Basic Logic algebras and lattice-ordered groups as algebras of binary relations

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- **Algebras of binary relations**
- **Embedding ℓ-groups**
- **Embedding BL-algebras**
- **Finite representable generalized BL-algebras**

Algebras of binary relations

Let us recall some standard results:

M. Stone: Every Boolean algebra is isomorphic to a subalgebra of all subsets of some set U, with $\cup, \cap, -, \emptyset, U$ as operations.

C. Holland: Every ℓ-group is isomorphic to a subalgebra of all order-automorphisms of a chain, with pointwise order and $\circ,^{-1}, id$ as operations.

Let R, S be binary relations ($\subseteq U^2$)

relation composition:

$$R \circ S = \{(u, v) : \exists w\ (u, w) \in R \text{ and } (w, v) \in S\}$$

inverse: $R^{-1} = \{(v, u) : (u, v) \in R\}$ and

identity: $id_U = \{(u, u) : u \in U\}$

A *representable relation algebra on* U is a set A of relations that is closed under $\cup, \cap, -, \circ,^{-1}, id_U$.

RRA = class of all algebras isomorphic to representable relation algebras

Tarski: RRA is a variety.

Monk '64: RRA is not finitely axiomatizable.

$\circ$ is like a multiplication

$\circ, \cap$ distribute over $\cup$ as in ℓ-groups

Naive question: Can we embed ℓ-groups into representable relation algebras?

Well, we don't need complementation.

If $R \circ R^{-1} = id_U = R^{-1} \circ R$ then R is a permutation.

So any ℓ-group element would have to map to a permutation.

But this is incompatible with preserving the order of the ℓ-group since distinct permutation are disjoint as relations.

Also, if $R, S \subseteq id_U$ then $R \circ S = R \cap S$.

But this is certainly no true in ℓ-groups.

So forget about ${}^{-1}, id_U$ and instead look at the "residuated lattice reducts" of relation algebras.

residuals: $R\backslash S = \{(u, v) : R \circ \{(u, v)\} \subseteq S\}$ and
$R/S = \{(u, v) : \{(u, v)\} \circ S \subseteq R\}$

Definition: A *residuated lattice of (binary) relations* is a set A of relations that is closed under $\cup, \cap, \circ, \backslash, /$ and contains a relation 1 such that $1 \circ R = R \circ 1 = R$ for all $R \in A$.

(Note that 1 is usually not the identity relation.)

RLR denotes the quasivariety of all residuated lattices of relations.

Problem 1. Is RLR a variety?

It is obvious that every residuated lattice of relations is a distributive residuated lattice.

Problem 2. Is the converse also true?

Andreka [1991] proved a general result that implies RLR is not finitely axiomatizable.

Since distributive residuated lattices form a finitely axiomatizable variety, the answer to Problem 2 would be no if RLR is a variety.

Embedding ℓ-groups

LG denotes the variety of *lattice-ordered groups* (residuated lattices that satisfy $x(x\backslash 1) = 1$, so $x^{-1} = x\backslash 1$).

They are distributive residuated lattices.

Question: Is $\mathsf{LG} \subseteq \mathsf{RLR}$?

This is answered by the following result.

Theorem. *Every ℓ-group is isomorphic to a residuated lattice of relations, hence $\mathsf{LG} \subseteq \mathsf{RLR}$.*

Proof. Let $G = \langle \mathrm{Aut}(\Omega), \vee, \wedge, \circ, id_\Omega, \backslash, / \rangle$ be the ℓ-group of order-automorphisms of a chain Ω.

Note that $\vee, \wedge$ are calculated pointwise.

By Holland's embedding theorem, it suffices to embed G into a residuated lattice of relations on Ω.

For $g \in G$, let $R_g = \{(u,v) : u \leq g(v)\}$.

$R_g \cap R_h = R_{g \wedge h}$ since

$$\begin{aligned}
& (u,v) \in R_g \cap R_h \\
\iff \quad & u \leq g(v) \text{ and } u \leq h(v) \\
\iff \quad & u \leq \min\{g(v), h(v)\} = (g \wedge h)(v) \\
\iff \quad & (u,v) \in R_{g \wedge h}
\end{aligned}$$

$R_g \cup R_h = R_{g \vee h}$ is similar, using max.

