A Survey of Residuated Lattices

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Abstract. Residuation is a fundamental concept of ordered structures and categories. In this survey we consider the consequences of adding a residuated monoid operation to lattices. The resulting residuated lattices have been studied in several branches of mathematics, including the areas of lattice-ordered groups, ideal lattices of rings, linear logic and multi-valued logic. Our exposition aims to cover basic results and current developments, concentrating on the algebraic structure, the lattice of varieties, and decidability.

We end with a list of open problems that we hope will stimulate further research.

1 Introduction

A binary operation $\cdot$ on a partially ordered set $\langle P, \leq \rangle$ is said to be *residuated* if there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

$$x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x \backslash z.$$

The operations $\backslash$ and $/$ are referred to as the *right and left residual* of $\cdot$, respectively. It follows readily from this definition that $\cdot$ is residuated if and only if it is order preserving in each argument and, for all $x, y, z \in P$, the inequality $x \cdot y \leq z$ has a largest solution for x (namely z/y) and for y (namely $x \backslash z$). In particular, the residuals are uniquely determined by $\cdot$ and $\leq$.

The system $\mathbf{P} = \langle P, \cdot, \backslash, /, \leq \rangle$ is called a *residuated partially ordered groupoid* or *residuated po-groupoid*. We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element e, say, and the partial order is a lattice order. In this case we add the monoid unit and the lattice operations to the similarity type to get a purely algebraic structure $\mathbf{L} = \langle L, \vee, \wedge, \cdot, e, \backslash, / \rangle$ called a *residuated lattice-ordered monoid* or *residuated lattice* for short.

The class of all residuated lattices will be denoted by $\mathcal{RL}$. It is easy to see that the equivalences that define residuation can be captured by equations and thus $\mathcal{RL}$ is a finitely based variety. Our aim in this paper is to cover basic results and current developments, concentrating on the algebraic structure, the lattice of varieties and decidability results.

The defining properties that describe the class $\mathcal{RL}$ are few and easy to grasp. At the same time, the theory is also sufficiently robust to have been studied in several branches of mathematics, including the areas of lattice-ordered groups, ideal lattices of rings, linear logic and multi-valued logic. Historically speaking, our study draws from the work of W. Krull in [Kr24] and that of Morgan Ward and R. P. Dilworth, which appeared in a series of important papers [Di38], [Di39], [Wa37], [Wa38], [Wa40], [WD38] and [WD39]. Since that time, there has been substantial research regarding some specific classes of residuated structures, see for example [AF88], [Mu86], [Ha98] and [NPM99].

We conclude the introduction by summarizing the contents of the paper. Section 2 contains basic results on residuation and the universal algebraic background needed in the remainder of the paper. In Section 3 we develop the notion of a normal subalgebra and show that $\mathcal{RL}$ is an "ideal variety", in the sense that it is an equational class in which congruences correspond to "convex normal" subalgebras, in the same way that group congruences correspond to normal subgroups. Further, we provide an element-wise description of the convex normal subalgebra generated by an arbitrary subset. In Section 4, we study varieties of residuated lattices with distributive lattice reducts. As an application of the general theory developed in the preceding sections, we produce an equational basis for the important subvariety that is generated by all residuated chains. We conclude Section 4 by introducing the classes of generalized MV-algebras and generalized BL-algebras. These objects generalize MV-algebras and BL-algebras in two directions: the existence of a lower bound is not stipulated and the commutativity assumption is dropped. Thus, bounded commutative generalized MV-algebras are reducts of MV-algebras, and likewise for BL-algebras. Section 5 is concerned with the variety and subvarieties of cancellative residuated lattices, that is, those residuated lattices whose monoid reducts are cancellative. We construct examples that show that in contrast to ℓ-groups, the lattice reducts of cancellative residuated lattices need not be distributive. Of particular interest is the fact that the classes of cancellative integral generalized MV-algebras and cancellative integral generalized BL-algebras coincide, and are precisely the negative cones of ℓ-groups, hence the latter form a variety. We prove that the map that sends a subvariety of ℓ-groups to the corresponding class of negative cones is a lattice isomorphism of the subvariety lattices, and show how to translate equational bases between corresponding subvarieties. Section 6 is devoted to the study of the lattice of subvarieties of residuated lattices. We prove that there are only two cancellative commutative varieties that cover the trivial variety, namely the varieties generated by the integers and the negative integers (with zero). On the other hand, we show that there are uncountably many non-cancellative atoms of the subvariety lattice. In Section 7 we give details of a result of Ono and Komori [OK85] that shows the equational theory of $\mathcal{RL}$ is decidable. We also mention several related decidability results about subvarieties, and summarize the currently know results in a table. In the last section we list open problems that we hope will stimulate further research.

2 Basic Results

Let $\cdot$ be a residuated binary operation on a partially ordered set $\langle P, \leq \rangle$ with residuals $\backslash$ and $/$. Intuitively, the residuals serve as generalized division operations, and x/y is read as "x over y" while $y \backslash x$ is read as "y under x". In either case, x is considered the *numerator* and y is the *denominator*. When doing calculations, we tend to favor $\backslash$ since it is more closely related to applications in logic. However, any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e. replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y \backslash x$). It follows directly from the definition above that a statement is equivalent to its mirror image, and we often state results in only one form.

As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally $\vee$ and $\wedge$. We also define $x^1 = x$ and $x^{n+1} = x^n \cdot x$.

The existence of residuals has the following basic consequences.

Proposition 2.1. *Let* $\mathbf{P}$ *be a residuated po-groupoid.*

(i) *The operation $\cdot$ preserves all existing joins in each argument; i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subset P$ then $\bigvee_{x \in X, y \in Y} x \cdot y$ exists and*

$$\left(\bigvee X\right) \cdot \left(\bigvee Y\right) = \bigvee_{x \in X, y \in Y} x \cdot y.$$

(ii) *The residuals preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e. if $\bigvee X$ and $\bigwedge Y$ exist for $X, Y \subset P$ then for any $z \in P$, $\bigwedge_{x \in X} x \backslash z$ and $\bigwedge_{y \in Y} z \backslash y$ exist and*

$$\left(\bigvee X\right) \Big\backslash z = \bigwedge_{x \in X} x \backslash z \quad \text{and} \quad z \Big\backslash \left(\bigwedge Y\right) = \bigwedge_{y \in Y} z \backslash y.$$

A *residuated po-monoid* $\langle P, \cdot, e, \backslash, /, \leq \rangle$ is a residuated po-groupoid with an identity element e and an associative binary operation.

Proposition 2.2. *The following identities (and their mirror images) hold in any residuated po-monoid.*

(i) $(x \backslash y) z \leq x \backslash yz$

(ii) $x \backslash y \leq zx \backslash zy$

(iii) $(x\backslash y)(y\backslash z) \le x\backslash z$

(iv) $xy\backslash z = y\backslash(x\backslash z)$

(v) $x\backslash(y/z) = (x\backslash y)/z$

(vi) $(x\backslash e)y \le x\backslash y$

(vii) $x(x\backslash x) = x$

(viii) $(x\backslash x)^2 = x\backslash x$

If a residuated po-groupoid $\mathbf{P}$ has a bottom element we usually denote it by 0. In this case $\mathbf{P}$ also has a top element $0\backslash 0$, denoted by 1, and, for all $x \in P$, we have

$$x0 = 0 = 0x \quad \text{and} \quad 0\backslash x = 1 = x\backslash 1.$$

An equational basis for $\mathcal{RL}$ is given, for example, by a basis for lattice identities, monoid identities, $x(x\backslash z \wedge y) \le z$, $y \le x\backslash(xy \vee z)$ and the mirror images of these identities. Note that the operations $\cdot, \backslash, /$ are performed before $\vee, \wedge$. The two inequalities above follow directly from the fact that $\backslash$ is a right-residual of $\cdot$. Conversely, if the inequalities above hold, then $xy \le z$ implies $y \le x\backslash(xy \vee z) = x\backslash z$, and $y \le x\backslash z$ implies $xy = x(x\backslash z \wedge y) \le z$. Hence the identities indeed capture the property of being residuated. The argument just given is a special case of the following simple but useful observation (and its dual).

Lemma 2.3. *Let r, s, t be terms in a (semi)lattice-ordered algebra, and denote the sequence of variables $y_1, \dots, y_n$ by $\underline{y}$. Then the quasi-identity*

$$x \le r(\underline{y}) \Rightarrow s(x, \underline{y}) \le t(x, \underline{y})$$

is equivalent to the identity

$$s(x \wedge r(\underline{y}), \underline{y}) \le t(x \wedge r(\underline{y}), \underline{y}).$$

Combined with the property of residuation, the lemma above allows many quasi-identities to be translated to equivalent identities. For example the class of residuated lattices with cancellative monoid reducts forms a variety (see Section 5).

An interesting special case results when the underlying monoid structure of a residuated po-monoid is in fact a group. In this case, $x\backslash y$ is term definable as $x^{-1}y$ (and likewise $x/y = xy^{-1}$), so this class coincides with the class of *po-groups*. If we add the requirement that the partial order is a lattice, then we get the variety $\mathcal{LG}$ of *lattice-ordered groups*. This shows that lattice-ordered groups (or ℓ-groups for short) are term-equivalent to a subvariety of $\mathcal{RL}$. In the language of residuated lattices, this subvariety is defined relative to $\mathcal{RL}$ by the identity $x(x\backslash e) = e$.

