RUDIN-KEISLER POSETS OF COMPLETE BOOLEAN ALGEBRAS

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Abstract. The Rudin-Keisler ordering of ultrafilters is extended to complete Boolean algebras and characterised in terms of elementary embeddings of Boolean ultrapowers. The result is applied to show that the Rudin-Keisler poset of some atomless complete Boolean algebras is nontrivial.

1. Introduction

All concepts and notations not defined below can be found in [3].

Let B be a Boolean algebra, and let $\mathbb{P}_B$ denote the set of all partitions of B (i.e. maximal sets of pairwise disjoint elements). Note that $\mathbb{P}_B$ is ordered by the refinement relation: $\tau \le \sigma$ if for all $x \in \tau$ there exists a $y \in \sigma$ such that $x \le y$. Let $\hat{\sigma} = \bigcup\{\tau : \tau \le \sigma\}$ be the set of nonzero elements of B that are below some element of σ. Since σ is a partition, each $x \in \hat{\sigma}$ is less than or equal to a unique $y \in \sigma$, so there is a natural map j_σ from $\hat{\sigma}$ to σ given by $j_\sigma(x) = y$. For a map $s : \sigma \to Y$ we define $\hat{s} = s \circ j_\sigma$, and occasionally we also abbreviate dom$s$ by s^{d}.

For $\sigma \in \mathbb{P}_B$ we let $\mathcal{P}(\sigma)$ be the powerset Boolean algebra over the set σ. If all joins of subsets of σ exist in B (e.g. if B is $|\sigma|$-complete) then we identify $\mathcal{P}(\sigma)$ with the complete subalgebra of B that is completely generated by σ.

For powerset Boolean algebras, the Rudin-Keisler ordering of ultrafilters is defined on $D \in \mathrm{Uf}(\mathcal{P}(X))$, $E \in \mathrm{Uf}(\mathcal{P}(Y))$ by $D \le E$ if there exists a function $f : Y \to X$ such that

$$\text{for all } S \in \mathcal{P}(X), \qquad S \in D \text{ implies } f^{-1}[S] \in E. \qquad (*)$$

We also write $D \le_f E$ if $(*)$ holds. Note that this implication implies its converse, since $S \notin D$ implies $X \setminus S \in D$, hence $f^{-1}[X \setminus S] = Y \setminus f^{-1}[S] \in E$ and therefore $f^{-1}[S] \notin E$.

The duality between sets and powerset Boolean algebras implies the following equivalent definition: $D \le E$ iff there exists a complete homomorphism $\alpha : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that $\alpha[D] \subseteq E$.

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We wish to extend this ordering to complete (but not neccessarily atomic) Boolean algebras. Given a filter D in a complete Boolean algebra B, and a partition σ of B, we let $D_\sigma = D \cap \mathcal{P}(\sigma)$. Note that if D is an ultrafilter of B, then D_σ is an ultrafilter of $\mathcal{P}(\sigma)$. The idea of the definition below is to reduce the ordering of ultrafilters of B and C, to the usual Rudin-Keisler ordering of the induced ultrafilters on complete and atomic subalgebras of B and C. However, we need an additional concurrancy condition to ensure some nice properties of this extended ordering.

Definition 1.1. Let B, C be complete Boolean algebras, $D \in \mathrm{Uf}(B)$ and $E \in \mathrm{Uf}(C)$. We say that $D \leq E$ if there exists a map $g : \mathbb{P}_B \to \mathbb{P}_C$ and a family of maps $f_\sigma : g(\sigma) \to \sigma$ ($\sigma \in \mathbb{P}_B$) such that

(i) for all $S \subseteq \sigma$, $\sum S \in D$ implies $\sum f_\sigma^{-1}[S] \in E$, (i.e. $D_\sigma \leq_{f_\sigma} E_{g(\sigma)}$ for all $\sigma \in \mathbb{P}_B$) and,

(ii) the family of f_σ satisfies the following *concurrancy condition*

$$\forall \tau, \sigma \in \mathbb{P}_B,\ \tau \leq \sigma \text{ implies } \sum\{y \in g(\tau) \otimes g(\sigma) : \hat{f}_\tau(y) \leq \hat{f}_\sigma(y)\} \in E.$$

Here $\otimes$ is the meet operation in $\mathbb{P}_B$, i.e. $\sigma \otimes \tau$ is the greatest common refinement of σ and τ, given by $\{xy : x \in \sigma, y \in \tau\} \setminus \{0\}$. To make the connection with the previous version for powerset algebras, we have the following observation.