$R_g \circ R_h = R_{g \circ h}$ since

$$\begin{aligned}
& (u,v) \in R_g \circ R_h \\
\iff \quad & \exists w \; [(u,w) \in R_g \text{ and } (w,v) \in R_h] \\
\iff \quad & \exists w \; [u \leq g(w) \text{ and } w \leq h(v)] \\
\iff \quad & u \leq g(h(v)) \qquad\qquad (w = h(v) \text{ for } \Longleftarrow) \\
\iff \quad & (u,v) \in R_{g \circ h}
\end{aligned}$$

$R_g \backslash R_h = R_{g \backslash h}$ since

$$\begin{aligned}
& (u,v) \in R_g \backslash R_h \\
\iff \quad & R_g \circ \{(u,v)\} \subseteq R_h \\
\iff \quad & \forall w \; [(w,u) \in R_g \implies (w,v) \in R_h] \\
\iff \quad & \forall w \; [w \leq g(u) \implies w \leq h(v)] \\
\iff \quad & g(u) \leq h(v) \\
\iff \quad & u \leq g^{-1}(h(v)) = (g \backslash h)(v) \\
\iff \quad & (u,v) \in R_{g \backslash h}
\end{aligned}$$

$R_g / R_h = R_{g/h}$ since

$$\begin{aligned}
& (u,v) \in R_g / R_h \\
\iff \quad & \{(u,v)\} \circ R_h \subseteq R_g \\
\iff \quad & \forall w \; [(v,w) \in R_h \implies (u,w) \in R_g] \\
\iff \quad & \forall w \; [v \leq h(w) \implies u \leq g(w)] \\
\iff \quad & \forall w \; [h^{-1}(v) \leq w \implies g^{-1}(u) \leq w] \\
\iff \quad & g^{-1}(u) \leq h^{-1}(v) \text{for } \Longrightarrow)
\end{aligned}$$

$\iff \quad u \leq g(h^{-1}(v)) = (g/h)(v)$

$\iff \quad (u, v) \in R_{g/h}$

Finally, $R_{id} = \{(u, v) : u \leq v\} =$ "$\leq$" is an identity element since

$R_g \circ R_{id} = R_{g \circ id} = R_g = R_{id} \circ R_g$.

Therefore $\{R_g : g \in G\}$ is a residuated lattice of relations that is isomorphic to G. $\square$

Embedding BL-algebras

Theorem. *Every BL-algebra is isomorphic to some algebra of relations*

Proof. The MV-algebra on $[0, 1]$ is isomorphic to $\{M_r : r \in [0, 1]\}$ where $M_r = \{(u, v) \in (0, 1]^2 : v \leq u - 1 + r\}$.

The Gödel algebra on $[0, 1]$ is isomorphic to $\{G_r : r \in [0, 1]\}$ where $G_r = \{(u, v) \in (0, 1]^2 : v \leq \min\{u, r\}\}$.

The product algebra on $[0, 1]$ is isomorphic to $\{P_r : r \in [0, 1]\}$ where $P_r = \{(u, v) \in (0, 1]^2 : v \leq r \cdot u\}$.

To complete the proof it suffices to show that RLR is closed under ordinal sums of integral members.

Suppose $A, B \in \mathsf{RLR}$, with A integral, $A \subseteq \mathcal{P}(U^2)$ and $B \subseteq \mathcal{P}(V^2)$, where U and V are disjoint.

Define $C = A \cup \{R \cup 1^A : R \in B$. Then it is easy to check that $C \cong A \oplus B$.

Note that integrality of A is required to ensure that $1^A \cup 1^B$ is an identity of C. $\square$

Finite representable generalized BL-algebras

Generalized basic logic algebras, or *GBL-algebras* for short, are residuated lattices that satisfy

$x \wedge y = ((x \wedge y)/y)y \quad$ and $\quad x \wedge y = y(y\backslash(x \wedge y))$.

The variety of GBL-algebras contains LG, as well as the variety of *basic hoops* (defined by adding $xy = yx$ and $x \wedge y = (x/y)y$) to RL).

A residuated lattice is *integral* if the identity 1 is the top element.

This condition holds for basic hoops since $x \wedge 1 = (x/1)1 = x$.

A GBL-algebra is called a *GBL-chain* if it is linearly ordered.