Other well known subvarieties of $\mathcal{RL}$ include *integral residuated lattices* ($\mathcal{IRL}$, defined by $x \le e$), *commutative residuated lattices* ($\mathcal{CRL}$, defined by $xy = yx$), *Brouwerian algebras* ($\mathcal{B}r\mathcal{A}$, defined by $x \wedge y = xy$), and generalized Boolean algebras ($\mathcal{GBA}$, defined by $x \wedge y = xy$ and $x \vee y = (x\backslash y)\backslash y$).

We briefly discuss further connections between residuated lattices and existing classes of algebras. Recall that a *reduct* of an algebra is an algebra with the same universe but with a reduced list of fundamental operations. The opposite notion of an *expansion* is given by expanding the list of fundamental operations with new operations defined on the same universe. A *subreduct* is a subalgebra of a reduct. Subreducts obtained from $\mathcal{IRL}$ by omitting the $\vee$ operation have been studied as *generalized hoops*, and adding commutativity gives *hoops*. Omitting $\vee, \wedge, \backslash$ produces partially ordered left-residuated integral monoids (polrims). Subreducts of commutative $\mathcal{IRL}$ with only $\backslash, e$ as fundamental operations are *BCK-algebras*. If we expand bounded residuated lattices by considering the bounds as constant operations, we obtain the variety of residuated $0, 1$-lattices. It has subvarieties corresponding to the variety of Boolean algebras, Stone algebras, Heyting algebras, MV-algebras and intuitionistic linear logic algebras. Further expansion with unary complementation gives relation algebras and residuated Boolean monoids.

The classes mentioned above have been studied extensively in their own right. Most of them have also been studied in logical form, such as propositional logic, intuitionistic logic, multi-valued logic, BCK logic and linear logic. In this survey we consider only the algebraic viewpoint and restrict ourselves mainly to $\mathcal{RL}$ and (some of) its subvarieties.

Universal Algebraic Background. In order to describe the algebraic structure and properties of residuated lattices, we recall here some general terminology.

An n-ary operation f is compatible with a binary relation θ if

$$\text{for all } \langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \theta \text{ we have } f(a_1, \ldots, a_n)\, \theta f(b_1, \ldots, b_n).$$

An equivalence relation θ on (the underlying set of) an algebra $\mathbf{A}$ is a *congruence* if each fundamental operation (and hence each term-definable operation) of $\mathbf{A}$ is compatible with θ. The equivalence class of a is denoted by $[a]_\theta$, and the quotient algebra of $\mathbf{A}$ with respect to θ is denoted by $\mathbf{A}/\theta$. The collection of all congruences of $\mathbf{A}$ is written $\mathrm{Con}(\mathbf{A})$. It is an algebraic lattice[1] with intersection as meet.

An algebra $\mathbf{A}$ is *congruence permutable* if $\theta \circ \phi = \phi \circ \theta$ for all $\theta, \phi \in \mathrm{Con}(\mathbf{A})$. A variety is congruence permutable if each member has this property. By a result of Mal'cev (see [BS81]), this is equivalent to the existence of a ternary term p such that the variety satisfies the identities $p(x, y, y) = x = p(y, y, x)$. It is not difficult to show that, for residuated lattices, the term $p(x, y, z) = x/(z\backslash y \wedge e) \wedge z/(x\backslash y \wedge e)$ has this property, hence $\mathcal{RL}$ is congruence permutable.

[1] An element c in a complete lattice is *compact* if for all subsets S, $c \le \bigvee S$ implies $c \le s_1 \vee \cdots \vee s_n$ for some $s_1, \ldots, s_n \in S$. A complete lattice is *algebraic* if every element is a join of compact elements.

An algebra is *congruence regular* if each congruence is determined by any one of its congruence classes (i.e. for all $\theta, \phi \in \mathrm{Con}(A)$ if $[a]_\theta = [a]_\phi$ for some $a \in A$ then $\theta = \phi$).

Any algebra that has a group reduct is both congruence permutable and congruence regular. For algebras with a constant e, a weaker version of congruence regularity is *e-regularity*: for all $\theta, \phi \in \mathrm{Con}(A)$, $[e]_\theta = [e]_\phi$ implies $\theta = \phi$. We will see below that residuated lattices are e-regular, but not regular.

3 Structure theory

Unless noted otherwise, the results in this section are due to Blount and Tsinakis [BT]. The presentation using ideal terms is new, and the proofs have been revised considerably. A general framework for the notion of ideals in universal algebras was introduced by Ursini [Ur72] (see also Gumm and Ursini in [GU84]).

A term $t(u_1, \ldots, u_m, x_1, \ldots, x_n)$ in the language of a class $\mathcal{K}$ of similar algebras with a constant e is called an *ideal term of* $\mathcal{K}$ *in* $x_1, \ldots, x_n$ if $\mathcal{K}$ satisfies the identity $t(u_1, \ldots, u_m, e, \ldots, e) = e$. We also write the term as $t_{u_1,\ldots,u_m}(x_1, \ldots, x_n)$ to indicate the distinction between the two types of variables.

Examples of ideal terms for $\mathcal{RL}$ are $\lambda_u(x) = (u\backslash xu) \wedge e$, $\rho_u(x) = (ux/u) \wedge e$, referred to as the *left and right conjugates of* x *with respect to* u, as well as $\kappa_u(x, y) = (u \wedge x) \vee y$, and the fundamental operations $x \diamond y$ for $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$.

A subset H of $\mathbf{A} \in \mathcal{K}$ is a *$\mathcal{K}$-ideal of* $\mathbf{A}$ if for all ideal terms t of $\mathcal{K}$, and all $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_n \in H$ we have $t^{\mathbf{A}}(a_1, \ldots, a_m, b_1, \ldots, b_n) \in H$. Note that we use the superscript $\mathbf{A}$ to distinguish the term function $t^{\mathbf{A}}$ from the (syntactic) term t that defines it.

Clearly any e-congruence class is a $\mathcal{K}$-ideal. A class $\mathcal{K}$ is called an *ideal class* if in every member of $\mathcal{K}$ every ideal is an e-congruence class. We will prove below that this is the case for residuated lattices, and that the ideals of a residuated lattice are characterized as those subalgebras that are closed under the ideal terms λ, ρ and κ.

In analogy with groups, a subset S of a residuated lattice $\mathbf{L}$ is called *normal* if $\lambda_u(x), \rho_u(x) \in S$ for all $u \in L$ and all $x \in S$. The closed interval $\{u \in L : x \leq u \leq y\}$ is denoted by $[x, y]$. As for posets, we call S *convex* if $[x, y] \subseteq S$ for all $x, y \in S$. Note that for a sub*lattice* S the property of being convex is equivalent to $\kappa_u(x, y) \in S$ for all $u \in L$ and $x, y \in S$. Thus a convex normal subalgebra is precisely a subalgebra of $\mathbf{L}$ that is closed under the $\mathcal{RL}$-ideal terms λ, ρ and κ.

Now it follows immediately that every ideal is a convex normal subalgebra, and since we observed earlier that every e-congruence class is an ideal, we have shown that every e-congruence class is a convex normal subalgebra. As we will see below, the converse requires a bit more work.

The term $d(x, y) = x\backslash y \wedge y\backslash x \wedge e$ is useful for the description of congruences. Note that in ℓ-groups, d gives the negative absolute value of $x - y$. Alternatively one could use the opposite term $d'(x, y) = x/y \wedge y/x \wedge e$.

Lemma 3.1. *Let* $\mathbf{L}$ *be a residuated lattice. For any congruence* θ *of* $\mathbf{L}$, *we have* $a\theta b$ *if and only if* $d(a,b)\theta e$.

Proof. From $a\theta b$ one infers

$$e = (a\backslash a \wedge b\backslash b \wedge e)\,\theta\,(a\backslash b \wedge b\backslash a \wedge e) = d(a,b).$$

Conversely, $d(a,b)\ \theta\ e$ implies $a\ \theta\ ad(a,b) \leq a(a\backslash b) \leq b$, hence $[a]_\theta \leq [b]_\theta$, and similarly $[b]_\theta \leq [a]_\theta$. Therefore we have $a\theta b$. ∎

In the next result, $L^- = \{x \in L : x \leq e\}$ denotes the *negative part* of L.

Corollary 3.2. $\mathcal{RL}$ *is an e-regular variety. In fact, if* θ *and* ϕ *are congruences on a residuated lattice* $\mathbf{L}$, *then* $[e]_\theta \cap L^- = [e]_\phi \cap L^-$ *implies* $\theta = \phi$.

To see that $\mathcal{RL}$ is not regular, it suffices to consider the 3-element Brouwerian algebra $\{0 < a < e\}$ with $xy = x \wedge y$, since it has two congruences with $\{0\}$ as congruence class.

Lemma 3.3. *Suppose* M *is a convex normal submonoid of* $\mathbf{L}$. *For any* $a, b \in L$, $d(a,b) \in M$ *if and only if* $d'(a,b) \in M$.

Proof. Assume $d(a,b) \in M$. By normality $\rho_a(d(a,b)) \in M$. But

$$\rho_a(d(a,b)) \leq a(a\backslash b)/a \wedge e \leq b/a \wedge e \leq e \in M,$$

and similarly $\rho_b(d(a,b)) \leq a/b \wedge e \leq e$. Since M is convex and closed under $\cdot$, we have $(a/b \wedge e)(b/a \wedge e) \in M$, and this element is below $d'(a,b) \leq e$. Again by convexity, $d'(a,b) \in M$. The reverse implication is similar, with ρ replaced by λ. ∎

Lemma 3.4. *Let* H *be a convex normal subalgebra of* $\mathbf{L}$, *and define*

$$\theta_H = \{\langle a, b\rangle : d(a,b) \in H\}.$$

Then θ_H *is a congruence of* L *and* $H = [e]_{\theta_H}$.

