# MINIMAL EXPANSIONS OF SEMILATTICES

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Abstract. We determine the minimal extension of the sequence $\langle 0, 1, 1, \ldots, 1, 2\rangle$. This completes and extends the work of K. M. Koh, started in 1970, and solves Problem 15 in the survey on $p_n$-sequences and free spectra [GK92]. The results involve the investigation of some minimal expansions of semilattices.

## 1. Introduction

Let $\mathbf{A}$ be any algebra, and let $p_n = p_n(\mathbf{A})$ be the number of essentially $n$-ary term operations on $\mathbf{A}$. This sequence is closely connected with the *free spectrum* of $\mathbf{A}$, and has received considerable attention in universal algebra (see [GK92] for an extensive survey).

A finite sequence $\mathbf{a} = \langle a_0, a_1, \ldots, a_k\rangle$ is *represented by an algebra* $\mathbf{A}$ if $p_n(\mathbf{A}) = a_n$ for all $n \leq k$. If the collection of $p_n$-sequences of all algebras that represent $\mathbf{a}$ has a (pointwise) minimum member $\mathbf{p}$, then $\mathbf{a}$ is said to have the *minimal extension property* (MEP), and $\mathbf{p}$ is called the *minimal extension of* $\mathbf{a}$. This notion was introduced by G. Grätzer in 1970, and since then the MEP has been proved for many sequences (cf. [GK92]). It is noteworthy that still no finite sequence is known without this property.

Some authors have considered the MEP restricted to certain classes of algebras. In particular, J. Dudek considered the MEP of the sequence $\langle 0, 1, 2\rangle$ in the class $\mathcal{C}$ of algebras whose clone is generated by two commutative binary operations. He showed in [Du83] that if this sequence has the MEP in $\mathcal{C}$, then it is represented by the two-element distributive lattice $\mathbf{D}_2$. More recently, in [Du97], he proved that there exist further two four-element algebras $\mathbf{N}_2$ and $\mathbf{A}_4$ such that the $p_n$-sequence of any algebra $\mathbf{A} \in \mathcal{C}$ is pointwise greater or equal to $p_n(\mathbf{D}_2)$, $p_n(\mathbf{N}_2)$ or $p_n(\mathbf{A}_4)$.

Hence, what remained in this case was to compare the $p_n$-sequences of the three concrete algebras. It was known that $p_2 = 2$ for all three

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algebras, $p_3(\mathbf{N}_2) = 10 > 9 = p_3(\mathbf{D}_2)$, and $p_4(\mathbf{N}_2) = 114 = p_4(\mathbf{D}_2)$. Using a computer, and some optimizations to make the computation feasible, Burris and Willard [BW96] showed that $p_5(\mathbf{N}_2) = 2586 < 6894 = p_5(\mathbf{D}_2)$, thus completing the proof that $\langle 0, 1, 2\rangle$ has no MEP in the class $\mathcal{C}$.

In [Ki03], A. Kisielewicz proved that it is generally the case that for every MEP problem there exists a finite set of finite algebras such that it suffices to compare the $p_n$-sequences of these algebras.

The present paper is a result of our attempt to find an example of a finite sequence without MEP in a general case, following the approach of Dudek, Burris and Willard. Note that the sequence $\langle 0, 1, 2\rangle$ has the MEP (in the class of all algebras), represented by any rectangular band $\mathbf{B}$ with one essentially binary operation (in which case $p_n(\mathbf{B}) = 0$ for all $n > 2$).

A good candidate seemed to be the sequences $\langle 0, 1, 1, \ldots, 1, 2\rangle$ of length $m + 1$. For $\langle 0, 1, 1, 2\rangle$, i.e. $m = 3$, K. M. Koh [Ko72] proved that there are two algebras $\mathbf{S}_3$ and $\mathbf{T}_3$, with 5 elements each, such that any algebra $\mathbf{A}$ that represents $\langle 0, 1, 1, 2\rangle$ has $p_n(\mathbf{A}) \geq p_n(\mathbf{S}_3)$ for all $n$ or $p_n(\mathbf{A}) \geq p_n(\mathbf{T}_3)$ for all $n$. The problem of comparing the two $p_n$-sequences of $\mathbf{S}_3$ and $\mathbf{T}_3$ has been open for three decades, and appears as Problem 15 in [GK92]. Koh considered also a general case for $\langle 0, 1, 1, \ldots, 1, 2\rangle$, but here he has obtained weaker results ([Ko72], Theorem 19 [GK92]), and does not produce the set of algebras representing minimal $p_n$-sequences.

The small numbers $p_n$ on the one hand (considerably narrowing the possibilities to be considered), and the apparent difficulties in establishing definite results on the other hand, suggested that the situation in this case might be similar to that considered by Dudek, Burris and Willard. Yet in the end we proved that all these sequences have the MEP. This not only solves Problem 15 in [GK92], but also generalizes the solution to the case of $\langle 0, 1, 1, \ldots, 1, 2\rangle$.

In particular, we prove that there are two algebras $\mathbf{S}_m$ and $\mathbf{T}_m$ with $m + 2$ elements, such that any algebra $\mathbf{A}$ that represents the sequence $\langle 0, 1, 1, \ldots, 1, 2\rangle$ of length $m + 1$ has a homomorphic image from a subreduct onto either $\mathbf{S}_m$ or $\mathbf{T}_m$. This implies that either $p_n(\mathbf{A}) \geq p_n(\mathbf{S}_m)$ for all $n$, or $p_n(\mathbf{A}) \geq p_n(\mathbf{T}_m)$ for all $n$. For $m = 3$, this result is due to K. M. Koh [Ko72]. In the second half of this paper we then prove that $p_n(\mathbf{S}_m) \leq p_n(\mathbf{T}_m)$ for all $n$.

It is our pleasure to thank Ralph McKenzie for a suggestion that simplified our proofs in the last section. We also thank the participants of the Vanderbilt Algebra Seminar for many helpful comments.