Proposition 1.2. *Suppose B, C and D, E are as above, and $\alpha : B \to C$ is a complete homomorphism such that $\alpha[D] \subseteq E$. Then $D \leq E$.*

Proof. Let $g : \mathbb{P}_B \to \mathbb{P}_C$ be defined by $g(\sigma) = \alpha[\sigma] \setminus \{0\}$. The completeness of α is needed to ensure that $\sum g(\sigma) = 1$, and since α is meet-preserving, it is injective on families of disjoint elements that are not mapped to 0. Hence we can define an inverse $f_\sigma : g(\sigma) \to \sigma$ by $f_\sigma(y) = x$ iff $y = \alpha(x)$. Let $S \subseteq \sigma$, and suppose $\sum S \in D$. Then $\sum f_\sigma^{-1}[S] = \sum \alpha[S] = \alpha(\sum S) \in E$.

Finally, the concurrency condition holds in a somewhat stronger form: for $\tau \leq \sigma \in \mathbb{P}_B$, we have $g(\tau) \leq g(\sigma)$ and for all $y \in g(\tau)$, $\hat{f}_\tau(y) \leq \hat{f}_\sigma(y)$. □

The Rudin-Keisler ordering for complete Boolean algebras reduces to the usual ordering in case B, C are powerset algebras. In one direction this follows immediately from the above proposition.

In the other direction, suppose $B = \mathcal{P}(X)$, $C = \mathcal{P}(Y)$ and we are given a map $g : \mathbb{P}_B \to \mathbb{P}_C$, and maps f_σ such that $D_\sigma \leq_{f_\sigma} E_{g(\sigma)}$. Consider the smallest partition $\sigma_X = \{\{x\} : x \in X\}$ in $\mathbb{P}_B$ and the corresponding smallest partition $\sigma_Y \in \mathbb{P}_C$. The required map $f : Y \to X$ is induced by the map $f_{\sigma_X} \circ j_{g(\sigma_X)}$ restricted to σ_Y, via the obvious isomorphism between a set and its collection of singleton subsets. Hence $D \leq E$ in the usual Rudin-Keisler order.

Problem 1.3. For which algebras does the converse of Proposition 1.2 hold? Note that it does hold for powerset algebras.

The relation $\leq$ is a quasi-order on the class of all ultrafilters on complete Boolean algebras. We write $D \approx E$ if $D \leq E$ and $E \leq D$. When we restrict ourselves to a single algebra B, the partially ordered set of equivalence classes $\mathrm{Uf}(B)/\!\approx$ is denoted by $\mathrm{RK}(B)$.

2. Characterisation by elementary embeddings

For the RK-order on powerset Boolean algebras, Blass [1] proved the following characterisation theorem:

Theorem 2.1. *Let $D \in \mathrm{Uf}(\mathcal{P}(X))$ and $E \in \mathrm{Uf}(\mathcal{P}(Y))$. The following are equivalent:*

(i) $D \leq E$
(ii) *for every structure M, there exists an elementary embedding from the ultrapower M^X/D to M^Y/E.*

Since we will generalise this result to the extended RK-order, we briefly recall the details of this fundamental result. Assuming $f : Y \to X$ is the function that establishes $D \leq E$, one can define a map $e : M^X/D \to M^Y/E$ by $e(s/D) = (s \circ f)/E$, and this map is an elementary embedding since if ϕ is a formula in the language of M, and $s_1, \ldots, s_n \in M^X$ then

$$\begin{aligned} & M^X/D \models \phi[s_1/D, \ldots, s_n/D] \\ \text{iff } & \{x \in X : M \models \phi[s_1(x), \ldots, s_n(x)]\} \in D \\ \text{iff } & f^{-1}[\{x \in X : M \models \phi[s_1(x), \ldots, s_n(x)]\}] \in E \\ \text{iff } & \{y \in Y : M \models \phi[s_1(f(y)), \ldots, s_n(f(y))]\} \in E \\ \text{iff } & M^Y/E \models \phi[e(s_1/D), \ldots, e(s_n/D)]. \end{aligned}$$

The converse requires the following definition:

Definition 2.2. For any set A, we let $\bar{A}$ be the complete structure on A, defined as the model in which every relation R is the interpretation of some relation symbol, say $\bar{R}$, and and every function f is the interpretation of some function symbol, say $\bar{f}$, respectively.

