice.

Meets in $\mathrm{CN}(L)$ coincide with intersections.

Theorem. [Blount Tsinakis] *For any residuated lattice L, $\mathrm{CN}(L)$ is isomorphic to $\mathrm{Con}(L)$, via the mutually inverse maps $H \mapsto \theta_H$ and $\theta \mapsto [e]_\theta$.*

The Generation of convex normal subalgebras.

Recall that H is a convex normal subalgebra of a residuated lattice L provided it is closed under the RL-ideal terms κ, λ, ρ and the basic operations of L.

For a subset S of L, let $\mathrm{cn}(S)$ denote the intersection of all convex normal subalgebras containing S.

When $S = \{s\}$, we write $\mathrm{cn}(s)$ rather than $\mathrm{cn}(\{s\})$.

Clearly $\mathrm{cn}(S)$ can also be generated from S by iterating the ideal terms and basic operations.

The next result shows that we may compute $\mathrm{cn}(S)$ by applying these terms in a particular order. Let

$$\Delta(S) = \{s \wedge 1/s \wedge 1 : s \in S\}$$
$$\Gamma(S) = \{\lambda_{u_1} \circ \rho_{u_2} \circ \lambda_{u_3} \cdots \rho_{u_{2n}}(s) : n \in \omega, u_i \in L, s \in S\}$$
$$\Pi(S) = \{s_1 \cdot s_2 \cdots s_n : n \in \omega, s_i \in S\} \cup \{1\}.$$

Thus $\Gamma(S)$ is the normal closure of S, and

$\Pi(S)$ is the submonoid generated by S.

Note also that if $S \subseteq L^-$ then $\Delta(S) = S$.

Theorem. [Blount Tsinakis] *The convex normal subalgebra generated by a subset S in a residuated lattice L is*

$$\mathrm{cn}(S) = \{a \in L : x \leq a \leq x\backslash e \text{ for some } x \in \Pi\Gamma\Delta(S)\}.$$

An element a of L is *negative* if $a \leq 1$.

The following corollary describes explicitly how the negative elements of a one-generated convex normal subalgebra are obtained.

For $\mathbf{u} = \langle u_1, \ldots, u_{2n} \rangle$ let $\gamma_{\mathbf{u}} = \lambda_{u_1} \circ \rho_{u_2} \circ \lambda_{u_3} \circ \cdots \circ \rho_{u_{2n}}$.

Corollary. *Let L be a residuated lattice and $r, s \in L^-$. Then $r \in \mathrm{cn}(s)$ if and only if for some m, n there exist $\mathbf{u}_i \in L^{2n}$ $(i = 1, \ldots, m)$ such that*

$$\gamma_{\mathbf{u}_1}(s) \cdot \gamma_{\mathbf{u}_2}(s) \cdots \gamma_{\mathbf{u}_m}(s) \leq r.$$

A totally ordered residuated lattice is called a *residuated chain*.

The variety generated by all residuated chains is denoted by RL^C.

The following result provides a finite equational basis for RL^C.

A similar basis was obtained independently by [van Alten 01] for the subvariety of integral members of RL^C, and for pseudo-MTL-algebras by [Kühr 03].

An algebra is *subdirectly irreducible* if it has a smallest non-trivial congruence.

[Birkhoff 41] Every algebra is embedded in a product of subdirectly irreducible homomorphic images, hence a variety is generated by its subdirectly irreducible members.

Theorem. [Blount Tsinakis]
RL^C *is the variety of all residuated lattices that satisfy*

$$1 = \lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y).$$

Proof. The identity holds in any residuated chain, since if $x \leq y$, then $1 \leq (x \vee y)\backslash y$, so $1 \leq \rho_v((x \vee y)\backslash y) \leq 1$.

Let $\mathcal{V}$ be the variety of residuated lattices defined by this identity.

We have just seen that $\mathsf{RL}^C \subseteq \mathcal{V}$.

To show $\mathcal{V} \subseteq \mathsf{RL}^C$, it suffices to show that all subdirectly irreducible members of $\mathcal{V}$ are totally ordered.