## 2. Algebras representing $\langle 0, 1, \ldots, 1, 2\rangle$

Throughout this section, let $m \geq 2$ be fixed, and let $\mathbf{A}$ be an algebra such that $p_0(\mathbf{A}) = 0$, $p_i(\mathbf{A}) = 1$ for $i = 1, \ldots, m-1$, and $p_m(\mathbf{A}) = 2$. In particular this implies that $\mathbf{A}$ has no constant operations, and only the identity function as unary operation.

As we have noted, for $m = 2$, any rectangular band with an essentially binary operation represents the minimal extension of $\langle 0, 1, 2\rangle$. So we consider $m > 2$. In this case $\mathbf{A}$ has a unique essentially binary operation, which we denote by $x \cdot y$ or $xy$.

**Lemma 2.1.** *If $m > 2$, then $\langle A, \cdot\rangle$ is a semilattice.*

*Proof.* Since $\mathbf{A}$ has no constant operations, it follows that $xx$ is essentially unary, and since the identity operation is the only essentially unary operation on $\mathbf{A}$, the idempotent law $xx = x$ holds. Commutativity of $\cdot$ follows from the fact that $\mathbf{A}$ has exactly one binary operation.

Now consider the operation $g(x, y, z) = (xy)z$. Substituting $x$ for $y$, we get $g(x, x, z) = (xx)z = xz$. Since $xz$ depends on both $x$ and $z$, it follows that $g$ depends on the variable $z$, as well as on at least one of $x$ or $y$. But $g$ is symmetric in the first two variables since $\cdot$ is commutative, so we conclude that $g$ depends on both $x$ and $y$. This shows that $g$ is essentially ternary. The assumption that $m > 2$ implies there is at most one other essentially ternary operation on $\mathbf{A}$. Therefore two of the operations $g(x, y, z)$, $g(y, z, x)$, $g(z, x, y)$ must be identical, and by symmetry we can assume it is the first two. Hence $(xy)z = (yz)x = x(yz)$, where the second equality follows by commutativity. □

On the basis of this lemma, we will refer to an algebra that represents the length $m+1$ sequence $\langle 0, 1, \ldots, 1, 2\rangle$ as an *m-ary semilattice expansion*. The argument in the above proof, which shows that $g$ is essentially ternary, will be used repeatedly in the following form.

**Lemma 2.2.** *Suppose $h(x_1, \ldots, x_n)$ is an operation that satisfies the identity*

$$h(x_1, x_2, x_3, \ldots, x_k, x_{k+1}, \ldots, x_n) = h(x_2, x_3, \ldots, x_k, x_1, x_{k+1}, \ldots, x_n)$$

*for some $k \leq n$. If $h$ depends on $x_i$ for **some** $i \leq k$, then $h$ depends on $x_j$ for **all** $j \leq k$.*

It follows from Lemma 2.1 that the semilattice operation $\cdot$ gives rise to an essentially $n$-ary operation $x_1 \cdot x_2 \cdots x_n$ for all $n > 1$. Since we are assuming that $p_m = 2$, there exists exactly one other essentially $m$-ary term operation of $\mathbf{A}$ which we denote by $f$. This operation

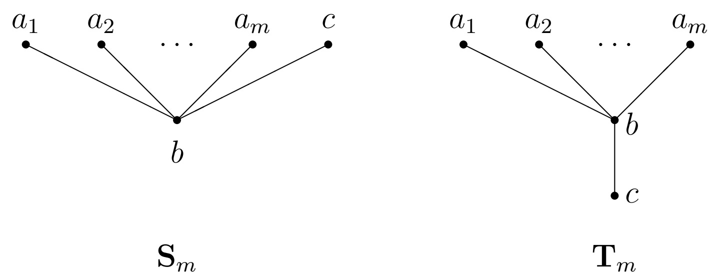


FIGURE 1. The semilattice orders of $\mathbf{S}_m$ and $\mathbf{T}_m$

must be symmetric in all its variables, otherwise there would be further essentially $m$-ary term operations.

We are interested in the structure of algebras $\mathbf{A}$ that represent the $(m+1)$-tuple $\langle 0, 1, \dots, 1, 2\rangle$ and have as few essentially $n$-ary term operations as possible for $n > m$. Hence we will assume from now on that $\mathbf{A}$ only has the fundamental operations $\cdot$ and $f$.

Since $f$ is distinct from the $m$-ary semilattice operation, there exist $a_1, \dots, a_m, b, c \in A$ such that

$$a_1 \cdot a_2 \cdots a_m = b \neq c = f(a_1, \dots, a_m).$$

We now consider the term $f(x_1, \dots, x_m)x_1x_2 \cdots x_m$. Since it is symmetric, it is essentially $m$-ary, and therefore equals either $x_1x_2 \cdots x_m$ or $f(x_1, \dots, x_m)$. In the first case it follows that $bc = c$, hence $c \leq b$, and in the second case $bc = b$, hence $b \leq c$.

In the remainder of this section we prove that the subalgebra of $\mathbf{A}$ that is generated by $a_1, \dots, a_m$ has a homomorphic image with $m+2$ elements $a'_1, \dots, a'_m, b', c'$ and that the algebraic structure of this image is completely determined apart from the dichotomy mentioned above (see Figure 1).

**Lemma 2.3.** *For any $i \in \{2, \dots, m\}$, $\mathbf{A}$ satisfies the identity*

$$f(x_1x_2 \cdots x_i, x_2x_3 \cdots x_{i+1}, \dots, x_mx_1 \cdots x_{i-1}) = x_1 \cdots x_m$$

*where addition of indices is calculated modulo $m$.*

*Proof.* For $i = m$, the equation holds since $f(x, \dots, x) = x$. Let $h_i(x_1, \dots, x_m)$ denote the term on the left side of the above identity.