Now, given an elementary embedding e from $\bar{X}^X$ to $\bar{X}^Y$, the map f is obtained by choosing any representative of $e(id_X/D)$, since for any $S \subseteq X$

$$\begin{aligned} & S \in D \\ \text{iff } & \{x \in X : \bar{X} \models \bar{S}[id_X(x)]\} \in D \\ \text{iff } & \bar{X}^X/D \models \bar{S}[id_X/D] \\ \text{iff } & \bar{X}^Y/E \models \bar{S}[e(id_X/D)] \\ \text{iff } & \{y \in Y : \bar{X} \models \bar{S}[f(y)]\} \in E \\ \text{iff } & f^{-1}[S] \in E. \end{aligned}$$

In order to generalise this result to the extended RK-order, we replace the ultrapowers above by Boolean ultrapowers. Recall that the (unbounded) Boolean power $M[B]$ of a model M over a complete Boolean algebra B can be constructed as a direct limit of powers M^σ, where $\sigma \in \mathbb{P}_B$ (see e.g. [5]). If B is a powerset algebra $\mathcal{P}(X)$, this construction reduces to the ordinary power M^X. Similarly, for any ultrafilter D of B, the Boolean ultrapower $M[B]/D$ is (isomorphic to) a direct limit of ultrapowers M^σ/D_σ, and when $B = \mathcal{P}(X)$, then $M[B]/D \cong M^X/D$. We include some of the details here, since they are relevant to the results of this section.

Definition 2.3. Let M be a structure for some language L, and let B be a complete Boolean algebra, with D a filter in B. The structure $M[B]/D$ has as universe the set $(\bigcup_{\rho\in\mathbb{P}_B} M^\rho)/\theta_D$, where θ_D is the equivalence relation defined by

$$s\theta_D t \quad \text{iff} \quad \sum\{x \in s^{\mathsf{d}} \otimes t^{\mathsf{d}} : \hat{s}(x) = \hat{t}(x)\} \in D.$$

Given an n-ary relation R on M, and $s_1/D \dots s_n/D \in M[B]/D$, we have

$$M[B]/D \models R[s_1/D \dots s_n/D] \quad \text{iff} \tag{1}$$

$$\sum\{x \in s_1^{\mathsf{d}} \otimes \cdots \otimes s_n^{\mathsf{d}} : M \models R[\hat{s}_1(x) \dots \hat{s}_n(x)]\} \in D. \tag{2}$$

Thus $M[B]/D$ is also a structure of the language L, usually called the (unbounded) reduced Boolean power of M (with respect to B, D). If we take D to be the trivial filter $\{1\}$, we get the unbounded Boolean power $M[B]$, and if we take D to be an ultrafilter, we get a Boolean ultrapower.

By an easy induction on the structure of formulas, it follows that if D is an ultrafilter then (1) and (2) remain equivalent when R is replaced by any formula.

Theorem 2.4. *Let B, C be complete Boolean algebras, $D \in \mathrm{Uf}(B)$ and $E \in \mathrm{Uf}(C)$. The following are equivalent:*

- (i) *$D \le E$,*
- (ii) *for any model M, there is an elementary embedding of $M[B]/D$ into $M[C]/E$,*
- (iii) *there is an elementary embedding of $\bar{B}[B]/D$ into $\bar{B}[C]/E$.*

Proof. Obviously (ii) implies (iii).