Let L be a subdirectly irreducible member of $\mathcal{V}$.

We show 1 is join-irreducible and deduce that L is totally ordered.

Consider $a, b \in L$ such that $a \vee b = 1$.

We will show that $\mathrm{cn}(a) \cap \mathrm{cn}(b) = \{1\}$.

Since we assumed L is subdirectly irreducible, it will follow that either $a = 1$ or $b = 1$.

Claim: If $x \vee y = 1$ then $\lambda_u(\rho_v(x)) \vee y = 1$. Here we make use of the defining identity for $\mathcal{V}$.

Assuming $x \vee y = 1$, we have

$$\lambda_u(x) \vee y = \lambda_u((x \vee y)\backslash x) \vee \rho_1((x \vee y)\backslash y) = 1.$$

Similarly $\rho_v(x) \vee y = 1$. Applying the second result to the first establishes the claim.

By the preceding corollary, the negative members of $\mathrm{cn}(a)$ are bounded below by finite products of iterated conjugates of a.

In any residuated lattice, if $a_i \vee b_j = 1$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ then $a_1 a_2 \cdots a_m \vee b_1 b_2 \cdots b_n = 1$.

Therefore $a \vee b = 1$ implies $a' \vee b' = 1$ for any $a' \in \mathrm{cn}(a) \cap L^-$ and $b' \in \mathrm{cn}(b) \cap L^-$.

Hence $\mathrm{cn}(a) \cap \mathrm{cn}(b) = \{1\}$.

Now choosing $u = v = 1$, the identity yields
$1 = ((x \vee y)\backslash x \wedge 1) \vee ((x \vee y)\backslash y \wedge 1)$.

So by join-irreducibility of 1 we have either
$1 = ((x \vee y)\backslash x \wedge 1)$ or $1 = ((x \vee y)\backslash y \wedge 1)$.

The first case implies $y \leq x$, and the second implies $x \leq y$. $\square$

Translation of positive universal sentences

Given a class $\mathcal{K}$, it is an interesting problem to obtain an explicit equational basis for $\mathrm{Var}(\mathcal{K})$.

If $\mathcal{K}$ is defined by positive universal sentences, we can apply the description of congruences to this problem.

Theorem. [Galatos, J. 02] *For any positive universal sentence φ there is a set of identities $E(\varphi)$ such that $L \models \varphi$ iff $L \models E(\varphi)$ for any subdirectly irreducible $L \in$ RL.*

We briefly indicate how the identities are constructed.

Any identity $s = t$ is equivalent to $1 \leq s\backslash t$ and $1 \leq t\backslash s$.

Any finite conjunction of identities $1 \leq s_i$ ($i = 1, \ldots, n$) is equivalent to the identity $1 = 1 \wedge s_1 \wedge \cdots \wedge s_n$.

So we may assume that φ is a universal disjunction of equations of the form $1 = t_i$ ($i = 1, \ldots, n$).

Let $C_0 = \{x_i : i \in \omega\}$ be a countable set of variables

$C_{n+1} = \{\lambda_{x_i}(\rho_{x_j}(t)) : i, j \in \omega,\ t \in C_n\}$

$C = \bigcup_{i \in \omega} C_i$, the set of iterated conjugates.

Let $E(\varphi) = \{e = \gamma_1(t_1) \vee \cdots \vee \gamma_n(t_n) : \gamma_1, \ldots, \gamma_n \in C\}$.

For the commutative case, the set $E(\varphi)$ can be replaced by the single equation $1 = t_1 \vee \ldots \vee t_n$.

The preceding result can be used to find a finite basis for the join of any two finitely based subvarieties of CRL (or FL_e or BL).

So for example it is straight forward to find equational bases for joins of MV, GA and PA.

Discriminator varieties

[Baaz 96] introduced the *projection* $\Delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$ as an additional operation in BL-algebras (it is not definable).

Δ maps fuzzy logic back into classical logic in a very strict way: anything that is not absolutely true is completely false.