Assume the result has been proved for $m \geq i > n$ and consider $i = n > 1$. Since $h_n(x_1, x_2, \dots, x_m) = h_n(x_2, \dots, x_m, x_1)$, it follows from Lemma 2.2 that $h_n$ is essentially $m$-ary, hence it is either $f(x_1, \dots, x_m)$ or $x_1 \cdots x_m$. In the first case, replacing $x_j$ by $x_jx_{j+1} \cdots x_{j+n-1}$ for $j = 1, \dots, m$, yields $h_k(x_1, \dots, x_m) = h_n(x_1, \dots, x_m) = f(x_1, \dots, x_m)$, for some $k > n$. But by the inductive hypothesis, $h_k(x_1, \dots, x_m) =$

$x_1 \cdots x_m$, which contradicts the assumption that $f$ is distinct from the $m$-ary semilattice operation. Therefore $h_n(x_1, \ldots, x_m) = x_1 \cdots x_m$. $\square$

**Lemma 2.4.** *The algebra* $\mathbf{A}$ *satisfies the identity*

$$f(x_2, x_2, x_3, \ldots, x_m) = x_2 x_3 \cdots x_m.$$

*Proof.* Let $t$ be the term $f(x_2, x_2, x_3, \ldots, x_m)$. It suffices to show that $t$ depends on all its variables, since $x_2 x_3 \cdots x_m$ is the only essentially $(m-1)$-ary operation on $A$.

By the preceding lemma, $f(x_1x_2, x_2x_3, \ldots, x_mx_1) = x_1 \cdots x_m$, hence

$$f(x_3x_2, x_2x_3, \ldots, x_{m-1}x_m, x_mx_3) = x_2x_3 \cdots x_m.$$

The term on the right depends on $x_2$, while the term on the left has $x_2$ in the first two arguments and is a substitution instance of $t$. Therefore $t$ depends on $x_2$.

Similarly, the term on the right depends on $x_m$, while for $m \geq 4$ the term on the left has $x_m$ in last two arguments only. Therefore $t$ depends on at least one of its last two arguments, and by symmetry of $f$, it must then depend on $x_3, x_4, \ldots, x_m$.

The case $m = 3$ requires separate consideration. Let $t = f(y, y, x)$. As before, $t$ depends on $y$. Assuming it does not depend on $x$, we have $f(y, y, x) = y$. Now consider the term $f(x, y, z)x$. It depends on all variables, since $f(x, y, y)x = yx$ depends on $x$ and $y$, hence $f(x, y, z)x$ depends on $x$ and on $y$ or $z$, and by symmetry it must depend on both. Therefore $f(x, y, z)x$ is either $xyz$ or $f(x, y, z)$. In either case $f(x, y, z)x = f(x, y, z)z$, and replacing $z$ by $y$ we get $yx = f(x, y, y)x = f(x, y, y)y = y$, which is impossible. Hence $f(y, y, x)$ depends on both variables. $\square$

**Lemma 2.5.** *Let* $s_1, \ldots, s_m$ *be semilattice terms using only the variables* $x_1, \ldots, x_n$ *for some* $n < m$*, and suppose that each of the variables appears in some* $s_i$*. Then* $f(s_1, \ldots, s_m) = x_1 \cdots x_n$ *holds in* $\mathbf{A}$.

*Proof.* Assume by way of contradiction that $f(s_1, \ldots, s_m)$ does not depend on $x_i$ for some $i \in \{1, \ldots, n\}$. Replacing $x_i$ by $x$ and all other variables by $y$, we obtain a term $t(x, y)$ which does not depend on $x$, hence $t(x, y) = y$ holds in $\mathbf{A}$.

Now $t(xy, y)$ is of the form $f(xy, \ldots, xy, y, \ldots, y)$, with at least one subterm $xy$, since $x_i$ appeared in some $s_j$. Note that either $xy$ or $y$ appears more than once in this expression, since $f$ has more than two arguments. By the preceding lemma $t(xy, y) = xy$, which contradicts $t(x, y) = y$. This shows that $f(s_1, \ldots, s_m)$ is essentially $n$-ary and hence equals $x_1 \cdots x_n$. $\square$

It follows from this lemma that if $S$ is an $(m-1)$-generated subsemilattice of $\mathbf{A}$, then $f$ restricted to $S$ is just the $m$-ary semilattice operation.

**Lemma 2.6.** *Every term of* $\mathbf{A}$ *depends on all its variables. Hence every term with less than $m$ variables is equivalent to the meet of these variables.*

*Proof.* Consider any term containing a variable $x$. Replacing all other variables by $y$, we get a term $t(x, y)$ with less than $m$ variables. Applying the lemma above inductively shows that $t(x, y) = xy$, so this term depends on $x$. It follows that the original term also depends on $x$. $\square$

For the next lemma, we define the length $|s|$ of a semilattice term $s$ to be the number of distinct variables that occur in it.

**Lemma 2.7.** *Let $s_1, \ldots, s_m$ be semilattice terms using only the variables $x_1, \ldots, x_m$, suppose that each of the variables appears in some $s_i$, and that at least one of the $s_i$ is not a variable. Then $f(s_1, \ldots, s_m) = x_1 \cdots x_m$ holds in* $\mathbf{A}$.

*Proof.* By symmetry, we may assume $x_1, x_2$ appear in $s_1$. The preceding lemma implies that the term $f(s_1, \ldots, s_m)$ depends on all its $m$ variables, hence either $\mathbf{A} \models f(s_1, \ldots, s_m) = x_1 \cdots x_m$ or $\mathbf{A} \models f(s_1, \ldots, s_m) = f(x_1, \ldots, x_m)$. Suppose to the contrary that the second identity holds, and let $x_k$ be a variable that appears in $s_j$ (for some fixed $j$) but not in $s_1$. Replacing $x_2$ by $s_j$, $x_j$ by $s_2$ and $x_i$ by $s_i$ for $i \neq 2, j$, we deduce that

$$f(s_1', \ldots, s_m') = f(s_1, s_j, s_3, \ldots, s_{j-1}, s_2, s_{j+1}, \ldots, s_m) = f(x_1, \ldots, x_m)$$

where $s_1'$ now includes the variables $x_1, x_2, x_k$. Repeating this step for each variable $x_k$ not in $s_1'$, we see that $\mathbf{A}$ satisfies the identity $f(s_1'', \ldots, s_m'') = f(x_1, \ldots, x_m)$, where $s_1'' = x_1 \cdots x_m$.