Assume (i) holds, and let g and f_σ be the associated maps for this inequality. Define $e : M[B]/D \to M[C]/E$ by $e(s/D) = (s \circ f_{s^{\mathsf{d}}})/E$. It suffices to check that this map is elementary: Let $\phi(x_1, \dots, x_n)$ be any formula in the language of M, and $s_1/D, \ldots, s_n/D \in M[B]/D$. Then

$$\begin{aligned}
&M[B]/D \models \phi[s_1/D, \ldots, s_n/D] \\
\text{iff } &\sum\{x \in s_1^{\mathsf{d}} \otimes \cdots \otimes s_n^{\mathsf{d}} : M \models \phi[\hat{s}_1(x), \ldots, \hat{s}_n(x)]\} \in D \\
\text{iff } &\sum f_\tau^{-1}[\{x \in \tau : M \models \phi[\hat{s}_1(x), \ldots, \hat{s}_n(x)]\}] \in E, \text{ where } \tau = s_1^{\mathsf{d}} \otimes \cdots \otimes s_n^{\mathsf{d}} \\
\text{iff } &\sum\{y \in g(\tau) : M \models \phi[\hat{s}_1(f_\tau(y)), \ldots, \hat{s}_n(f_\tau(y))]\} \in E \\
\text{iff } &\sum\{y \in g(s_1^{\mathsf{d}}) \otimes \cdots \otimes g(s_n^{\mathsf{d}}) : M \models \phi[s_1(\hat{f}_{s_1^{\mathsf{d}}}(y)), \ldots, s_n(\hat{f}_{s_n^{\mathsf{d}}}(y))]\} \in E \\
\text{iff } &M[C]/E \models \phi[e(s_1/D), \ldots, e(s_n/D)]
\end{aligned}$$

where the second last "iff" is justified by the concurrancy condition on the f_σ: Since $\tau \leq s_i^{\mathsf{d}}$, it follows by concurrancy that

$$\sum\{y \in g(\tau) \otimes g(s_i^{\mathsf{d}}) : \hat{f}_\tau(y) \leq \hat{f}_{s_i^{\mathsf{d}}}(y)\} \in E$$

for each $i = 1, \ldots, n$, hence

$$\sum\{y \in g(\tau) \otimes g(s_1^{\mathsf{d}}) \otimes \cdots \otimes g(s_n^{\mathsf{d}}) : \hat{s}_i(\hat{f}_\tau(y)) = s_i(\hat{f}_{s_i^{\mathsf{d}}}(y)) \text{ for all } i\} \in E.$$

Now assume (iii) holds, and let e be the given elementary embedding. Consider the identity map $\mathrm{id}_\sigma : \sigma \to \sigma \subseteq B$, with the codomain extended to the set B. Then id_σ/D is in $\bar{B}[B]/D$, so $e(\mathrm{id}_\sigma/D)$ is an equivalence class in $\bar{B}[C]/E$. For each $\sigma \in \mathbb{P}_B$, choose $f_\sigma \in e(\mathrm{id}_\sigma/D)$, and let $g(\sigma) = \mathrm{dom} f_\sigma$. We first argue that although f_σ maps into $\bar{B}$, we can assume that it's range is entirely within σ: Let $\bar{\sigma}$ be the relation symbol of $\bar{B}$ such that $\bar{B} \models \bar{\sigma}[x]$ iff $x \in \sigma$. Since $\sum\{x \in \sigma : \bar{B} \models \bar{\sigma}(\mathrm{id}_\sigma(x))\} = 1 \in D$, we have that $\bar{B}[B] \models \bar{\sigma}[\mathrm{id}_\sigma/D]$, hence $\bar{B}[C] \models \bar{\sigma}[e(\mathrm{id}_\sigma/D)]$. But this means that $\sum\{y \in g(\sigma) : \bar{B} \models \bar{\sigma}(f_\sigma(y))\} = c \in E$. Therefore $f_\sigma(y) \in \sigma$ whenever $y \leq c$. Choose any fixed $b \in \sigma$ and define $f'_\sigma : g(\sigma) \to \sigma$ by

$$f'_\sigma(y) = \begin{cases} f_\sigma(y) & \text{if } y \leq c \\ b & \text{otherwise} \end{cases}$$

then $f'_\sigma/E = f_\sigma/E$, so we can replace f_σ by f'_σ.