In the logical setting Δ has the meaning of “very true”.

He then proved many nice consequences for BL_Δ.

To see the reason for this, we relate Δ to another another concept from Universal Algebra.

An algebra A is a *discriminator algebra* if there is a 3-ary term that satisfies

$$t(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}$$

Disciminator algebras were first studied by Foster in the 1950s.

[McKenzie 75] found a simple equational basis for the variety generated by the class of all algebras in which a fixed term t acts as the discriminator.

McKenzie also proved that if the algebras have two constants $0 \neq 1$, then any universal sentence φ can be translated to a term φ^* such that φ is equivalent to $\varphi^* = 1$ in all discriminator algebras.

This allows results from logic to be translated to equational logic.

Theorem. *A BL-algebra has the projection Δ iff it has a discriminator term t.*

Define $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ as in Boolean algebras.

Then $x \leftrightarrow y = 1$ iff $x = y$.

Now let $t(x, y, z) = z\Delta(x \leftrightarrow y) \vee x\,(\neg\Delta(x \leftrightarrow y))$.

Conversely, given t, define $\Delta(x) = \neg t(1, x, 0)$.

In fact this can be immediately generalized to bounded residuated lattices (without commutativity or integrality or other BL axioms).

Let $\Delta(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$

and define $x \leftrightarrow y = x\backslash y \wedge y\backslash x$, and $\neg x = x\backslash 0$.

Again, $x \leftrightarrow y \geq 1$ iff $x = y$, and $\neg 0 = T$ (top), $\neg 1 = 0$.

Now let $t(x, y, z) = z\Delta(x \leftrightarrow y) \vee x\,(\neg\Delta(x \leftrightarrow y) \wedge 1)$.

Conversely, given t, define $\Delta(x) = \neg t(1, x \wedge 1, 0) \wedge 1$.

Theorem. *A variety of bounded commutative residuated lattices is a discriminator variety iff Δ is a term function in all simple members iff $\Delta(x) = (x \wedge 1)^n$ for some fixed n in all simple members.*

Decomposition of generalized BL-algebras

The results in this section are due to [Galatos 03]. They clarify the connection between GBL, and its subvarieties of integral GBL-algebras and lattice-ordered groups.

Recall that $\cdot$ is *right-complemented* if $x \le y$ implies $\exists c(x = yc)$

or equivalently, with residuals, if $x \le y$ implies $x = y(y\backslash x)$.

In any residuated lattice this becomes $x \wedge y = y(y\backslash(x \wedge y))$.

Assuming integrality, it reduces to right-divisibility $x \wedge y = y(y\backslash x)$.

But what if we don't assume integrality? We still get distributivity, but not prelinearity since ℓ-groups are complemented (= right- and left-complemented).

GBL is the variety of $\cdot$-complemented residuated lattices.

For a residuated lattice L, define

invertible elements = $G(L) = \{x \in L : x(x\backslash 1) = 1 = (1/x)x\}$

integral elements = $I(L) = \{x \in L : x\backslash 1 = 1 = 1/x\}$.

Theorem. *For any* GBL*-algebra L, $G(L)$ and $I(L)$ are subalgebras, with $G(L) \in$* LG*, $I(L) \in$* IGBL*, and the map $f : G(L) \times I(L) \to L$ given by $f(x, y) = xy$ is an isomorphism.*

Corollary. *Any finite* GBL*-algebra is integral.*

For classes of algebras $\mathcal{K}_1$, $\mathcal{K}_2$ define

$$\mathcal{K}_1 \times \mathcal{K}_2 = \{A \times B : A \in \mathcal{K}_1, B \in \mathcal{K}_2\}.$$

Corollary. $\mathsf{GBL} = \mathsf{LG} \vee \mathsf{IGBL} = \mathsf{LG} \times \mathsf{IGBL}$ *and* $\mathsf{GMV} = \mathsf{LG} \vee \mathsf{IGMV} = \mathsf{LG} \times \mathsf{IGMV}$

Here GMV is defined as all residuated lattices that are *division complemented*: $x \leq y$ implies $\exists c, d(y = x/c$ and $y = d\backslash x)$

or by identities $x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$.