Now choose any $x_k$ in $s_2''$, and replace $x_k$ by $s_1''$, $x_1$ by $s_k''$, and $x_i$ by $s_i''$ for $i \neq 1, k$. This produces the equation

$$\begin{aligned} &f(x_1 \cdots x_m, x_1 \cdots x_m, s_3''', \ldots, s_m''') \\ &= f(s_k'', s_2'', \ldots s_{k-1}'', s_1'', s_{k+1}'' \ldots, s_m'') = f(x_1, \ldots, x_m). \end{aligned}$$

Since the first term reduces to $x_1 \cdots x_m$ by Lemma 2.4, this is a contradiction. $\square$

It follows from this result that if $f(a_1, \ldots, a_m) \neq a_1 \cdots a_m$ then the elements $a_1, \ldots, a_m$ are pairwise incomparable. In fact, the next lemma implies that the structure of any $m$-generated subalgebra $\mathbf{B}$ of $\mathbf{A}$ is completely determined by the semilattice structure of $\mathbf{B}$ and the value of the term $f(x_1, \ldots, x_m)x_1 \cdots x_m$ applied to the $m$ generators.

**Lemma 2.8.** *The following identities hold in* $\mathbf{A}$.

(i) $f(x_1, \ldots, x_m)x_i = f(x_1, \ldots, x_m)x_1 \cdots x_m$ *for any* $i \in \{1, \ldots, m\}$,

(ii) $f(f(x_1, \ldots, x_m), t_2, t_3, \ldots, t_m) = f(x_1, \ldots, x_m)x_1 \cdots x_m$ *for any semilattice term* $t_2$ *and any terms* $t_3, \ldots, t_m$ *with variables from* $x_1, \ldots, x_m$.

*Proof.* (i) By Lemma 2.6 it follows that $\mathbf{A}$ satisfies either the identity $f(x_1, \ldots, x_m)x_i = f(x_1, \ldots, x_m)$ or the identity $f(x_1, \ldots, x_m)x_i = x_1 \cdots x_m$, and both of them easily imply the desired identity.

(ii) Since the terms $t_3, \ldots, t_m$ use at most $m$ variables, we may assume by Lemma 2.6 that each term is either of the form $f(x_1, \ldots, x_m)$ or a meet of variables. If one of the $t_i$ is $f(x_1, \ldots, x_m)$, then the first $f$ in the left-hand-side of the identity has a repeated argument, hence by Lemma 2.4 this side reduces to the product of its arguments, and by (i) it equals $f(x_1, \ldots, x_m)x_1 \cdots x_m$.

So we may assume that $t_3, \ldots, t_m$ are also semilattice terms. By Lemma 2.6 we have either

$$\text{(1)} \qquad \mathbf{A} \models f(f(x_1, \ldots, x_m), x_2, \ldots, x_m) = x_1 \cdots x_m \text{ or}$$

$$\text{(2)} \qquad \mathbf{A} \models f(f(x_1, \ldots, x_m), x_2, \ldots, x_m) = f(x_1, \ldots, x_m).$$

In case (1) holds, replacing $x_1$ by $f(x_1, \ldots, x_m)$ produces

$$\begin{aligned} &f(f(f(x_1, \ldots, x_m), x_2, \ldots, x_m), x_2, \ldots, x_m) \\ &= f(x_1, \ldots, x_m)x_2 \cdots x_m = f(x_1, \ldots, x_m)x_1 \cdots x_m, \end{aligned}$$

while the left hand side simplifies to $f(x_1 \cdots x_m, x_2, \ldots, x_m) = x_1 \cdots x_m$ by the preceding lemma. Hence $\mathbf{A}$ satisfies $f(x_1, \ldots, x_m)x_1 \cdots x_m = x_1 \cdots x_m$ in this case.

Now assume to the contrary that

$$(*) \qquad f(f(x_1, \ldots, x_m), t_2, \ldots, t_m) = f(x_1, \ldots, x_m).$$

Using an instance of (1), with $x_1$ replaced by $f(x_1, \ldots, x_m)$ and $x_i$ replaced by $t_i$ for $i > 1$, we see that

$$\begin{aligned} &f(x_1, \ldots, x_m) = f(f(x_1, \ldots, x_m), t_2, \ldots, t_m) && \text{by}(*) \\ &= f(f(f(x_1, \ldots, x_m), t_2, \ldots, t_m), t_2, \ldots, t_m) && \text{by}(*) \\ &= f(x_1, \ldots, x_m)t_2 \cdots t_m && \text{instance of (1)} \\ &= f(x_1, \ldots, x_m)x_1 \cdots x_m = x_1 \cdots x_m && \text{by Lemma 2.8(i).} \end{aligned}$$

The second last $=$ holds since the $t_i$ are semilattice terms, and the last $=$ follows from the identity of the previous paragraph. This contradiction shows that the result holds under the assumption of (1).

In case (2) holds, we have

$$f(x_1,\dots,x_m)x_1\cdots x_m = f(x_1,\dots,x_m)x_3\cdots x_m \qquad \text{by Lemma 2.8 (i)}$$
$$= f(f(x_1,\dots,x_m), f(x_1,\dots,x_m), x_3,\dots,x_m) \qquad \text{by Lemma 2.4}$$
$$= f(f(x_1, f(x_1,\dots,x_m), x_3,\dots,x_m), f(x_1,\dots,x_m), x_3,\dots,x_m)$$
$$= f(x_1, f(x_1,\dots,x_m), x_3,\dots,x_m) \quad \text{by (2), with } f(x_1,\dots,x_m) \text{ for } x_2$$
$$= f(x_1,\dots,x_m) \qquad \text{symmetric version of (2)}$$

where the middle equality also follows from a symmetric version of (2) applied to the first argument of $f$.