Proof. Clearly θ_H is reflexive and symmetric. Assuming $d(a,b), d(b,c) \in H$, we have

$$d(a,b)d(b,c) \wedge d(b,c)d(a,b) \leq (a\backslash b)(b\backslash c) \wedge (c\backslash b)(b\backslash a) \wedge e \leq d(a,c) \leq e$$

by 2.2 (iii). Since H is convex, $d(a,c) \in H$, and therefore θ_H is an equivalence relation.

Assuming $d(a,b) \in H$ and $c \in L$, it remains to show that $d(c\diamond a, c\diamond b), d(a\diamond c, b\diamond c) \in H$ for $\diamond \in \{\cdot, \wedge, \vee, \backslash, /\}$. Since H is convex and $d(x,y) \leq e \in H$, it suffices to construct elements of H that are below these two expressions.

By 2.2 (ii) $a\backslash b \leq ca\backslash cb$, hence $d(a,b) \leq d(ca, cb)$.

By 2.2 (i) and (iv) $\lambda_c(d(a,b)) \leq c\backslash(a\backslash b)c \wedge c\backslash(b\backslash a)c \wedge e \leq d(ac, bc)$.

Since $(a \wedge c) \cdot d(a,b) \leq ad(a,b) \wedge cd(a,b) \leq b \wedge c$, we have $d(a,b) \leq (a \wedge c)\backslash(b \wedge c)$, and similarly $d(a,b) \leq (b \wedge c)\backslash(a \wedge c)$. Therefore $d(a,b) \leq d(a \wedge c, b \wedge c) \leq e$. The computation for $\vee$ is the same.

By 2.2 (iii) $a\backslash b \leq (c\backslash a)\backslash(c\backslash b)$, whence $d(a,b) \leq d(c\backslash a, c\backslash b)$. Using the same lemma we get $a\backslash b \leq (a\backslash c)/(b\backslash c)$ and therefore $d(a,b) \leq d'(a\backslash c, b\backslash c)$. It follows that $d'(a\backslash c, b\backslash c)$ is in H, so by Lemma 3.3 the same holds for d. The computation for / is a mirror image of the above.

Finally, $H = [e]_{\theta_H}$ holds because H is a convex subalgebra. Indeed, $a \in H$ implies $d(a,e) \in H$, and the reverse implication holds since $d(a,e) \leq a \leq d(a,e)\backslash e$. ■

The collection of all convex, normal subalgebras of a residuated lattice $\mathbf{L}$ will be denoted by $\mathrm{CN}(\mathbf{L})$. This is easily seen to be an algebraic lattice, with meet in $\mathrm{CN}(\mathbf{L})$ given by intersections.

Theorem 3.5. [BT] *For any residuated lattice* $\mathbf{L}$, $\mathrm{CN}(\mathbf{L})$ *is isomorphic to* $\mathrm{Con}(\mathbf{L})$, *via the mutually inverse maps* $H \mapsto \theta_H$ *and* $\theta \mapsto [e]_\theta$.

Proof. By the preceding lemma, θ_H is a congruence, and we remarked earlier that $[e]_\theta$ is a convex, normal subalgebra. Since the given maps are clearly order-preserving, it suffices to show they are inverses of each other. The lemma above already proved $H = [e]_{\theta_H}$. To show that $\theta = \theta_{[e]_\theta}$, we let $H = [e]_\theta$ and observe that $[e]_{\theta_H} = H = [e]_\theta$. Since $\mathcal{RL}$ is e-regular, the result follows. ■

The Generation of Ideals. Recall that $\mathbf{H}$ is a convex normal subalgebra of a residuated lattice $\mathbf{L}$ provided it is closed under the $\mathcal{RL}$-ideal terms κ, λ, ρ and the fundamental operations of $\mathbf{L}$. For a subset S of L, let $\mathrm{cn}(S)$ denote the intersection of all convex normal subalgebras containing S. When $S = \{s\}$, we write $\mathrm{cn}(s)$ rather than $\mathrm{cn}(\{s\})$. Clearly $\mathrm{cn}(S)$ can also be generated from S by iterating the ideal terms and fundamental operations. The next result shows that we may compute $\mathrm{cn}(S)$ by applying these terms in a particular order. Let

$$\begin{aligned}
&\Delta(S) = \{s \wedge e/s \wedge e : s \in S\}\\
&\Gamma(S) = \{\lambda_{u_1} \circ \rho_{u_2} \circ \lambda_{u_3} \circ \cdots \circ \rho_{u_{2n}}(s) : n \in \omega, u_i \in L, s \in S\}\\
&\Pi(S) = \{s_1 \cdot s_2 \cdots s_n : n \in \omega, s_i \in S\} \cup \{e\}.
\end{aligned}$$

Thus $\Gamma(S)$ is the normal closure of S, and $\Pi(S)$ is the submonoid generated by S. Note also that if $S \subseteq L^-$ then $\Delta(S) = S$.

Theorem 3.6. [BT] *The convex normal subalgebra generated by a subset* S *in a residuated lattice* $\mathbf{L}$ *is*

$$\mathrm{cn}(S) = \{a \in L : x \leq a \leq x\backslash e \text{ for some } x \in \Pi\Gamma\Delta(S)\}.$$

Proof. Let $\langle S\rangle$ denote the right hand side above. Since this is clearly a subset of $\mathrm{cn}(S)$, we need to show that $S\subseteq\langle S\rangle$ and that $\langle S\rangle$ is a convex normal subalgebra.

Suppose $s\in S$, and consider $x=s\wedge e/s\wedge e\in\Pi\Gamma\Delta(S)$. Then $x\le s$ and $s\le(e/s)\backslash e\le x\backslash e$, hence $s\in\langle S\rangle$. This shows $S\subseteq\langle S\rangle$.

Now let $a,b\in\langle S\rangle$. Then there are $x,y\in\Pi\Gamma\Delta(S)$ such that $x\le a\le x\backslash e$ and $y\le b\le y\backslash e$. Note that $\Pi\Gamma\Delta(S)\subseteq L^-$ since $\Delta(S)\subseteq L^-$ and L^- is closed under conjugation and multiplication. Hence $xy\le x,y$ and therefore $x\backslash e,y\backslash e\le xy\backslash e$ and $xy\le a,b\le xy\backslash e$. Since $xy\in\Pi\Gamma\Delta(S)$, it follows that $\langle S\rangle$ is convex and closed under $\vee$ and $\wedge$. Also, using 2.2 (i), (iv) we have

$$xyxy\le ab\le(xy\backslash e)(xy\backslash e)\le xy\backslash(xy\backslash e)=xyxy\backslash e$$

hence $\langle S\rangle$ is closed under $\cdot$. Note that even when S is empty, we have $e\in\langle S\rangle$ since e is the empty product in $\Pi(\emptyset)$.

To see that $\langle S\rangle$ is normal, we need to show that $\lambda_u(a),\rho_u(a)\in\langle S\rangle$ for any $u\in L$ and $a\in S$. We first observe, using 2.2(i), that

$$\lambda_u(p)\lambda_u(q)\le(u\backslash pu)(u\backslash qu)\wedge e\le u\backslash pu(u\backslash qu)\wedge e\le u\backslash pqu\wedge e=\lambda_u(pq).$$

Since $x\in\Pi\Gamma\Delta(S)$, there exist $z_1,\dots,z_n\in\Gamma\Delta(S)$ such that $x=z_1z_2\cdots z_n$. Let $z=\lambda_u(z_1)\cdots\lambda_u(z_n)$, and note that $z\in\Pi\Gamma\Delta(S)$. By the observation above, $z\le\lambda_u(x)\le\lambda_u(a)\le\lambda_u(x\backslash e)$. Moreover, $z\lambda_u(x\backslash e)\le\lambda_u(x(x\backslash e))\le e$, hence $\lambda_u(x\backslash e)\le z\backslash e$. Therefore $\lambda_u(a)\in\langle S\rangle$. The argument for ρ_u is similar.

Finally, we have to prove that $\langle S\rangle$ is closed under $\backslash$ and $/$. By assumption, $x\le a$ and $b\le y\backslash e$, hence $a\backslash b\le x\backslash(y\backslash e)=yx\backslash e$ by 2.2 (iv). Also $a\le x\backslash e$ and $y\le b$ imply $yxa\le b$. Therefore $a\lambda_a(yx)\le a(a\backslash yxa)\le yxa\le b$, and it follows that $\lambda_a(yx)\le a\backslash b$. By the same observation above, there exists $z\in\Pi\Gamma\Delta(S)$ such that $z\le yx\lambda_a(yx)\le a\backslash b\le z\backslash e$.

For $/$, we note that $x\le a$ and $b\le y\backslash e$ imply $xyb\le a$, whence $xy\le a/b$. For the upper bound we have $a/b\le u$, where $u=(x\backslash e)/y$. We claim that $u\le x\rho_u(y)\backslash e$, then the (mirror image of the) observation above shows there exists $z\in\Pi\Gamma\Delta(S)$ such that $z\le xyx\rho_u(y)\le a/b\le z\backslash e$. Indeed, $x\rho_u(y)u\le x(uy/u)u\le xuy\le e$ by definition of u. This completes the proof. ∎

An element a of $\mathbf{L}$ is *negative* if $a\in L^-$, and a subset of L is *negative* if it is a subset of L^-.