[Galatos 03] also shows that IGMV satisfies prelinearity: $1 \leq (x\backslash y \vee y\backslash x) \wedge (x/y \vee y/x)$.

Combined with the decomposition result one obtains the following.

Corollary. *The subdirectly irreducible members of* CGMV *are linearly ordered, hence* CGMV *is a subvariety of* RL^C.

Categorical equivalence of ℓ-groups with their negative cones

We now consider the categorical equivalence between lattice ordered groups and cancellative integral generalized BL-algebras [J, Tsinakis 02] [BCGJT] [van Alten], as a special case of the very general Morita equivalence in universal algebra [McKenzie 96].

Negative Cones of ℓ-Groups. Recall that the *negative part* of a residuated lattice L is $L^- = \{x \in L : x \leq 1\}$.

The *negative cone of L* is defined as $\langle L^-, \vee, \wedge, \cdot, e, /^{L^-}, \backslash^{L^-} \rangle$,

$$\text{where } a/^{L^-} b = a/b \wedge e \text{ and } a\backslash^{L^-} b = a\backslash b \wedge e.$$

It is easy to check that L^- is again a residuated lattice.

For a class $\mathcal{K}$ of residuated lattices, $\mathcal{K}^-$ denotes the class of negative cones of members of $\mathcal{K}$.

Using a standard construction to embed certain cancellative monoids in their groups of fractions one can prove the following.

Theorem. [Bahls Cole Galatos J, Tsinakis 03] LG^- *is a variety, defined by the identities* $xy/y = x = y\backslash yx$ *and* $(x/y)y = x \wedge y = y(y\backslash x)$. *Alternatively, the last two identities can be replaced by* $x/(y\backslash x) = x \vee y = (x/y)\backslash x$.

Corollary. *The variety* $\mathsf{CLG}^- = \mathrm{Var}(\mathbb{Z}^-)$ *is defined by the identities* $xy = yx$, $x = y\backslash yx$ *and* $x \wedge y = y(y\backslash x)$. *Alternatively, the last identity can be replaced by* $x \vee y = (y\backslash x)\backslash x$.

The Subvarieties of LG and LG^-. We now extend the map $^- : \mathsf{LG} \to \mathsf{LG}^-$ to subclasses of LG, and in particular to the lattice of subvarieties $\mathbf{L}(\mathsf{LG})$.

We show that the image of a variety is always a variety, that every subvariety of LG^- is obtained in this way and that the map is order preserving.

The proof is syntactical and shows how equational bases can be translated back and forth.

Independently, [van Alten 01] also discovered a basis for LG^-, by proving that it is term-equivalent to the variety of cancellative generalized hoops.

From Subvarieties of LG^- to Subvarieties of LG. In this direction, the translation is derived essentially from the definition of the negative cone.

For a residuated lattice term t, we define a translated term t^- by

$$x^- = x \wedge 1 \qquad (st)^- = s^- t^- \qquad 1^- = 1$$
$$(s/t)^- = s^-/t^- \wedge 1 \quad (s \vee t)^- = s^- \vee t^-$$
$$(s\backslash t)^- = s^-\backslash t^- \wedge 1 \quad (s \wedge t)^- = s^- \wedge t^-.$$

Theorem. *Let $\mathcal{V}$ be a subvariety of LG^-, defined by a set $\mathcal{E}$ of identities and let $\mathcal{W}$ be the subvariety of LG defined by the set of identities $\mathcal{E}^- = \{s^- = t^- : (s = t) \in \mathcal{E}\}$. Then $\mathcal{W}^- = \mathcal{V}$.*

As an example, consider the variety NLG^- that is defined by the identity $x^2y^2 \leq yx$ relative to LG^-.

The corresponding identity for the variety NLG of normal valued ℓ-groups is $(x \wedge 1)^2(y \wedge 1)^2 \leq (y \wedge 1)(x \wedge 1)$.