Now assume to the contrary that

$$(**) \qquad f(f(x_1,\dots,x_m), t_2,\dots,t_m) = x_1\cdots x_m.$$

Using the identity just proven, with $x_1$ replaced by $f(x_1,\dots,x_m)$ and $x_i$ replaced by $t_i$ for $i > 1$, we obtain

$$x_1\cdots x_m = f(f(x_1,\dots,x_m), t_2,\dots,t_m) \qquad \text{by } (**)$$
$$= f(f(x_1,\dots,x_m), t_2,\dots,t_m) f(x_1,\dots,x_m)t_2\cdots t_m$$
$$= x_1\cdots x_m f(x_1,\cdots,x_m)t_2\cdots t_m \qquad \text{by } (**)$$
$$= f(x_1\cdots x_m)x_1\cdots x_m \qquad \text{by Lemma 2.8(i).}$$

This contradiction completes the proof. □

**Lemma 2.9.** *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$ *that is generated by elements* $a_1,\dots,a_m \in A$*, let* $b = a_1\cdots a_m$*,* $c = f(a_1,\dots,a_m)$*, and assume that* $b \neq c$*. Then the equivalence relation* $\theta$ *with equivalence classes* $\{a_1\}$*, …,* $\{a_m\}$*,* $\{c\}$ *and* $[b] = B \setminus \{a_1,\dots,a_m,c\}$ *is a congruence relation on* $\mathbf{B}$*.*

*Proof.* It suffices to show that if $u, v \in [b]$ then for all $w, w_2,\dots,w_m \in B$ we have $uw\theta vw$ and $f(u, w_2,\dots,w_m)\theta f(v, w_2,\dots,w_m)$.

Each $w \in B$ is obtained from a term $t(x_1,\dots,x_m)$ applied to the generators $a_1,\dots,a_m$. By the preceding lemmas, we may assume this term evaluates to either $a_1,\dots,a_m, c$, a meet of generators, or $bc$. Earlier we already observed that the last expression is either $b$ or $c$, hence $[b]$ contains only meets of generators. Now consider $u, v \in [b]$ and $w \in B$. If $w = c = bc$ then $uw = vw = c$, and otherwise $uw, vw \in [b]$, by Lemma 2.8(i). Similarly, using (ii) of the same lemma, it follows that if $bc = c \in \{w_2,\dots,w_m\} \subseteq B$ then $f(u, w_2,\dots,w_m) = f(v, w_2,\dots,w_m) = c$ and otherwise $f(u, w_2,\dots,w_m), f(v, w_2,\dots,w_m) \in [b]$. Therefore $\theta$ has the substitution property. □

Consider the quotient algebra $\mathbf{B}/\theta$ of the subalgebra $\mathbf{B}$ from the preceding lemma. As observed earlier, the elements $a_1,\dots,a_m$ are pairwise incomparable, and $bc$ is either $b$ or $c$. By Lemma 2.8 the operation $f$ is completely determined by the value of $bc$, hence $\mathbf{B}/\theta$ is one of two possible nonisomorphic $m+2$-element algebras. These two algebras will be denoted by $\mathbf{S}_m$ and $\mathbf{T}_m$. The underlying set of elements for both of them is $\{\{a_1\}, \ldots, \{a_m\}, [b], \{c\}\}$, but for simplicity we rename the elements $a_1, \ldots, a_m, b, c$. The operation $f$ is defined by

$$f(x_1, \ldots, x_m) = \begin{cases} c & \text{if } \{x_1, \ldots, x_m\} = \{a_1, \ldots, a_m\} \\ x_1 x_2 \cdots x_m & \text{otherwise.} \end{cases}$$

The difference between the algebras is in the semilattice structure. For $\mathbf{S}_m$ the semilattice has height two with minimal element $b$, and for $\mathbf{T}_m$ it has height 3 with minimal element $c$ and unique cover $b$ (see Figure 1).

**Corollary 2.10.** *Let $\mathbf{A}$ be an $m$-ary semilattice expansion with $m > 2$ and $p_m(\mathbf{A}) = 2$. Then either $p_n(\mathbf{A}) \geq p_n(\mathbf{S}_m)$ for all $n$, or $p_n(\mathbf{A}) \geq p_n(\mathbf{T}_m)$ for all $n$.*

For the case $m = 3$, these algebras appear in [Ko71], and generate the varieties $\mathcal{K}_1$ and $\mathcal{K}_2$ in [GK92].

For the case $m = 2$, an analogous result involves in addition the two-element lattice $\mathbf{D}_2$, and follows directly from Dudek's results mentioned in the introduction. The algebras $\mathbf{N}_2$ and $\mathbf{A}_4$ in [Du97] are respectively, $\mathbf{S}_2$ and $\mathbf{T}_2$ in our notation. The proofs and results in the remaining sections hold also for $m = 2$.

## 3. The $p_n$-sequence of $\mathbf{T}_m$

In this section we prove that the value of $p_n(\mathbf{T}_m)$ is given by the number of antichains in a certain poset. In the subsequent section we then show that $p_n(\mathbf{S}_m)$ is strictly less than $p_n(\mathbf{T}_m)$ for all $n > m$.

Let $X = \{x_1, \ldots, x_n\}$ be a set of $n$ distinct variables. An *$m$-subpartition of $X$* is any partition into $m$ (disjoint nonempty) blocks of some subset of $X$. The collection of all $m$-subpartitions of $X$ is denoted by $\mathrm{SPart}_m(X)$. We define a relation $\sqsubseteq$ on $\mathrm{SPart}_m(X)$ by $\rho \sqsubseteq \sigma$ iff there exists a bijection $\phi : \rho \to \sigma$ such that $Y \subseteq \phi(Y)$ for each block $Y \in \rho$.

**Lemma 3.1.** *The relation $\sqsubseteq$ is a partial order on $\mathrm{SPart}_m(X)$.*

*Proof.* It is clearly reflexive and transitive. Assume $\rho \sqsubseteq \sigma$ and $\sigma \sqsubseteq \rho$ for some $\rho, \sigma \in \mathrm{SPart}_m(X)$, with corresponding bijections $\phi : \rho \to \sigma$ and $\psi : \sigma \to \rho$. Then $Y \subseteq \phi(Y) \subseteq \psi(\phi(Y))$, and since blocks are disjoint it follows that $Y = \psi(\phi(Y))$, whence $Y = \phi(Y)$. This means $\phi = \sigma$, so $\sqsubseteq$ is antisymmetric. $\square$

For a subset $Y = \{y_1, \dots, y_k\}$ of $X$, $\overline{Y}$ denotes the term $y_1 \cdots y_k$, and for an $m$-subpartition $\rho = \{Y_1, \dots, Y_m\}$, we let $f(\rho) = f(\overline{Y}_1, \dots, \overline{Y}_m)$. Let $\alpha = \{\rho_1, \dots, \rho_k\}$ be an antichain in this partial order, and let $Y$ be the (possibly empty) set of variables that do not occur in (a block of) these subpartitions. Consider the term

$$t_\alpha = f(\rho_1)f(\rho_2)\cdots f(\rho_k)\overline{Y}.$$

Note that the empty antichain corresponds to the term $x_1 \cdots x_n$.