The following corollary describes explicitly how the negative elements of a one-generated convex normal subalgebra are obtained. It is used later to find an equational basis for the variety generated by totally ordered residuated lattices. We use the notation $\gamma_{\mathbf{u}}$ where $\mathbf{u}=\langle u_1,\dots,u_{2n}\rangle$ to denote $\lambda_{u_1}\circ\rho_{u_2}\circ\lambda_{u_3}\circ\cdots\circ\rho_{u_{2n}}$.

Corollary 3.7. *Let $\mathbf{L}$ be a residuated lattice and $r,s\in L^-$. Then $r\in\mathrm{cn}(s)$ if and only if for some m,n there exist $\mathbf{u}_i\in L^{2n}$ $(i=1,\dots,m)$ such that*

$$\gamma_{\mathbf{u}_1}(s)\cdot\gamma_{\mathbf{u}_2}(s)\cdots\gamma_{\mathbf{u}_m}(s)\le r.$$

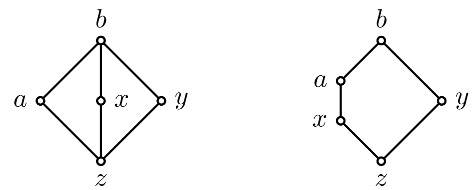


Figure 1.

The final result in this section is due to N. Galatos [Ga00]. It is especially useful for finite residuated lattices, where it shows that congruences are determined by the negative central idempotent elements.

The *center* of a residuated lattice $\mathbf{L}$ is the set

$$Z(\mathbf{L}) = \{x \in L : ux = xu \ \text{ for all } \ u \in L\}.$$

This is easily seen to be a join-subsemilattice and a submonoid of $\mathbf{L}$. There is a close relationship between negative idempotent elements of $Z(\mathbf{L})$ and ideals in $\mathbf{L}$. Let $\mathrm{CI}(\mathbf{L}) = \{x \in Z(\mathbf{L})^- : x = x^2\}$ be the set of all negative central idempotent elements.

Corollary 3.8. [Ga00] *$\langle \mathrm{CI}(\mathbf{L}), \vee, \cdot\rangle$ is dually embedded in $\mathrm{CN}(\mathbf{L})$ via the map*

$$x \mapsto \{a \in \mathbf{L} : x \le a \le x\backslash e\}.$$

If $\mathbf{L}$ is finite, this map is a dual isomorphism.

4 Distributive Residuated Lattices

The variety of residuated lattices that satisfy the distributive law $x \wedge (y \vee z) = (x\wedge y)\vee(x\wedge z)$ is denoted by $\mathcal{DRL}$. In this section we consider conditions on residuated lattices that imply the distributive law, as well as some results about subvarieties of $\mathcal{DRL}$. We begin with a generalization of a result by Berman [Be74]. A *downward join-endomorphism* on a lattice $\mathbf{L}$ is a join-preserving map $f : \mathbf{L} \to \mathbf{L}$ such that $f(x) \le x$ for all $x \in L$. An upward meet-endomorphism is defined dually.

Proposition 4.1. *The following are equivalent in any lattice $\mathbf{L}$.*

(i) *$\mathbf{L}$ is distributive.*

(ii) *For all $a, b \in L$ with $a \le b$ there exists a downward join-endomorphism that maps b to a.*

(iii) *The dual of* (ii): *For all* $a, b \in L$ *with* $a \leq b$ *there exists an upward meet-endomorphism that maps* a *to* b.

Proof. To see that (i) implies (ii), it suffices to check that (by distributivity) the map $x \mapsto a \wedge x$ is the required downward join-endomorphism. Conversely, suppose (ii) holds. We will show that $\mathbf{L}$ has no sublattice isomorphic to the lattices M_3 or N_5 in Figure 1, whence it follows by a well known result of Dedekind that $\mathbf{L}$ is distributive. Suppose to the contrary that either of them is a sublattice of $\mathbf{L}$. For the elements labeled a and b, let f be the downward join-endomorphism that maps b to a. Then $f(y) \leq y$ and $a = f(b) = f(x \vee y) = f(x) \vee f(y)$, hence $f(y) \leq a$. It follows that $f(y) \leq y \wedge a = z$. Since $z \leq x$ and $f(x) \leq x$, we have $a = f(x) \vee f(y) \leq x$, which contradicts our assumptions in Figure 1.

The proof of (i)$\Leftrightarrow$(iii) is dual to the one above. ∎

Corollary 4.2. *In* $\mathcal{RL}$, *any of the following imply the distributive law.*

(i) $x\backslash x = e$ *and* $x\backslash(y \vee z) = x\backslash y \vee x\backslash z$

(ii) $x(x\backslash y \wedge e) = x \wedge y$

(iii) $x\backslash xy = y$, $xy = yx$ *and* $x(y \wedge z) = xy \wedge xz$

Proof. Let $\mathbf{L} \in \mathcal{RL}$ and consider $a \leq b \in L$. For (i), the map $f(x) = a(b\backslash x)$ is a downward join-endomorphism that maps b to a, and in case (ii) the map $f(x) = x(b\backslash a \wedge e)$ has the same properties. The equations in (iii) imply that $f(x) = a\backslash xb$ is an upward meet-endomorphism that maps a to b. ∎

Note that either of the conditions (i) or (ii) above imply the well known result that ℓ-groups have distributive lattice reducts.

A totally ordered residuated lattice is referred to as a *residuated chain*, and the variety generated by all residuated chains is denoted by $\mathcal{RL}^C$. Since chains are distributive lattices, this is a subvariety of $\mathcal{DRL}$. The following result provides a finite equational basis for $\mathcal{RL}^C$. A similar basis was obtained independently by C. J. van Alten [vA1] for the subvariety of integral members (defined by $x \leq e$) of $\mathcal{RL}^C$.

Theorem 4.3. [BT] $\mathcal{RL}^C$ *is the variety of all residuated lattices that satisfy*

$$e = \lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y).$$

Proof. It is easy to check that the identity holds in $\mathcal{RL}^C$. Let $\mathcal{V}$ be the variety of residuated lattices defined by this identity, and let $\mathbf{L}$ be a subdirectly irreducible member of $\mathcal{V}$. To show that $\mathbf{L}$ is totally ordered, we first observe that it suffices to prove e is join-irreducible. Indeed, choosing $u = v = e$, the identity yields $e = ((x \vee y)\backslash x \wedge e) \vee ((x \vee y)\backslash y \wedge e)$, so by join-irreducibility we have either $e = ((x \vee y)\backslash x \wedge e)$ or $e = ((x \vee y)\backslash y \wedge e)$. The first case implies $y \leq x$, and the second implies $x \leq y$.

Now to prove the join-irreducibility of e, consider $a, b \in L$ such that $a \vee b = e$. We will show that $\mathrm{cn}(a) \cap \mathrm{cn}(b) = \{e\}$, and since we assumed $\mathbf{L}$ is subdirectly irreducible, it will follow that either $a = e$ or $b = e$.

Claim: If $a_1, \ldots, a_m, b_1, \ldots, b_n \in L^-$ *and* $a_i \vee b_j = e$ *for all* i, j *then*

$$a_1 a_2 \cdots a_m \vee b_1 b_2 \cdots b_n = e.$$

This is an immediate consequence of the observation that $x \vee y = x \vee z = e$ implies $x \vee yz = e$ in any ℓ-groupoid with unit (since $e = (x \vee y)(x \vee z) = x^2 \vee xz \vee yx \vee yz \leq x \vee yz \leq e$).

Claim: If $x \vee y = e$ *then* $\lambda_u(\rho_v(x)) \vee y = e$. Here we make use of the identity that was assumed to hold in $\mathbf{L}$. Given $x \vee y = e$, we have $\lambda_u(x) \vee y = \lambda_u((x \vee y)\backslash x) \vee \rho_e((x \vee y)\backslash y) = e$, and similarly $\rho_v(x) \vee y = e$. Applying the second result to the first establishes the claim.

By Corollary 3.7 the negative members of $\mathrm{cn}(a)$ are bounded below by finite products of iterated conjugates of a. By the two preceding claims, $a \vee b = e$ implies $a' \vee b' = e$ for any $a' \in \mathrm{cn}(a) \cap L^-$ and $b' \in \mathrm{cn}(b) \cap L^-$, hence $\mathrm{cn}(a) \cap \mathrm{cn}(b) = \{e\}$. ∎

Since $(x \vee y)\backslash z = x\backslash z \wedge y\backslash z$ holds in $\mathcal{RL}$, and $x \leq u\backslash xu$ whenever $ux \leq xu$ we deduce the following result.

Corollary 4.4. *An equational basis for the variety* $\mathcal{CRL}^{\mathcal{C}}$*, generated by all commutative residuated chains, is given by* $xy = yx$ *and* $e = (y\backslash x \wedge e) \vee (x\backslash y \wedge e)$.