Next we show that for all $S \subseteq \sigma$, $\sum S \in D$ iff $\sum f_\sigma^{-1}[S] \in E$. Let $\bar{S}$ be the relation symbol of $\bar{B}$ such that $\bar{B} \models \bar{S}[x]$ iff $x \in S$. Then

$$\begin{aligned}
&\sum S \in D \\
\text{iff } &\sum \{x \in \sigma : \bar{B} \models \bar{S}[\mathrm{id}_\sigma(x)]\} \in D \\
\text{iff } &\bar{B}[B]/D \models \bar{S}[\mathrm{id}_\sigma/D] \\
\text{iff } &\bar{B}[C]/E \models \bar{S}[e(\mathrm{id}_\sigma/D)] \\
\text{iff } &\bar{B}[C]/E \models \bar{S}[f_\sigma/E] \\
\text{iff } &\sum \{y \in g(\sigma) : \bar{B} \models \bar{S}[f_\sigma(y)]\} \in E \\
\text{iff } &\sum f^{-1}[\{x \in \sigma : \bar{B} \models \bar{S}[x]\}] \in E \\
\text{iff } &\sum f^{-1}[S] \in E.
\end{aligned}$$

Finally we prove the concurrancy condition: Let $\tau \leq \sigma$, and let $\bar{R}$ be a relation symbol for the graph of $j_\sigma{\restriction}_\tau$, i.e., $\bar{B} \models \bar{R}[b,c]$ iff $b \in \tau$, $c \in \sigma$ and $b \leq c$. Then $\bar{B}[B]/D \models \bar{R}[\mathrm{id}_\tau/D, \mathrm{id}_\sigma/D]$ since $\sum\{x \in \tau : \bar{B} \models R[\mathrm{id}_\tau(x), \hat{\mathrm{id}}_\sigma(x)]\} = 1$. Therefore $\bar{B}[C]/E \models \bar{R}[f_\tau/E, f_\sigma/E]$, which means that $\sum\{y \in g(\tau) \otimes g(\sigma) : \bar{B} \models \bar{R}[\hat{f}_\tau(y), \hat{f}_\sigma(y)]\} \in E$. This is equivalent to the concurrancy condition. □

Remark 2.5. In the definition of $D \leq E$, it suffices to consider partitions from a dense subsemilattice S of $\mathbb{P}_B$. This follows from the characterisation theorem above since if $f \in B^\sigma$ for some $\sigma \in \mathbb{P}_B$, then there exists $\tau \in S$ with $\tau \leq \sigma$, and we may replace f by $\hat{f}{\restriction}_\tau$.

Example 2.6. Let $A = \prod_{i \in I} B_i$ be a product of complete Boolean algebras. Recall that each factor B_i is isomorphically embedded into the relative subalgebra $A{\restriction}e_i$, where e_i is the I-tuple for which $e_i(i) = 1_{B_i}$ and $e_i(j)$ is 0_{B_j} in all other coordinates $j \neq i$. We denote this relative embedding of B_i into A by γ_i. Observe that $\pi_i \circ \gamma_i$ is the identity function on B_i, and although γ_i is not a homomorphism, it does preserve all existing joins and meets.

For a family of partitions $\sigma_i \in \mathbb{P}_{B_i}$ $(i \in I)$ we define the *partition product* $\mathbb{X}_{i \in I} \sigma_i$ to be $\bigcup_{i \in I} \gamma_i[\sigma_i]$. This is easily seen to be a partition of A, and the set of all partition products forms a dense subsemilattice of $\mathbb{P}_A$.

Recall from [3] the definition of a relative subalgebra $B{\restriction}u$ of a Boolean algebra B with $u \in B$. If $D \in \mathrm{Uf}(B)$ and $u \in D$, we let $Du = \{x \cdot u : x \in D\}$. Note that Du is an ultrafilter in $B{\restriction}u$. With the characterisation theorem at hand, we get the following result.

Proposition 2.7. *Let B, C be complete Boolean algebras, and $D \in \mathrm{Uf}(B)$, $E \in \mathrm{Uf}(C)$. The following are equivalent:*

(i) $D \leq E$,

(ii) *there exist* $u \in D$ *and* $v \in E$ *such that* $Du \leq Ev$,

(iii) *for some* $u \in D$ *and some complete subalgebra* C' *of* C, *we have* $Du \leq C' \cap E$.