The next lemma lists some facts about $\mathbf{T}_m$, as may be checked by straightforward verification.

**Lemma 3.2.** *The following statements hold in* $\mathbf{T}_m$*:*

(i) *For any* $i = 1, \dots, m$*, a term* $t$ *evaluates to* $a_i$ *iff all variables in* $t$ *are assigned* $a_i$*.*

(ii) *If one of the subterms in a term* $t$ *evaluates to* $c$*, then* $t$ *evaluates to* $c$ *(i.e.* $c$ *is a zero).*

(iii) *For terms* $t_1, \dots, t_m$*, if* $t_i$ *and* $t_j$ *have some variable in common for some* $i \neq j$*, then* $\mathbf{T}_m \models f(t_1, \dots, t_m) = t_1 \cdots t_m$*.*

(iv) *For semilattice terms* $s_1, \dots, s_m$*, if* $x$ *is a variable that appears in at least one of these terms, then* $\mathbf{T}_m \models f(s_1, \dots, s_m)x = f(s_1, \dots, s_m)$*.*

(v) $\mathbf{T}_m \models f(f(x_1, \dots, x_m)y_1, y_2, \dots, y_m)$
$= f(x_1 \cdots x_m y_1, y_2, \dots, y_m)f(x_1, \dots, x_m)$*.*

(vi) *If* $\rho \sqsubseteq \sigma$ *in* $\mathrm{SPart}_m(X)$*, then* $f(\rho)f(\sigma) = f(\rho)\overline{Y}$*, where* $Y = \bigcup \sigma \setminus \bigcup \rho$ *(if this is empty, the factor* $\overline{Y}$ *is omitted).*

**Theorem 3.3.** *There is a bijective correspondence between the antichains of* $(\mathrm{SPart}_m(X), \sqsubseteq)$ *and the essentially* $n$*-ary term functions on* $\mathbf{T}_m$*. In fact, for each antichain* $\alpha$*, the term* $t_\alpha$ *is a normal form for the term function* $t_\alpha^{\mathbf{T}_m}$*.*

*Proof.* We have to show that each term function can be obtained from a term $t_\alpha$, and that for distinct antichains $\alpha$ and $\beta$, the term functions $t_\alpha^{\mathbf{T}_m}$ and $t_\beta^{\mathbf{T}_m}$ are different.

For the first part we observe that by Lemma 3.2 (v), any term $t$ can be rewritten to a term that has no nested occurrences of the operation $f$, and by 3.2 (iii) we may assume that the collection of sets of variables that occur as arguments in any $f$-subterm form an $m$-subpartition of $X$. Thus $t$ is of the form $f(\rho_1)f(\rho_2)\cdots f(\rho_k)\overline{Y}$ for some $\rho_1, \rho_2, \dots, \rho_k \in \mathrm{SPart}_m(X)$. Moreover, by 3.2 (iv), we may assume that the variables in $Y$ do not appear in any of the blocks of the $m$-subpartitions. Finally, by 3.2 (vi), we may delete any factors $f(\rho_i)$ that are not minimal with respect to $\sqsubseteq$ restricted to $\{\rho_1, \dots, \rho_k\}$, as long as any variables that do

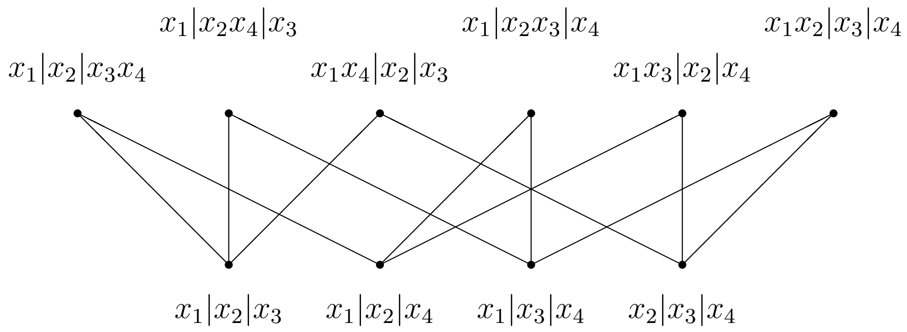


FIGURE 2. $\mathrm{SPart}_3\{x_1, x_2, x_3, x_4\}$

not appear in the minimal subpartitions are added to the set $Y$. Hence the collection of subpartitions form an antichain.

Suppose now that $\alpha$ and $\beta$ are distinct antichains. Then one of them, say $\alpha$, contains a subpartition $\rho$ that is not greater or equal to any subpartition $\sigma$ in $\beta$ with respect to $\sqsubseteq$. Choose an assignment of $a_1, \ldots, a_m$ to the variables of $\rho$ such that $f(\rho) = c$, and assign $b$ to all other variables. Then $t_\alpha$ evaluates to $c$, but none of the factors of $t_\beta$ evaluate to $c$, hence the two terms induce distinct term functions. $\square$

The poset $\mathrm{SPart}_3\{x_1, x_2, x_3, x_4\}$ is shown in Figure 2, and it has $2^6 + (2^4 - 1) + 4 \cdot (2^3 - 1) + 6 = 113$ antichains, hence $p_4(\mathbf{T}_3) = 113$. While it is possible to give upper and lower bounds for the number of antichains in the posets corresponding to larger values of $m, n$ it seems unlikely that an exact formula can be obtained.