Generalized MV-Algebras and BL-Algebras. Recall that a residuated lattice is said to be *integral* if e is its top element. The variety of all integral residuated lattices is denoted by $\mathcal{IRL}$. A $0,1$-*lattice* is a bounded lattice with additional constant operations 0 and 1 denoting the bottom and top element, respectively. In a bounded residuated lattice it suffices to include 0 in the similarity type since $1 = 0\backslash 0$. For algebraic versions of logic it is natural to assume the existence of 0, since this usually denotes the logical constant *false*. In addition, it is often the case that the operation $\cdot$ is commutative, hence the residuals are related by the identity $x\backslash y = y/x$, and $x\backslash y$ is usually written $x \to y$. In logic this operation may be interpreted as a (generalized) implication.

Commutative residuated $0,1$-lattices have been studied extensively in both algebraic and logical form under various names (e.g., linear logic algebras, BCK lattices, full Lambek algebras with exchange, residuated commutative ℓ-monoids, residuated lattices). The aim of this section is to point out that several results in this area are true in the more general setting of residuated lattices, without assuming commutativity or the existence of bounds.

The algebraic version of classical propositional logic is given by *Boolean algebras*, and for intuitionistic logic it is given by *Heyting algebras*. The class $\mathcal{HA}$ of all Heyting algebras is a variety of residuated $0,1$-lattices that is axiomatized by $x \cdot y = x \wedge y$, and the class $\mathcal{BA}$ of all Boolean Algebras is axiomatized by the additional identity $(x \to 0) \to 0 = x$. This is of course a version of the classical law of double negation.

If we do not assume the existence of a constant 0, the corresponding classes of algebras are called *Brouwerian algebras* ($\mathcal{B}r\mathcal{A}$) and *generalized Boolean algebras* ($\mathcal{GBA}$). In the latter case, the law of double negation is rewritten as $(x \to y) \to y = x \vee y$, to avoid using the constant 0. Note that the identity $xy = x \wedge y$ implies distributivity, commutativity and integrality.

A *basic logic algebra* (or BL-algebra for short) is a commutative residuated $0,1$-lattice in which the identities $x(x \to y) = x \wedge y$ and $(x \to y) \vee (y \to x) = e$ hold. Taking $x = e$ it follows that BL-algebras are integral, and by Theorem 4.2 (ii) the first identity implies the distributive law. From the second identity and Corollary 4.4 it follows that subdirectly irreducible BL-algebras are totally ordered, but we will not assume this identity in the generalizations we consider below. For more background on BL-algebras we refer to [Ha98] and [NPM99].

The algebraic version of Łukasiewicz's multi-valued logic is given by MV-algebras. In our setting we may define MV-algebras as commutative residuated $0,1$-lattices that satisfy the identity $(x \to y) \to y = x \vee y$, though they are often defined in a slightly different but term-equivalent similarity type (see e.g. [COM00]). The class of all MV-algebras is denoted by $\mathcal{MV}$.

Standard examples of MV-algebras are Boolean algebras, and the $[0,1]$-algebra defined on the unit interval, with $x \cdot y = \max(0, x+y-1)$ and $x \to y = \min(1, 1-x+y)$.

The latter example can be generalized to abelian ℓ-groups as follows. If $\mathbf{G} = \langle G, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is an abelian ℓ-group and a a positive element, then $\Gamma(\mathbf{G}, a) = \langle [e, a], \wedge, \vee, \circ, a, \to, e\rangle$ is an MV-algebra, where

$$x \circ y = xy/a \vee e, \quad x \to y = ya/x \wedge a.$$

Chang [Ch59] proved that if $\mathbf{M}$ is a totally ordered MV-algebra then there is an abelian ℓ-group $\mathbf{G}$ and a positive element a of it such that $\mathbf{M} \cong \Gamma(\mathbf{G}, a)$. Moreover, Mundici [Mu86] generalized the result to all MV algebras and proved that Γ is an equivalence between the category of MV-algebras and the category of abelian ℓ-groups with strong unit. A good reference for MV-algebras is [COM00].

As before, we may consider the corresponding classes of generalized algebras where we assume neither commutativity nor the existence of 0. The class $\mathcal{GBL}$ of all *generalized BL-algebras* is the subvariety of $\mathcal{RL}$ defined by $y(y\backslash x \wedge e) = x \wedge y = (x/y \wedge e)y$, and the class $\mathcal{GMV}$ of all *generalized MV-algebras* is the subvariety defined by $x/(y\backslash x \wedge e) = x \vee y = (x/y \wedge e)\backslash x$. As for BL-algebras, the members of $\mathcal{GBL}$ are distributive by Theorem 4.2 (ii). Note that both of these classes include the variety of all ℓ-groups. Also, if we assume that e is the top element, the resulting integral subvarieties $\mathcal{IGBL}$ and $\mathcal{IGMV}$ can be defined by the simpler identities $y(y\backslash x) = x \wedge y = (x/y)y$ and $x/(y\backslash x) = x \vee y = (x/y)\backslash x$, respectively (and these identities imply integrality).

Lemma 4.5.

(i) *In $\mathcal{RL}$ the GBL identity $y(y\backslash x \wedge e) = x \wedge y$ is equivalent to the quasi-identity $x \leq y \Rightarrow x = y(y\backslash x)$.*

(ii) *The GMV identity $x/(y\backslash x \wedge e) = x \vee y$ is equivalent to the quasi-identity $x \leq y \Rightarrow y = x/(y\backslash x)$.*

Proof. The quasi-identity in (i) is equivalent to the identity $x \wedge y = y(y\backslash(x \wedge y))$. Since $\backslash$ distributes over $\wedge$ in the numerator, we may rewrite this as $x \wedge y = y(y\backslash x \wedge y\backslash y))$. Taking $x = e$ in the quasi-identity shows that all positive elements are invertible. In particular, since $e \leq y\backslash y = (y\backslash y)^2$ by 2.2 (viii), we have that

$$e = (y\backslash y)((y\backslash y)\backslash e) = (y\backslash y)^2((y\backslash y)\backslash e) = y\backslash y.$$

The proof for (ii) is similar. ■

Residuated groupoids that satisfy the quasi-identity in (i) above and its mirror image are usually referred to as *complemented*, but since this term has a different meaning for lattices, we do not use it here.

The next result shows that $\mathcal{GMV}$ is subvariety of $\mathcal{GBL}$. In particular, it follows that GMV-algebras are distributive.

Theorem 4.6. [BCGJT] *Every GMV-algebra is a GBL-algebra.*

Proof. We make use of the quasi-equational formulation from the preceding lemma. Assume $x \leq y$ and let $z = y(y\backslash x)$. Note that $z \leq x$ and $y\backslash z \leq x\backslash z$, hence

$$\begin{array}{rcll} x\backslash z & = & ((y\backslash z)/(x\backslash z))\backslash(y\backslash z) & \\ & = & (y\backslash(z/(x\backslash z)))\backslash(y\backslash z) & \text{since } (u\backslash v)/w = u\backslash(v/w) \\ & = & (y\backslash x)\backslash(y\backslash z) & \text{since } z \leq x \Rightarrow x = z/(x\backslash z) \\ & = & (y(y\backslash x)\backslash z & \text{since } u\backslash(v\backslash w) = vu\backslash w \\ & = & z\backslash z. & \end{array}$$

Therefore $x = z/(x\backslash z) = z/(z\backslash z) = z$, as required. The proof of $x = (x/y)y$ is similar. ■

5 Cancellative residuated lattices

The results in this section are from [BCGJT].

A residuated lattice is said to be *cancellative* if its monoid reduct satisfies the left and right cancellation laws: for all x, y, z, $xy = xz$ implies $y = z$, and $xz = yz$ implies $x = y$.

The class of cancellative residuated lattices is denoted by $\mathcal{C}an\mathcal{RL}$. Recall that if a residuated lattice has a bottom element 0, then $0x = 0 = x0$. Hence any nontrivial cancellative residuated lattice is infinite.

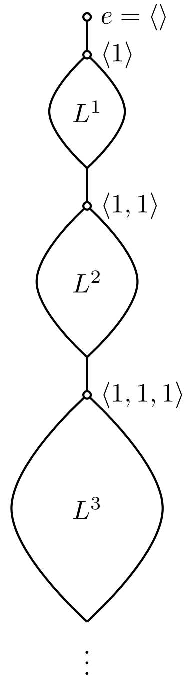


Figure 2: The structure of $\mathbf{L}^*$

Lemma 5.1. *A residuated lattice is left cancellative if and only if it satisfies the identity $x\backslash xy = y$.*

Proof. Since $\cdot$ distributes over $\vee$, the left cancellative law is equivalent to $xz \leq xy \Rightarrow z \leq y$, and this is in turn equivalent to $z \leq x\backslash xy \Rightarrow z \leq y$. Taking $z = x\backslash xy$, we see that this implication holds if and only if $x\backslash xy \leq y$. Since the reverse inequality holds in any residuated lattice, the result follows. ∎

Corollary 5.2. *$\mathcal{C}an\mathcal{RL}$ is a variety.*

In contrast to ℓ-groups, whose lattice reducts are distributive, the next result shows that cancellative residuated lattices only satisfy those lattice identities that hold in all lattices. In [Co] it is proved that even commutative cancellative residuated lattices are not necessarily distributive.

Given a lattice $\mathbf{L}$, we construct a simple integral cancellative residuated lattice $\mathbf{L}^*$ that has $\mathbf{L}$ as lattice subreduct.

Theorem 5.3. *Every lattice can be embedded into the lattice reduct of a simple integral member of* $\mathcal{C}an\mathcal{RL}$.