Proof. (i) implies (ii), and (i) implies (iii) follow immediately if we take $u = 1_B$, $v = 1_C$ and $C' = C$. To prove (ii) implies (i), we observe that for any structure M, $M[B]/D \cong M[B{\restriction}u]/Du$, and by the preceeding theorem, the latter is elementarily embedded in $M[C{\restriction}v]/Ev \cong M[C]/E$. Another application of the same theorem gives (i).

The implication from (iii) to (i) is proved similarly, using the additional fact that $M[C']/(C' \cap E)$ is elementarily embedded in $M[C]/E$. □

3. Extending RK-posets

In this section we look at conditions under which the RK-poset of one Boolean algebra is embedded in the RK-poset of another.

3.1. Relative subalgebras.

Lemma 3.1. *Let* B *be a Boolean algebra and* $C = B{\restriction}a$ *a relative subalgebra of* B. *If* D *is an ultrafilter of* C *then* $\bar{D} = \{x \in B : x \geq y$ *for some* $y \in D\}$ *is an ultrafilter of* B.

Proof. By definition, $\bar{D}$ is up-closed, and since D is meet-closed, the same holds true for $\bar{D}$. Therefore $\bar{D}$ is a filter. Given $x \in B$, we have $x \cdot a \in C$, hence $x \cdot a \in D$ or $-^a(x \cdot a) \in D$. Since $-^a(x \cdot a) = -x \cdot a$, we either have $x \in \bar{D}$ or $-x \in \bar{D}$, as required. □

Corollary 3.2. *If* C *is isomorphic to a relative subalgebra of* B, *then* $\mathrm{RK}(C)$ *is embeddable into* $\mathrm{RK}(B)$.

Proof. We can assume that $C = B{\restriction}a$ for some $a \in B$. Let $D, E \in \mathrm{Uf}(C)$. Then $D = \bar{D}a$ and $E = \bar{E}a$, so if $D \leq E$, then $\bar{D} \leq \bar{E}$ follows from Proposition 2.7(ii) $\Rightarrow$ (i).

Conversely, $\bar{D} \leq \bar{E}$ implies $D \leq E$ since relativization preserves the comparability of ultrafilters. □

3.2. Powers of complete Boolean algebras. For a set J and a complete Boolean algebra B, consider the direct power B^J. For ultrafilters D in B, and H in $\mathcal{P}(J)$, we define $D_H = \{s \in B^J : s^{-1}[D] \in H\}$.

If B is a powerset algebra, say $\mathcal{P}(I)$, then B^J is isomorphic to $\mathcal{P}(I \times J)$ and D_H is isomorphic to the product ultrafilter $D \times H$ (as defined in [2]). It is straightforward to check that D_H is an ultrafilter in this more general setting.

Lemma 3.3. *Suppose* B *is a complete Boolean algebra,* $D \in \mathrm{Uf}(B)$, $F \in \mathrm{Uf}(\mathcal{P}(I))$ *and* $H \in \mathrm{Uf}(\mathcal{P}(J))$. *If* $F \leq H$ *then* $D_F \leq D_H$.

Proof. Since F and H are ultrafilters in powerset algebras, we can use the original definition of the RK-order. Assume $F \leq H$, and let h be the function from J to I such that for all $S \subseteq I$, $S \in F$ implies $h^{-1}[S] \in H$. To show that $D_F \leq D_H$, it suffices by Proposition 1.2 to define a complete homomorphism $\alpha : B^I \to B^J$ such that $\alpha[D_F] \subseteq D_H$. Given $s \in B^I$, we let $\alpha(s) = s \circ h$. Since the operations in B^I are defined pointwise, this is a complete homomorphism, and for $s \in D_F$ we have $s^{-1}[D] \in F$, hence $(s \circ h)^{-1}[D] = h^{-1}[s^{-1}[D]] \in H$. □

The reverse implication requires a bit more work and an additional assumption. A filter D is said to be κ-complete if for any set $S \subset D$ with $|S| < \kappa$ we have $\prod S \in D$. For an ultrafilter in a complete Boolean algebra B, this is equivalent to the condition that for any $\sigma \in \mathbb{P}_B$ with $|\sigma| < \kappa$ we have $D \cap \sigma \neq \emptyset$ (see e.g. [4] 0.9).