## 4. Comparing the $p_n$-sequences of $\mathbf{S}_m$ and $\mathbf{T}_m$

In this section we prove that the $p_n$-sequence of $\mathbf{S}_m$ is strictly below the $p_n$-sequence of $\mathbf{T}_m$ when $n > m$. As before we consider terms $t$ with variables from $X = \{x_1, \ldots, x_n\}$. Since we want to count terms that induce essentially $n$-ary operations, we may assume that all variables of $X$ occur in $t$.

Let $u, v, w : X \to S_m$ be assignments, and denote their extension to the collection of all terms by the same symbols. The symbol $\hat{u}$ is defined to be the collection $\{u^{-1}\{a_1\}, \ldots, u^{-1}\{a_m\}\}$. Note that if $u(t) = c$ and $u$ is nonconstant then $\hat{u}$ is an $m$-subpartition of $X$.

For a term $t$ we let

$$S(t) = \{\hat{u} : u \text{ is a nonconstant assignment and } u(t) = c\}.$$

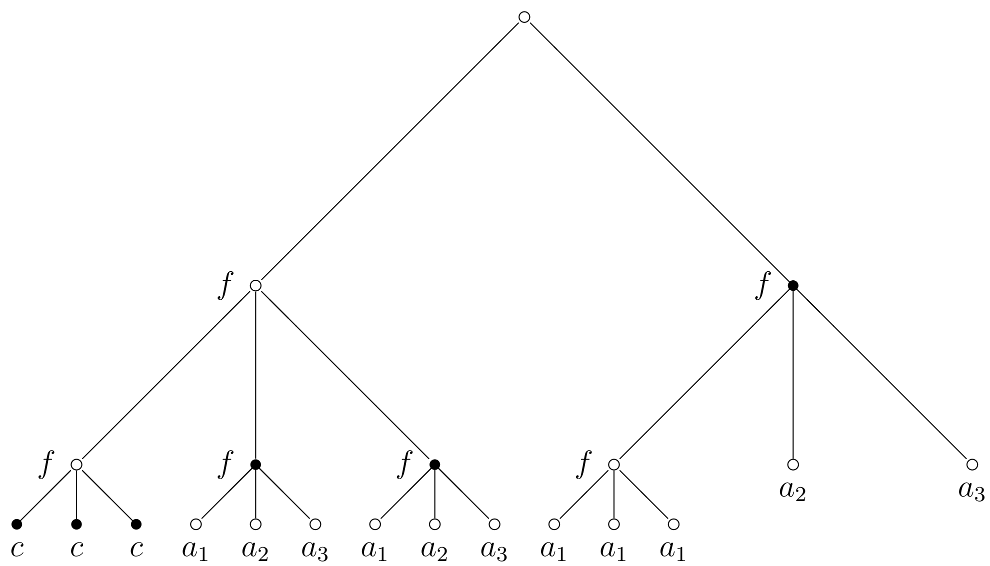


FIGURE 3. A term with an antichain of $u$-critical subterms

By the preceding observation, $S(t)$ is a subset of $\mathrm{SPart}_m(X)$, hence it is also partially ordered by the relation $\sqsubseteq$ introduced in Section 3.2. The next result follows from the fact that $u(t) = a_i$ if and only if $u(x) = a_i$ for all variables $x$ in $t$.

**Lemma 4.1.** *If $t$ and $t'$ are terms such that $S(t) = S(t')$, then $t$ and $t'$ induce the same term function on $\mathbf{S}_m$.*

Although $S(t)$ is not an antichain in general, the following facts are used to show that $S(t)$ is determined by the antichain of its minimal members.

A *$u$-critical subterm* (of $t$ for an assignment $u$) is a subterm $s$ such that $u(s) = c$ but $u(r) \neq c$ for all proper subterms $r$ of $s$. Note that if $s$ is a variable $x$, then it is $u$-critical iff $u(x) = c$. Otherwise, $s$ is necessarily of the form $s = f(r_1, \ldots, r_m)$ (since for a meet we have $u(r_1 r_2) = c$ iff $u(r_1) = c$ and $u(r_2) = c$), and it is $u$-critical if and only if $\{u(r_1), \ldots, u(r_m)\} = \{a_1, \ldots, a_m\}$.

Let $s_1, \ldots, s_k$ be a list of all $u$-critical subterms of a term $t$. By definition, neither of them is a subterm of another, and hence they form an antichain in the tree of all subterms of $t$ ordered by the "is a subterm of" relation (cf. Figure 3). Moreover, if $u$ is a nonconstant assignment with $u(t) = c$, then this is a maximal antichain, i.e. each occurrence of a variable in $t$ is an occurrence in some $u$-critical subterm $s$.

Indeed, consider a chain of subterms in $t$ containing this occurrence (which corresponds to a path from a leaf to the root in the tree), and let $s$ be the least subterm in this chain with $u(s) = c$. Such a term exists, since $t$ is on the top of the chain. If $s$ is a variable, then it is $u$-critical. Otherwise, it has the form $s = f(r_1, \dots, r_m)$, by the same argument as above, with $u(r_i) \neq c$ for some $i$. Since $u(s) = c$, $\{u(r_1), \dots, u(r_m)\} = \{a_1, \dots, a_m\}$, as required.

Note that what we proved is equivalent to that there is a term $t'(x_1, \dots, x_k)$ such that $t'(s_1, \dots, s_k) = t$. The term tree for $t'$ is obtained from the term tree of $t$ by cutting branches outgoing from the nodes corresponding to $u$-critical terms (in Figure 3 they are denoted by $\bullet$).

**Lemma 4.2.** *Let* $\{U_1, \dots, U_m\}, \{V_1, \dots, V_m\} \in S(t)$ *and suppose* $V_i \subseteq U_i$ *for* $i = 1, \dots, m$, *with at least one of the inclusions proper. Then* $\{U_1 \setminus V_1, \dots, U_m \setminus V_m\} \in S(t)$.