From Subvarieties of LG to Subvarieties of LG^-.

Since $\cdot$ and $^{-1}$ distribute over $\vee$ and $\wedge$, any LG identity is equivalent to a conjunction of two identities of the form $1 \le p(g_1, \ldots, g_n)$, where p is a lattice term and $g_1, \ldots, g_n$ are group terms.

Since ℓ-groups are distributive, this can be further reduced to a finite conjunction of inequalities of the form $1 \le g_1 \vee \cdots \vee g_n$.

For a term $t(x_1, \ldots, x_m)$ and a variable z distinct from $x_1, \ldots, x_m$, let

$$\bar{t}(z, x_1, \ldots, x_m) = t(z^{-1}x_1, \ldots, z^{-1}x_m).$$

Lemma. *Let $L \in$ LG. From any group term g one can (effectively) construct a RL term $\widehat{g}$ such that $(g \wedge e)^L|_{L^-} = \widehat{g}^{L^-}$.*

Theorem. *Let $\mathcal{V}$ be a subvariety of* LG*, defined by a set $\mathcal{E}$ of identities, which we may assume are of the form $1 \leq g_1 \vee \ldots \vee g_n$. Let*

$$\overline{\mathcal{E}} = \{1 = \widehat{g_1} \vee \ldots \vee \widehat{g_n} : 1 \leq g_1 \vee \ldots \vee g_n \text{ is in } \mathcal{E}\}.$$

Then $\overline{\mathcal{E}}$ is an equational basis for $\mathcal{V}^-$ relative to LG^-*.*

For example consider the variety RLG of *representable ℓ-groups* which (by definition) is generated by the class of linearly ordered groups.

An equational basis for this variety is given by $1 \leq x^{-1}yx \vee y^{-1}$ (relative to LG).

Applying the translation above, we obtain $1 = zx\backslash(zy/z)x \vee y\backslash z$ as an equational basis for RLG^-.

Corollary. *The map $\mathcal{V} \mapsto \mathcal{V}^-$ from $\mathbf{L}(\mathsf{LG})$ to $\mathbf{L}(\mathsf{LG}^-)$ is a lattice isomorphism, with the property that finitely based subvarieties of LG are mapped to finitely based subvarieties of LG^- and conversely.*

Categorical equivalence and the functor $L \mapsto L^-$

The connection between LG and LG^- is actually a special case of a categorical equivalence.

In the algebraic setting such equivalences were characterized by [McKenzie 96].

This is a generalization of Morita's celebrated theorem, which gives concrete conditions on two rings with unit that characterize when the varieties of left unitary modules over these rings are categorically equivalent.

Let A be an algebra, and let T be the set of all terms in the language of A.

Given a unary term σ we define a new algebra called the *σ-image* of A by

$$A(\sigma) = \langle \sigma(A), \{t_\sigma : t \in T\} \rangle$$

where $t_\sigma^{A(\sigma)}(x_1, \ldots, x_n) = \sigma^A(t^A(x_1, \ldots, x_n))$.

The second construction is the *matrix power* of A.

Let T_k be the set of k-ary terms. For a positive integer n we define

$$A^{[n]} = \langle A^n, \{m_t : t \in (T_{kn})^n \text{ for some } k > 0\} \rangle,$$

where $m_t : (A^n)^k \longrightarrow A^n$ is given by

$$m_t(\bar{x}_1, \ldots, \bar{x}_k)_i = t_i^A(x_{11}, \ldots, x_{1n}, \ldots, x_{k1}, \ldots, x_{kn}).$$

For a class $\mathcal{K}$ of algebras we let $\mathcal{K}(\sigma)$ and $\mathcal{K}^{[n]}$ be the classes of σ-images and n-th matrix powers respectively.

A term σ is *idempotent in* $\mathcal{K}$ if $\mathcal{K} \models \sigma(\sigma(x)) = \sigma(x)$.