GBL-chains generate the variety of *representable GBL-algebras*.

Pseudo BL-algebras are bounded integral GBL-algebras expanded with a constant 0 denoting the least element, and that satisfy *prelinearity*: $x\backslash y \vee y\backslash x = 1 = x/y \vee y/x$.

BL-algebras are commutative pseudo BL-algebras (in which case prelinearity can be derived from the basic hoop axioms).

Hajek [1998] proved that all subdirectly irreducible BL-algebras are BL-chains.

Also, by definition, BL-algebras are commutative and integral.

The lattice-reduct of any GBL-algebra is distributive (see e.g. J. and Tsinakis [2002])

But in general, GBL-algebras are neither integral nor commutative nor representable, (consider any nonrepresentable ℓ-group).

Note that if a GBL-algebra has a top element $\top$, then $\top = 1$:

From $x \wedge y = y(y\backslash(x \wedge y))$, we deduce

$$1 = 1 \wedge \top = \top(\top\backslash(1 \wedge \top)) = \top(\top\backslash 1)$$

hence $\top = \top 1 = \top\top(\top\backslash 1) = \top(\top\backslash 1) = 1$.

(More generally this shows 1 is a maximal idempotent in any GBL-algebra.)

Therefore any finite GBL-algebra is integral.

We now prove that all finite GBL-chains are in fact commutative and hence basic hoops.

This result also holds for pseudo BL-chains (since any finite GBL-chain can be expanded to a pseudo BL-chain by adding a constant 0 to denote the least element)

Problem 3. Are all finite GBL-algebras commutative?

The GBL identities are equivalent to the following property that is also called *divisibility*:

$x \leq y \implies (\exists z(x = zy)$ and $\exists z(x = yz))$.

As usual, the symbol $\prec$ denotes the covering relation,
and $\preceq$ denotes the covering-or-equal relation.

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and $\preceq$ denotes the covering-or-equal relation.

Lemma 1. *In any integral GBL-chain, if $a \prec b$ then for all c we have $ac \preceq bc$ and $ca \preceq cb$.*

Proof. We show the contrapositive. Suppose a, b, c, x are elements in an integral GBL-chain such that $ac < x < bc$.

By integrality $x \le c$, hence by divisibility $\exists z$ such that $x = zc$.

Now $zc < bc \implies b \not\le z$, so the linear order implies $z < b$.

Similarly, $ac < zc$ implies $a < z$.

Hence a is not covered by b.

The argument for $ca < x < cb$ is similar. □

Let L and M be integral residuated lattices with no elements in common.

Suppose further that the identity element of L is join-irreducible.

The *ordinal sum* of L and M is an integral residuated lattice defined on the set $(L \setminus \{1\}) \cup M$ as follows:

$\cdot$ restricted to L and M agrees with the original product on L and M respectively, and for $x \in L \setminus \{e\}$ and $y \in M$,

$$x \cdot y = x = y \cdot x.$$

The order on the ordinal sum also agrees with the original order on L and M, and all elements of $L \setminus \{e\}$ are below all elements of M.

Let a be an idempotent element ($aa = a$) of an integral residuated lattice L.

Then it is easy to check that $\uparrow a = \{x \in L : x \le a\}$ is a subalgebra of L.

Similarly, $\downarrow a = \{x \in L : x \le a\}$ is closed under $\cdot$ and the lattice operations, and

$\cdot$ is residuated by the operations

$$x \backslash^{\downarrow} y = x \backslash y \wedge a \quad \text{and} \quad x /^{\downarrow} y = x / y \wedge a.$$

The next result shows that for finite GBL-chains a acts as the identity element on $\downarrow a$.

Lemma 2. *If a finite GBL-chain L contains an idempotent $a \neq 0, 1$ then $\downarrow a$ is a residuated lattice with identity $1^{\downarrow} = a$, and L decomposes as the ordinal sum of $\uparrow a$ and $\downarrow a$.*

Proof. To conclude that $\downarrow a$ is a residuated lattice, it suffices to show that $ax = x = xa$ for all $x \leq a$.

This follows from the preceding lemma since L is a finite chain,

$aa = a$, $a0 = 0$, and the map $x \mapsto ax$ preserves $\preceq$.