Lemma 3.4. *Suppose B is a complete Boolean algebra, $D \in \mathrm{Uf}(B)$, $F \in \mathrm{Uf}(\mathcal{P}(I))$ and $H \in \mathrm{Uf}(\mathcal{P}(J))$. If D is $|I|^+$-complete then $D_F \leq D_H$ implies $F \leq H$.*

Proof. Suppose $D_F \leq D_H$. Then there exists a map $g : \mathbb{P}_{B^I} \to \mathbb{P}_{B^J}$ and maps $h_\gamma : g(\gamma) \to \gamma \in \mathbb{P}_{B^I}$ such that for all $S \subseteq \gamma$, $\sum S \in D_F$ implies $\sum h_\gamma^{-1}[S] \in D_H$.

Consider the partition $\sigma_I = \{\chi_{\{i\}} \in B^I : i \in I\}$ and the corresponding partition $\sigma_J \in \mathbb{P}_{B^J}$, where χ_K is the charateristic function of $K \subseteq I$ or J respectively. Let α be the complete homomorphism from $\mathcal{P}(\sigma_I)$ to $\mathcal{P}(g(\sigma_I))$ given by $\alpha(\sum S) = \sum h_{\sigma_I}^{-1}[S]$.

To show that $F \leq H$, we need to define a map $h : J \to I$ such that $S \in F$ implies $h^{-1}[S] \in H$ for all $S \subseteq I$. Given $j \in J$ and $i \in I$, let $h(j) = i$ iff $\pi_j(\alpha(\chi_{\{i\}})) \in D$. The map is well-defined for all $j \in J$ since we are assuming that D is $|I|^+$-complete, so the partition $\pi_j \circ \alpha[\sigma_I] \setminus \{0\}$ intersects D, and since D is a filter, this intersection is a singleton.

Let $S \in F$. This is equivalent to $\{i \in I : \chi_S(i) \in D\} \in F$, and hence to $\chi_S \in D_F$. It follows that $\alpha(\chi_S) \in D_H$ and therefore $\alpha(\chi_S)^{-1}[D] \in H$. The following equivalent statements show that $\alpha(\chi_S)^{-1}[D] = h^{-1}[S]$:

$j \in \alpha(\chi_S)^{-1}[D]$

iff

$\alpha(\chi_S)(j) \in D$

iff (since $\chi_S = \sum_{i \in S} \chi_{\{i\}}$)

$\sum_{i \in S} \alpha(\chi_{\{i\}})(j) \in D$

iff (since D is $|I|^+$-complete)

$\alpha(\chi_{\{i\}})(j) \in D$ for some $i \in S$

iff (since $\pi_j(s) = s(j)$)

$h(j) \in S$

iff

$j \in h^{-1}[S]$

□

Theorem 3.5. *Let B be a complete Boolean algebra, and suppose there exists a κ^+-complete ultrafilter in B. Then the poset* $\mathrm{RK}(\mathcal{P}(\lambda))$ *is order embeddable into the poset* $\mathrm{RK}(B^\kappa)$ *for any $\lambda \leq \kappa$.*

If B is homogeneous and contains a partition of size κ then $B^\kappa \cong B$. Hence if B has a κ^+-complete ultrafilter then $\mathrm{RK}(\mathcal{P}(\kappa))$ is order embeddable into $\mathrm{RK}(B)$.

An example of such a boolean algebra B is given by the collapsing algebra $\mathrm{Col}(\kappa^+, \lambda)$ if we assume that κ^+ is strongly inaccessible and $|\mathrm{Col}(\kappa^+, \lambda)|$-almost compact (see [4] Theorem 3.6), or if we assume that κ^+ is measurable.

Problem 3.6. Can the above theorem be proved in ZFC (i.e. without the large cardinal assumption about the existence of a κ^+-complete ultrafilter)?

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