It is *invertible in* $\mathcal{K}$ if there exist unary terms $t_1, \ldots, t_n$ and an n-ary term t (for some $n > 0$) such that

$$\mathcal{K} \models x = t(\sigma(t_1(x)), \ldots, \sigma(t_n(x))).$$

A central result of [McKenzie 96] is the following.

Theorem. *Two varieties $\mathcal{V}$ and $\mathcal{W}$ are categorically equivalent if and only if there is an $n > 0$ and an invertible idempotent term σ for $\mathcal{V}^{[n]}$ such that $\mathcal{W}$ is term-equivalent to $\mathcal{V}^{[n]}(\sigma)$.*

In the setting of ℓ-groups and their negative cones, we can see an instance of this result.

The term $\sigma(x) = x \wedge 1$ is certainly idempotent.

It is invertible (with $n = 2$) since $x = (x \wedge 1)(x^{-1} \wedge 1)^{-1}$ holds in all ℓ-groups.

Of course $L(\sigma)$ is not of the same type as L^{-}, but they are term equivalent.

In the other direction, every member of LG^{-} can be mapped to a τ-image of a matrix square that is term-equivalent to an ℓ-group.

In general, the term τ is given by

$$\tau(\bar{x}) = \langle \sigma t_1 t(\bar{x}), \ldots, \sigma t_n t(\bar{x}) \rangle.$$

This reduces to $\tau(\langle x, y \rangle) = \langle x/y, y/x \rangle$ for negative cones.

Categorical equivalence of GMV and ℓ-group expansions

[Chang 59] showed that every subdirectly irreducible MV-algebra can be obtained from an interval in an abelian ℓ-group.

[Mundici 86] generalized this result to a categorical equivalence between the category of MV-algebras and the category of abelian ℓ-groups with a strong unit.

$u \in G$, an ℓ-group, is a *strong unit* if $G = \bigcup_{1 \leq n} [u^{-n}, u^n]$.

[Dvurečenskij 02] removed commutativity and extended this equivalence to the larger catgories of pseudo MV-algebras and ℓ-groups with strong unit.

Recently [Galatos 03] extended it further to generalized MV-algebras (removing integrality and the existence of a least element) by replacing the strong unit by a nucleus-kernel composition.

A *closure operator* on a residuated lattice L is an order-preserving map $\gamma : L \to L$ such that $x \leq \gamma(x) = \gamma(\gamma(x))$.

A *interior operator* on L is the dual concept, i.e. an order-preserving map $\delta : L \to L$ such that $x \geq \delta(x) = \delta(\delta(x))$.

A *nucleus* is a closure operator such that $\gamma(x)\gamma(y) \leq \gamma(xy)$.

A *kernel* is an interior operator such that
$\delta(\delta(x)\delta(y)) = \delta(x)\delta(y)$, $\delta(1) = 1$ and
$\delta(x) \wedge y = \delta(\delta(x) \wedge y)$.

Let LG^* be the category with objects $\langle G, \beta \rangle$, where G is an ℓ-group and $\beta(x) = \gamma(\delta(x))$ for some nucleus γ and some kernel δ on G and $\beta[G] = G$. The morphisms are the homomorphisms for the expanded ℓ-groups.

Theorem. [Galatos 03] *The categories LG^* and GMV are equivalent.*

Given a term t, and a variable z that does not appear in t, let t_z be the term obtained by replacing every variable v in t by $v \vee z$.

Theorem. [Galatos 03] *An identity $s = t$ holds in IGMV iff the identity $s_z = t_z$ holds in LG^-.*

Combining this result with the earlier decomposition result for GMV Galatos obtains the following result.

Corollary. *An identity $s = t$ holds in GMV iff $s = t$ holds in LG and $s_z = t_z$ holds in LG^-.*

[Holland and McCleary 79] showed that LG is decidable, and this transfers to LG^- by the categorical equivalence discussed earlier.

Corollary. *The equational theories of IGMV and GMV are decidable.*

The decision procedure for ℓ-groups has recently been implemented on the web at **`www.chapman.edu/~jipsen/`** and has been modified to also handle the varieties IGMV and GMV.