Proof. Suppose $\mathbf{L}$ is any lattice. We may assume that $\mathbf{L}$ has a top element 1, since any lattice can be embedded in a lattice with a top. The lattice $\mathbf{L}^*$ is defined to be the ordinal sum of the cartesian power $\mathbf{L}^n$, with every element of L^n above every element of L^{n+1}, for $n = 0, 1, 2, \ldots$; (see Figure 2). The operation $\cdot$ is on $\mathbf{L}^*$ is simply concatenation of sequences, so the monoid reduct of $\mathbf{L}^*$ is the free monoid generated by L^1. This operation is obviously cancellative, and it is residuated since each L^k has a largest element $\mathbf{1}_k = \langle 1, 1, \ldots, 1\rangle$. Hence the largest solution $\mathbf{z}$ of

$$\langle x_1, \ldots, x_m\rangle \cdot \mathbf{z} \leq \langle y_1, \ldots, y_n\rangle$$

is either $\mathbf{z} = \langle y_{m+1}, \ldots, y_n\rangle$ if $m \leq n$ and $x_i \leq y_i$ for $1 \leq i \leq m$, or $\mathbf{z} = \mathbf{1}_k$, where $k = \max(n - m + 1, 0)$. (Note that $\mathbf{1}_0 = \langle\rangle = e$.) Thus $\backslash$ exists, and the argument for $/$ is similar.

Since $\mathbf{L}^* = \mathrm{cn}(\langle 1\rangle)$, the convex normal subalgebra generated by the coatom of $\mathbf{L}^*$, it follows from Theorem 3.5 that $\mathbf{L}^*$ is simple. ∎

Corollary 5.4. *The lattice of subvarieties of lattices is order-embeddable into* $\mathbf{L}(\mathcal{C}an\mathcal{RL})$.

Proof. Given any variety $\mathcal{V}$ of lattices, define $\widehat{\mathcal{V}} = \mathbf{V}(\{\mathbf{L}^* : \mathbf{L} \in \mathcal{V}\})$, where $\mathbf{V}(\mathcal{K})$ denotes the variety generated by the class $\mathcal{K}$. Since lattice varieties are closed under the operation of ordinal sum and adding a top element, the lattice reduct of $\mathbf{L}^*$ is in the same variety as $\mathbf{L}$. Since the map $\mathcal{V} \mapsto \widehat{\mathcal{V}}$ is clearly order preserving, it suffices to prove that it is one-one. If $\mathcal{V}$ and $\mathcal{W}$ are two distinct lattice varieties, then there is a lattice identity that holds, say, in $\mathcal{V}$ but not in $\mathcal{W}$. Since the lattice reducts of $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{W}}$ are members of $\mathcal{V}$ and $\mathcal{W}$ respectively, the same identity holds in $\widehat{\mathcal{V}}$, but fails in $\widehat{\mathcal{W}}$. ∎

Negative Cones of ℓ-Groups. Recall that the *negative part* of a residuated lattice $\mathbf{L}$ is $L^- = \{x \in L : x \leq e\}$. The *negative cone of* $\mathbf{L}$ is defined as $\mathbf{L}^- = \langle L^-, \vee, \wedge, \cdot, e, /^{\mathbf{L}^-}, \backslash^{\mathbf{L}^-}\rangle$, where

$$a/^{\mathbf{L}^-} b = a/b \wedge e \quad\text{and}\quad a\backslash^{\mathbf{L}^-} b = a\backslash b \wedge e.$$

It is easy to check that $\mathbf{L}^-$ is again a residuated lattice. For a class $\mathcal{K}$ of residuated lattices, $\mathcal{K}^-$ denotes the class of negative cones of members of $\mathcal{K}$.

The following standard construction shows that certain cancellative monoids can be embedded in their groups of fractions (see e.g. [Fu63]).

Lemma 5.5. *Let* $\mathbf{M}$ *be a cancellative monoid such that* $aM = Ma$ *for all* $a \in M$. *Then there exists a group* $\mathbf{G}$ *and an embedding* $a \mapsto \widehat{a}$ *from* $\mathbf{M}$ *to* $\mathbf{G}$ *such that every element of* $\mathbf{G}$ *is of the form* $\widehat{a}\widehat{b}^{-1}$ *for some* $a, b \in \mathbf{M}$.

Theorem 5.6. *For* $\mathbf{L} \in \mathcal{RL}$ *the following statements are equivalent.*

(i) $\mathbf{L}$ *is the negative cone of an* ℓ*-group.*

(ii) $\mathbf{L}$ *is a cancellative integral generalized MV-algebra.*

(iii) $\mathbf{L}$ *is a cancellative integral generalized BL-algebra.*

Proof. (i)⇒(ii): for all $\mathbf{L} \in \mathcal{LG}$ and all $a, b \in L^-$, $ab/^{\mathbf{L}^-} b = abb^{-1} \wedge e = a \wedge e = a$, and $a/^{\mathbf{L}^-}(b\backslash^{\mathbf{L}^-} a) = a(b^{-1}a \wedge e)^{-1} \wedge e = (aa^{-1}b \vee a) \wedge e = a \vee b$.

(ii)⇒(iii) follows from Theorem 4.6.

(iii)⇒(i): We use Lemma 5.5 to show that any cancellative integral generalized BL-algebra is in $\mathcal{LG}^-$.

Let $\mathbf{L}$ be a cancellative integral generalized BL-algebra. For any elements $a, b \in L$ we deduce by Lemma 4.5(i) that $a(a\backslash ba) = ba$ since $b \leq e$. Hence there exists an element b_a, namely $a\backslash ba$, such that $ba = ab_a$. It follows that $La \subseteq aL$ for all elements $a \in L$, and similarly $aL \subseteq La$. Therefore the underlying monoid of L satisfies the conditions of the preceding lemma, and can be embedded into a group $\mathbf{G}$ in the prescribed manner.

We consider the following standard order on G: for all $a, b \in G$, $a \leq_G b$ if and only if $ab^{-1} \in L$. It is well known (see e.g. [Fu63], p. 13) that $\leq_G$ is a compatible partial order on G whose negative elements are precisely the elements of L. We proceed to show that $\leq_G$ is an extension of the original order $\leq$ of L. More explicitly, we prove that for all $a, b \in L$,

$$a \leq b \quad \Leftrightarrow \quad a \leq_G b \quad \Leftrightarrow \quad ab^{-1} = a/b \quad \Leftrightarrow \quad b^{-1}a = b\backslash a. \tag{$*$}$$

Let $x \leq_G y$. Thus $xy^{-1} \in L$, and $x = xy^{-1}y \leq y$ because $xy^{-1} \leq e$. Assuming $x \leq y$, $xy^{-1}y = x = x \wedge y = (x/y)y$ by the generalized basic logic identity, whence cancelling y gives $xy^{-1} = x/y$. Now if $xy^{-1} = x/y$, then $x = xy^{-1}y = (x/y)y = x \wedge y$, hence $x \leq y$. Similarly $x \leq y$ is equivalent to $y^{-1}x = y\backslash x$. Finally if $xy^{-1} = x/y$, we have that $xy^{-1} \in L$ and thus $x \leq_G y$.

The preceding conclusion allows us to drop the subscript on $\leq_G$. It remains to show that $\leq$ is a lattice order. Since any ℓ-group satisfies the identity $x \vee y = (xy^{-1} \vee e)y$, it suffices to establish the existence of all joins of the form $g \vee e$ for $g \in G$.

Let $a, b \in L$ such that $g = ab^{-1}$. We claim that $ab^{-1} \vee e = (a \vee b)b^{-1}$, where the join on the right hand side is computed in $\mathbf{L}$. Since $a, b \leq a \vee b$, it follows that $e \leq (a \vee b)b^{-1}$ and $ab^{-1} \leq (a \vee b)b^{-1}$.

If we consider any other element of G, say cd^{-1} (where $c, d \in L$), such that both $e \leq cd^{-1}$ and $ab^{-1} \leq cd^{-1}$ hold, we have $a \leq cd^{-1}b = cd^{-1}bdd^{-1}$. Now we note that bd and d are elements of L such that $bd \leq d$. Thus $(*)$ shows that $d^{-1}bd = d\backslash bd = b_d$. Therefore $a \leq cb_dd^{-1}$, so that $ad \leq cb_d$. Similarly, working with $e \leq cd^{-1}$, we establish $bd \leq cb_d$, and hence $ad \vee bd \leq cb_d$. Since $\mathbf{L}$ is a residuated lattice, products distribute over joins, so we have $a \vee b \leq cb_dd^{-1} = cd^{-1}b$ and finally $(a \vee b)b^{-1} \leq cd^{-1}$, as desired. ∎

Corollary 5.7. *$\mathcal{LG}^-$ is a variety, defined by the identities $xy/y = x = y\backslash yx$ and $(x/y)y = x \wedge y = y(y\backslash x)$. Alternatively, the last two identities can be replaced by $x/(y\backslash x) = x \vee y = (x/y)\backslash x$.*

Corollary 5.8. *The variety $\mathbf{V}(\mathbb{Z}^-)$ is defined by the identities $xy = yx$, $x = y\backslash yx$ and $x \wedge y = y(y\backslash x)$. Alternatively, the last identity can be replaced by $x \vee y = (y\backslash x)\backslash x$.*

The Subvarieties of $\mathcal{LG}$ and $\mathcal{LG}^-$. We now extend the map $^- : \mathcal{LG} \to \mathcal{LG}^-$ to subclasses of $\mathcal{LG}$, and in particular to the lattice of subvarieties $\mathbf{L}(\mathcal{LG})$. We show that the image of a variety is always a variety, that every subvariety of $\mathcal{LG}^-$ is obtained in this way and that the map is order preserving. The proof is syntactical and shows how equational bases can be translated back and forth. We note that these results are related to R. McKenzie's general characterization of categorical equivalence [McK96]. For further discussion about this, we refer to [BCGJT]. Independently, C. J. van Alten [vA2] also discovered a basis for $\mathcal{LG}^-$, by proving that it is term-equivalent to the variety of cancellative generalized hoops. The correspondence between subvarieties of ℓ-groups and subvarieties of $\mathcal{LG}^-$ then follows from McKenzie's categorical equivalence.