To see that L decomposes as an ordinal sum,

note that if $x \leq a$ and $y \geq a$ then $yx = x$,

since $1x = x$, $ax = x$, and $\cdot$ is order preserving.

Similarly $xy = x$. □

Since ordinal sums of commutative integral GBL-algebras are commutative, it now only remains to show that every finite integral GBL-algebras without idempotents (other than 0 and 1) is commutative.

Lemma. *Let $A = \{a_0, a_1, \ldots, a_n\}$ be the elements of a finite GBL-algebra, with $a_0 = 1$, $a_n = 0$ and $a_i \succ a_{i+1}$ for $i < n$.*

Suppose that A has no idempotents other than $1, 0$, and that for some fixed $m \leq n$ and all $i + j < m$ we have $a_i \cdot a_j = a_{i+j}$.

Then $a_{m-k} \cdot a_k = a_m$ for all $k \leq m$.

Lemma 3. *Let $A = \{a_0, a_1, \ldots, a_n\}$ be the elements of a finite GBL-algebra, with $a_0 = 1$, $a_n = 0$ and $a_i \succ a_{i+1}$ for $i < n$.*

Suppose that A has no idempotents other than $1, 0$, and that for some fixed $m \leq n$ and all $i + j < m$ we have $a_i \cdot a_j = a_{i+j}$.

Then $a_{m-k} \cdot a_k = a_m$ for all $k \leq m$.

Proof. Assume the stated conditions hold, and let $k \leq m$.

If $k = 0$, then the conclusion follows immediately.

For $k = 1$, Lemma 1 implies that $a_{m-1} \cdot a_1$ is either a_{m-1} or a_m,

since $a_{m-1} \cdot a_0 = a_{m-1}$.

We claim that the first case is impossible since it implies that a_{m-1} is idempotent.

This follows from the observation that if $a_{m-1} \cdot a_1 = a_{m-1}$ then $a_{m-1} \cdot a_1 \cdot a_1 \cdots a_1 = a_{m-1}$, and

the product of $m - 1$ copies of a_1 is a_{m-1} by assumption.

Therefore $a_{m-1} \cdot a_1 = a_m$. (End of basis step)

Now suppose that $a_{m-(k-1)} \cdot a_{k-1} = a_m$.

Then $a_{m-k} \cdot a_k = a_{m-k} \cdot a_1 \cdot a_{k-1} = a_{m-k+1} \cdot a_{k-1} = a_m$.

So the desired result follows by induction on k. $\square$

The *n-element Wajsberg chain* is a basic hoop with elements $a_0 \succ a_1 \succ \cdots \succ a_{n-1}$ such that $a_i \cdot a_j = a_{\min(i+j,n-1)}$, hence commutative.

Theorem. *Every finite GBL-chain is commutative (hence a basic hoop).*

Proof. Suppose A is a GBL-chain that has elements $a_0 \succ a_1 \succ \cdots \succ a_n = 0$.

Any finite GBL-algebra is integral, hence $a_0 = 1$.

If $n = 1$, then A is the 2-element Wajsberg chain (= BA).

Now suppose $n > 1$. If a_i is idempotent for some $0 < i < n$,

then A decomposes by Lemma 2 into the ordinal sum of two smaller GBL-chains.

So we may assume that A has no idempotents other that $1, 0$.

Therefore by Lemma 1, $a_1 \cdot a_1 = a_2$.

If $n = 2$, then A is the 3-element Wajsberg chain, and

if $n > 2$, then the assumptions of Lemma 3 are satisfied with $m = 3$.

Using this lemma as the inductive step we see that

A has the structure of the $n+1$-element Wajsberg chain. $\square$

So the finite GBL-chains are just ordinal sums of Wajsberg chains.

This makes it easy to count the number nonisomorphic (G)BL-chains with n elements.

We just have to choose which of the $n-2$ elements between 1 and 0 are idempotents.

For each of the 2^{n-2} different choices we obtain a nonisomorphic (G)BL-chain.

Corollary. *For $n > 1$ there are 2^{n-2} GBL-chains with n elements.*

Since there are noncommutative representable integral GBL-algebras, we also have the following result.

Corollary. *The variety of representable GBL-algebras and the variety of integral representable GBL-algebras are not generated by their finite members (i.e. they do not have the finite model property).*

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References

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