*Proof.* Let $u, v$ be assignments that have corresponding $m$-subpartitions $\hat{u} = \{U_1, \dots, U_m\}$, $\hat{v} = \{V_1, \dots, V_m\}$, and let $w : X \to S_m$ be the assignment defined by

$$w(x) = \begin{cases} a_i & \text{if } x \in U_i \setminus V_i \\ c & \text{otherwise.} \end{cases}$$

Note that $w$ is nonconstant since $V_i \neq \emptyset$ and $U_i \setminus V_i \neq \emptyset$ for some $i$, so it suffices to prove that $w(t) = c$. And since $t = t'(s_1, \dots, s_k)$ for some term $t'$, where $s_1, \dots, s_k$ are $u$-critical subterms, it is enough to show that $w(s) = c$ for each $u$-critical subterm $s$.

If $s$ is a variable, then $w(s) = c$ since $u^{-1}\{c\} \subseteq w^{-1}\{c\}$. If $s$ is not a variable, then by remarks preceding this lemma, $s = f(r_1, \dots, r_m)$, and without loss of generality we may assume that $u(r_i) = a_i$ for $i = 1, \dots, m$. Since $V_i \subseteq U_i$, it follows that $v(r_i) = a_i$ or $v(r_i) = c$ for each $i$. Moreover, since $v(t) = c$, either $v(r_i) = c$ for all $i$ or $v(r_i) = a_i$ for each $i$. In the first case, by definition, $w(r_i) = a_i$ for all $i$, and therefore $w(s) = c$, as required. In the second case, $w(r_i) = c$ for all $i$, and therefore, again, $w(s) = c$. $\square$

Two $m$-subpartitions $\rho, \sigma$ of $X$ are said to be completely disjoint if $\bigcup \rho \cap \bigcup \sigma = \emptyset$. Note that the lemma above states that if a partition $\hat{u}$ properly contains a partition $\hat{v}$, then it can be decomposed into completely disjoint partitions $\hat{v}$ and $\hat{w}$. The following is the converse of this fact.

**Lemma 4.3.** *Let* $\{V_1, \ldots, V_m\}, \{W_1, \ldots, W_m\}$ *be two completely disjoint* $m$*-subpartitions in* $S(t)$*, and define* $U_i = V_i \cup W_i$ *for* $i = 1, \ldots, m$*. Then* $\{U_1, \ldots, U_m\} \in S(t)$*.*

*Proof.* Let $u, v, w$ be assignments such that $\hat{u} = \{U_1, \ldots, U_m\}$, $\hat{v} = \{V_1, \ldots, V_m\}$ and $\hat{w} = \{W_1, \ldots, W_m\}$. Since $\hat{v}$ and $\hat{w}$ are completely disjoint, the nontrivial $v$-critical subterms and the nontrivial $w$-critical subterms of $t$ are incomparable in the tree of subterms of $t$. We will prove that $u(s) = c$ for any such subterms. Since we also have $u(x) = c$ for any variable that is both $v$-critical and $w$-critical, it follows that $u(t) = c$.

So let $s = f(r_1, \ldots, r_m)$ be a nontrivial $v$-critical subterm (the argument for $w$ is similar). Then $v(s) = c$ and we may assume $v(r_i) = a_i$ for $i = 1, \ldots, m$. So $v(x) \neq c$ for all variables $x$ in $s$, hence $u(x) = v(x)$ for all these variables $x$. Clearly this implies $u(s) = v(s) = c$. □

**Theorem 4.4.** *For all* $n > m$*, we have* $p_n(\mathbf{S}_m) < p_n(\mathbf{T}_m)$*.*

*Proof.* For a term $t$, let $M(t)$ be the antichain of minimal members of $S(t)$. The preceding two lemmas imply that $S(t)$ is determined by $M(t)$.

Now consider the map given by $t^{\mathbf{S}_m} \mapsto M(t)$. It is well-defined by Lemma 4.1. Since $M(t)$ is an antichain of $\mathrm{SPart}_m(X)$, it follows from Theorem 3.3 that this map is an injection that proves $p_n(\mathbf{S}_m) \leq p_n(\mathbf{T}_m)$.

To prove that the inequality is strict we need to exhibit an antichain of $m$-subpartitions of $X = \{x_1, \ldots, x_n\}$, which is not of the form $M(t)$ for any term $t$ in $n$ variables. To this end, consider the antichain $N$ consisting of the following two $m$-subpartitions (with singleton blocks each): $x_1|\ldots|x_m$ and $x_1|\ldots|x_{m-1}|x_{m+1}$. This is well defined, since $n > m > 2$. Suppose now that $N = M(t)$. It follows, by definition of $M(t)$, that $t$ has subterms $f(x_1, \ldots, x_m)$, and $f(x_1, \ldots, x_{m-1}, x_{m+1})$. Also, in particular, $u(t) = c$ under the assignment $u(x_i) = a_i$ for $i \leq m$ and $x_i = c$, otherwise. Yet, for this assignment the second subterm evaluates to $b$, which yields $u(t) = b$, a contradiction. □

## References

[BW96] S. Burris and R. Willard, *Problem 17 of Grätzer and Kisielewicz*, Algebra Universalis 36 (1996), no. 4, 573–575.

[Du83] J. Dudek, *A characterization of distributive lattices*, in *Contributions to lattice theory (Szeged 1980)*, Colloq. Math. Soc. János Bolyai, 33, North-Holland 1983, pp. 325–335.

[Du97] J. Dudek, *The minimal extension of sequences II. On Problem 17 of Grätzer and Kisielewicz*, Periodica Math. Hung. 34 (1997), 177–183.

[GK92] G. Grätzer and A. Kisielewicz, *A survey of some open problems on $p_n$-sequences and free spectra of algebras and varieties*, Universal algebra and quasigroup theory (Jadwisin, 1989), 57–88, Res. Exp. Math., 19, Heldermann, Berlin, 1992.

[Ki03] A. Kisielewicz, *Complete systems of subalgebras*, Internat. J. Algebra and Comput., to appear.

[Ko71] K. M. Koh, *On the number of essentially $n$-ary polynomials of idempotent algebras*, Ph. D. Thesis, University of Manitoba.

[Ko72] K. M. Koh, *Idempotent algebras with one essentially binary polynomial*, Nanyang Univ. J. Part III 6 (1972), 18–27.

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