From Subvarieties of $\mathcal{LG}^-$ to Subvarieties of $\mathcal{LG}$. In this direction, the translation is derived essentially from the definition of the negative cone. For a residuated lattice term t, we define a translated term t^- by

$$\begin{array}{lll} x^- = x \wedge e & e^- = e & \\ (s/t)^- = s^-/t^- \wedge e & (s\backslash t)^- = s^-\backslash t^- \wedge e & \\ (st)^- = s^-t^- & (s \vee t)^- = s^- \vee t^- & (s \wedge t)^- = s^- \wedge t^-. \end{array}$$

Recall that $t^{\mathbf{L}^-}$ denotes the term-function defined by t in the algebra $\mathbf{L}^-$.

Lemma 5.9. *Let $\mathbf{L} \in \mathcal{RL}$ and consider any $\mathcal{RL}$ term t. For any $a_1, \ldots, a_n \in L$,*

$$t^{-\mathbf{L}}(a_1, \ldots, a_n) = t^{\mathbf{L}^-}(a_1 \wedge e, \ldots, a_n \wedge e).$$

Proof. By definition this is true for variables and the constant term e. Assume the statement holds for terms s and t. Since $(s/t)^- = s^-/t^- \wedge e$, we have

$$\begin{aligned} &(s/t)^{-\mathbf{L}}(a_1, \ldots, a_n) = (s^{-\mathbf{L}}(a_1, \ldots, a_n)/^{\mathbf{L}}t^{-\mathbf{L}}(a_1, \ldots, a_n)) \wedge e \\ &= (s^{\mathbf{L}^-}(a_1 \wedge e, \ldots, a_n \wedge e)/^{\mathbf{L}}t^{\mathbf{L}^-}(a_1 \wedge e, \ldots, a_n \wedge e)) \wedge e \\ &= (s/t)^{\mathbf{L}^-}(a_1 \wedge e, \ldots, a_n \wedge e) \end{aligned}$$

and similar inductive steps for $\backslash$, $\cdot$, $\vee$, $\wedge$ complete the proof. ∎

Lemma 5.10. *For any $\mathbf{L} \in \mathcal{RL}$, $\mathbf{L}^- \models s = t$ if and only if $\mathbf{L} \models s^- = t^-$.*

Proof. Suppose $\mathbf{L}^- \models s = t$, and let $a_1, \dots, a_n \in L$. By the preceding lemma, $s^{-\mathbf{L}}(a_1, \dots, a_n) = s^{\mathbf{L}^-}(a_1 \wedge e, \dots, a_n \wedge e) = t^{\mathbf{L}^-}(a_1 \wedge e, \dots, a_n \wedge e) = t^{-\mathbf{L}}(a_1, \dots, a_n)$, hence $\mathbf{L} \models s^- = t^-$. The reverse implication is similar and uses the observation that for $a_i \in L^-$, $a_i = a_i \wedge e$. ∎

Theorem 5.11. *Let $\mathcal{V}$ be a subvariety of $\mathcal{LG}^-$, defined by a set $\mathcal{E}$ of identities and let $\mathcal{W}$ be the subvariety of $\mathcal{LG}$ defined by the set of equations $\mathcal{E}^- = \{s^- = t^- : (s = t) \in \mathcal{E}\}$. Then $\mathcal{W}^- = \mathcal{V}$.*

Proof. Consider $\mathbf{M} \in \mathcal{W}^-$, which means there exists an $\mathbf{L} \in \mathcal{W}$ such that $\mathbf{M}$ is isomorphic to $\mathbf{L}^-$. Then $\mathbf{L} \models \mathcal{E}^-$, and by the previous lemma this is equivalent to $\mathbf{L}^- \models \mathcal{E}$, which in turn is equivalent to $\mathbf{L}^- \in \mathcal{V}$. Hence $\mathbf{M} \in \mathcal{V}$.

Conversely, let $\mathbf{M} \in \mathcal{V}$. Then there exists an ℓ-group $\mathbf{G}$ such that $\mathbf{M}$ is isomorphic to $\mathbf{G}^-$. ($\mathbf{G}$ is constructed as in Theorem 5.6.) Using the previous lemma again, we get that $\mathbf{G} \models \mathcal{E}^-$, hence $\mathbf{M} \in \mathcal{W}^-$. ∎

As an example, consider the variety $\mathcal{N}^-$ that is defined by the identity $x^2y^2 \leq yx$ relative to $\mathcal{LG}^-$. The corresponding identity for the variety $\mathcal{N}$ of normal valued ℓ-groups is $(x \wedge e)^2(y \wedge e)^2 \leq (y \wedge e)(x \wedge e)$.

From Subvarieties of $\mathcal{LG}$ to Subvarieties of $\mathcal{LG}^-$. Note that since $\cdot$ and $^{-1}$ distribute over $\vee$ and $\wedge$, any $\mathcal{LG}$ identity is equivalent to a conjunction of two identities of the form $e \leq p(g_1, \dots, g_n)$, where p is a lattice term and $g_1, \dots, g_n$ are group terms. Since ℓ-groups are distributive, this can be further reduced to a finite conjunction of inequalities of the form $e \leq g_1 \vee \cdots \vee g_n$.

For a term $t(x_1, \dots, x_m)$ and a variable z distinct from $x_1, \dots, x_m$, let

$$\bar{t}(z, x_1, \dots, x_m) = t(z^{-1}x_1, \dots, z^{-1}x_m).$$

Lemma 5.12. *Let $\mathbf{L}$ be an ℓ-group, and t an ℓ-group term. Then*

$$\mathbf{L} \models e \leq t(x_1, \dots, x_m) \text{ iff}$$
$$\mathbf{L} \models x_1 \vee \cdots \vee x_m \vee z \leq e \Rightarrow e \leq \bar{t}(z, x_1, \dots, x_m).$$

Proof. In the forward direction this is obvious. To prove the reverse implication, assume the right hand side holds and let $a_1, \dots, a_m \in L$. Define $c = a_1^{-1} \wedge \cdots \wedge a_m^{-1} \wedge e$ and $b_i = ca_i$ for $i = 1, \dots, m$. Then $c \leq e$ and $c \leq a_i^{-1}$, hence $b_i \leq e$. Now by assumption, $e \leq t(c^{-1}b_1, \dots, c^{-1}b_m) = t(a_1, \dots, a_m)$. ∎

Lemma 5.13. *Let $\mathbf{L} \in \mathcal{LG}$. For any group term g, there exist an $\mathcal{RL}$ term $\widehat{g}$ such that $(g \wedge e)^{\mathbf{L}}|_{L^-} = \widehat{g}^{\mathbf{L}^-}$.*

Proof. Essentially we have to rewrite group terms so that all the variables with inverses appear at the beginning of the term. This is done using conjugation: $xy^{-1} =$

$y^{-1}(yxy^{-1}) = y^{-1}(yx/y)$. Note that L^- is closed under conjugation by arbitrary elements, since $x \leq e$ implies $yxy^{-1} \leq e$. If we also have $y \leq e$, then $yx \in L^-$ and $yx \leq y$, hence $yx/^{\mathbf{L}^-} y = yx/^{\mathbf{L}} y$.

To describe the translation of an arbitrary group term, we may assume that it is of the form $p_1 q_1^{-1} p_2 q_2^{-1} \cdots p_n q_n^{-1}$ where the p_i and q_i are products of variables (without inverses). By using conjugation, we write this term in the form

$$q_1^{-1} q_2^{-1} \cdots q_n^{-1} (q_n(\cdots(q_2(q_1 p_1/q_1)p_2/q_2)\cdots)p_n/q_n).$$

So we can take $\widehat{g} = s\backslash t$ where

$$s = q_n \cdots q_2 q_1 \quad \text{and} \quad t = q_n(\cdots(q_2(q_1 p_1/q_1)p_2/q_2)\cdots)p_n/q_n.$$

■

Corollary 5.14. *Let $g_1, \ldots, g_n$ be group terms with variables among $x_1, \ldots, x_m$. For any ℓ-group $\mathbf{L}$,*

$$\mathbf{L}^- \models \widehat{g}_1 \vee \ldots \vee \widehat{g}_n = e \text{ iff } \mathbf{L} \models x_1 \vee \ldots \vee x_m \leq e \Rightarrow e \leq g_1 \vee \ldots \vee g_n.$$

For the next result, recall the discussion about identities in ℓ-groups, and the definition of $\overline{t}$ at the beginning of this subsection.