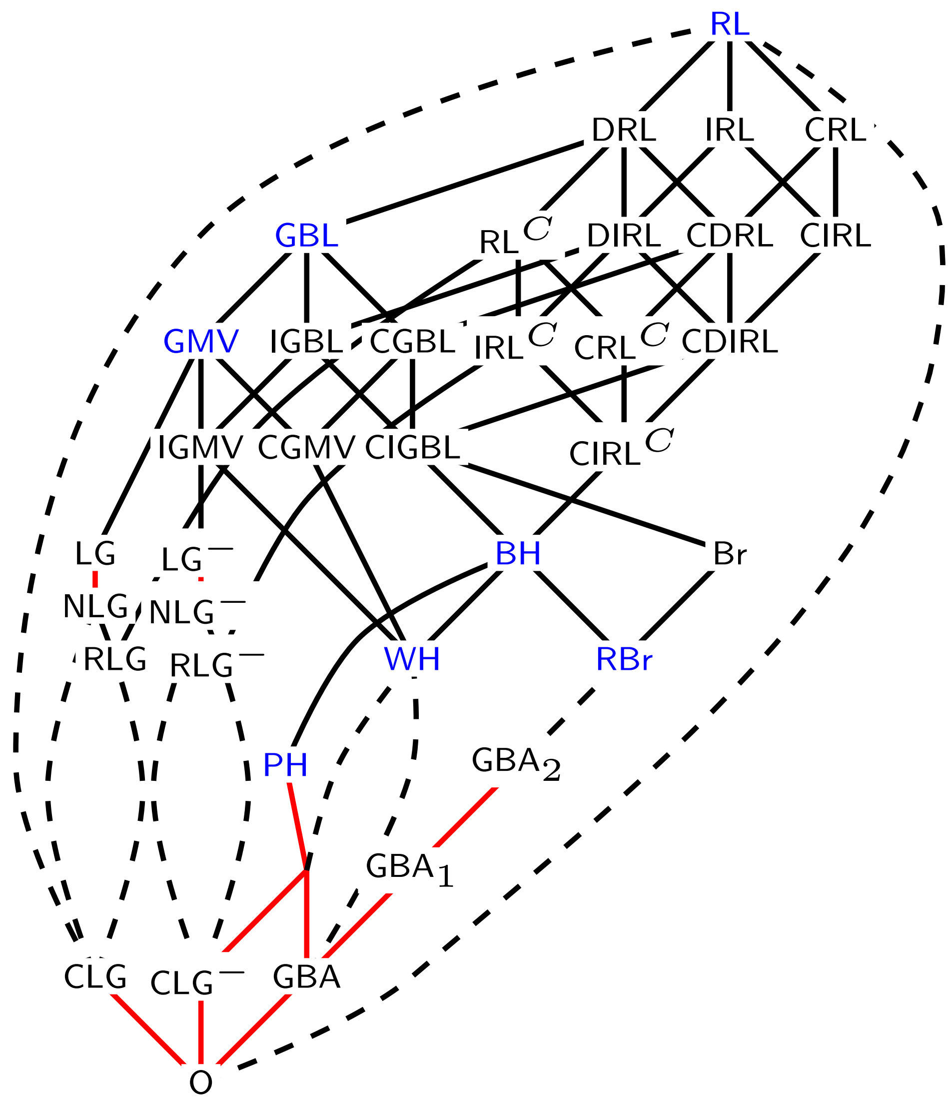

Figure 3: Some subvarieties of RL ordered by inclusion

Size	$n =$	3	4	5	6
FL	RL	3	20	149	1488
DFL	DRL	3	20	115	899
FL_e	CRL	3	16	100	794
FL_w	IRL	3	9	49	364
psMTL_{SI}	Fleas_{SI}	2	8	44	308
MTL_{SI}	CIRL^{C}_{SI}	2	6	22	94
	GBL_{SI}	2	4	8	16
	GMV_{SI}	1	1	1	1

*

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