Theorem 5.15. *Let $\mathcal{V}$ be a subvariety of $\mathcal{LG}$, defined by a set $\mathcal{E}$ of identities, which we may assume are of the form $e \leq g_1 \vee \ldots \vee g_n$. Let*

$$\overline{\mathcal{E}} = \{e = \widehat{\overline{g_1}} \vee \ldots \vee \widehat{\overline{g_n}} : e \leq g_1 \vee \ldots \vee g_n \text{ is in } \mathcal{E}\}.$$

Then $\overline{\mathcal{E}}$ is an equational basis for $\mathcal{V}^-$ relative to $\mathcal{LG}^-$.

Proof. By construction, any member of $\mathcal{V}^-$ satisfies all the identities in $\overline{\mathcal{E}}$. On the other hand, if $\mathbf{M} \in \mathcal{LG}^-$ is a model of the identities in $\overline{\mathcal{E}}$, then $\mathbf{M}$ is the negative cone of some $\mathbf{L} \in \mathcal{LG}$. From the reverse directions of Corollary 5.14 and Lemma 5.12 we infer that $\mathbf{L}$ satisfies the equations in $\mathcal{E}$, hence $\mathbf{M} \in \mathcal{V}^-$. ■

For example consider the variety $\mathcal{R}$ of *representable ℓ-groups* which (by definition) is generated by the class of totally ordered groups (see [AF88] for more details). An equational basis for this variety is given by $e \leq x^{-1}yx \vee y^{-1}$ (relative to $\mathcal{LG}$). Applying the translation above, we obtain $e = zx\backslash(zy/z)x \vee y\backslash z$ as as equational basis for $\mathcal{R}^-$.

Corollary 5.16. *The map $\mathcal{V} \mapsto \mathcal{V}^-$ from $\mathbf{L}(\mathcal{LG})$ to $\mathbf{L}(\mathcal{LG}^-)$ is a lattice isomorphism, with the property that finitely based subvarieties of $\mathcal{LG}$ are mapped to finitely based subvarieties of $\mathcal{LG}^-$ and conversely.*

Proof. Let $\mathcal{V}$ be any subvariety of $\mathcal{LG}$. By the above theorem, $\mathcal{V}^-$ is a subvariety of $\mathcal{LG}^-$, and by Theorem 5.11 every subvariety of $\mathcal{LG}^-$ is of this form. Consider $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{LG}$, and let $\mathcal{E} \supseteq \mathcal{E}'$ be equational bases for $\mathcal{V}$, $\mathcal{V}'$ respectively. Defining $\mathcal{E}^-$ as in the preceding theorem, we have $\mathcal{E}^- \supseteq \mathcal{E}'^-$, hence $\mathcal{V} \subseteq \mathcal{V}'$. Finally, the map is injective since every ℓ-group is determined by its negative cone. It follows that the map is a lattice isomorphism, and the translation $\mathcal{E} \mapsto \mathcal{E}^-$ clearly maps finite sets to finite sets. ∎

6 Lattices of Subvarieties

We now investigate the structure of $\mathbf{L}(\mathcal{RL})$. In particular we consider which varieties are atoms in this lattices or in some of the ideals generated by particular subvarieties. Since any nontrivial ℓ-group has a subalgebra isomorphic to $\mathbb{Z}$, one obtains the well-known result that $\mathbf{V}(\mathbb{Z})$ is the only atom of $\mathbf{L}(\mathcal{RL})$. Similarly the result below is a straight forward consequence of the observation that every nontrivial integral, cancellative, residuated lattice has a subalgebra isomorphic to $\mathbb{Z}^-$.

Theorem 6.1. $\mathbf{V}(\mathbb{Z}^-)$ *is the only atom in the lattice of integral subvarieties of* $\mathcal{C}an\mathcal{RL}$.

For the next result, note that the identity $x\backslash x = e$ holds in any cancellative residuated lattice.

Theorem 6.2. [BCGJT] $\mathbf{V}(\mathbb{Z})$ *and* $\mathbf{V}(\mathbb{Z}^-)$ *are the only atoms in the lattice of commutative subvarieties of* $\mathcal{C}an\mathcal{RL}$.

Proof. Let $\mathbf{L}$ be a nontrivial cancellative, commutative residuated lattice. If $\mathbf{L}$ is integral, then $\mathbf{V}(\mathbf{L})$ contains $\mathbb{Z}^-$ by the preceding theorem. So we may assume that $\mathbf{L}$ is not integral. Let $S = \{x \in L : x\backslash e = e\}$ and consider the least congruence θ that collapses all members of S to e. By cancellativity and 2.2 (iv) $e = (x\backslash e)\backslash(x\backslash e) = x(x\backslash e)\backslash e$, hence $x(x\backslash e) \in S$ for all $x \in L$. It follows that $\mathbf{L}/\theta$ satisfies the identity $x(x\backslash e) = e$ and thus is an ℓ-group. It suffices to prove that $\mathbf{L}/\theta$ is nontrivial since any nontrivial ℓ-group contains a subalgebra isomorphic to $\mathbb{Z}$.

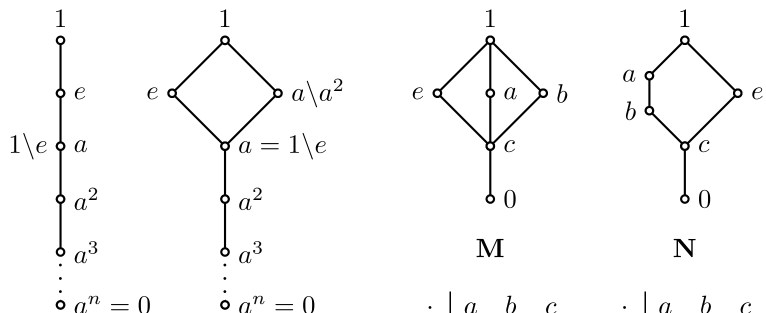
Note that $S \subseteq L^-$, and by commutativity, S is closed under conjugation: if $x \in S$ and $u \in L$ then

$$x = x \wedge e \leq u\backslash ux \wedge e = u\backslash xu \wedge e = \lambda_u(x) \leq e,$$

hence $e \leq \lambda_u(x)\backslash e \leq x\backslash e = e$, and it follows that $\lambda_u(x) \in S$. Furthermore, S is closed under $\cdot$ since if $x, y \in S$ then $xy\backslash e = y\backslash(x\backslash e) = y\backslash e = e$. So in the notation of Theorem 3.6 $\Pi\Gamma\Delta(S) = S$, whence $\mathrm{cn}(S) = \{u : x \leq u \leq x\backslash e \text{ for some } x \in S\}$. In particular, any negative element $u \in \mathrm{cn}(S)$ satisfies $x \leq u \leq e$, so $e \leq u\backslash e \leq x\backslash e = e$. Now choose $a \not\leq e$ and consider the negative element $b = e/a \wedge e$. Then $b \notin \mathrm{cn}(S)$ since $b\backslash e \geq e \vee a > e$. By Theorem 3.5 it follows that θ does not collapse all of $\mathbf{L}$, as required. ∎

It is not known whether there are noncommutative atoms below the variety of cancellative residuated lattices (see Problem 8.5).

Without the assumption of cancellativity, it is much easier to construct algebras that generate atoms in $\mathbf{L}(\mathcal{RL})$. An algebra is called *strictly simple* if it has no nontrivial congruences or subalgebras. It is easy to see that in a congruence distributive variety, any finite strictly simple algebra generates a variety that is an atom. In Figure 3 we list several finite strictly simple residuated lattices. In each case we give only the multiplication operation, since the residuals are determined by it. Also, in all cases $x0 = 0 = 0x$, $xe = x = ex$ and $x1 = x = 1x$ (when $x \neq e$).



$\mathbf{T}_n$ $\mathbf{T}'_n$ **M** **N**

$\cdot$	a	b	c
a	a	0	0
b	c	b	c
c	c	0	0

$\cdot$	a	b	c
a	c	0	0
b	0	0	0
c	0	0	0

Figure 3.

We now give an example of uncountably many residuated chains that generate distinct atoms of $\mathbf{L}(\mathcal{RL})$. Let S be any subset of ω. The algebra $\mathbf{C}_S$ is based on the set $\{0, a, b, e, 1\} \cup \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}$, with the following linear order:

$$0 < a < b < c_0 < c_1 < c_2 < \cdots < \cdots < d_2 < d_1 < d_0 < e < 1$$

The operation $\cdot$ is defined by $ex = x = xe$, if $x \neq e$ then $1x = x = x1$, and if $x \notin \{e, 1\}$ then $0x = 0 = x0$, $ax = 0 = xa$, and $bx = 0 = xb$. Furthermore, for all $i, j \in \omega$, $c_i c_j = 0$, $d_i d_j = b$,

$$c_i d_j = \begin{cases} 0 & \text{if } i < j \\ a & \text{if } i = j \text{ or } (i = j + 1 \text{ and } j \in S) \\ b & \text{otherwise} \end{cases} \qquad d_i c_j = \begin{cases} 0 & \text{if } i \geq j \\ b & \text{otherwise.} \end{cases}$$

This information is given in the form of an operation table in Figure 4. Depending on the chosen subset S, the elements $\mathbf{s}_i$ in the table are either a (if $i \in S$) or b (if $i \notin S$).

$\cdot$	1	e	d_0	d_1	d_2	d_3	$\ldots$	$\ldots$	c_3	c_2	c_1	c_0	b	a	0
1	1	1	d_0	d_1	d_2	d_3	$\ldots$	$\ldots$	c_3	c_2	c_1	c_0	b	a	0
e	1	e	d_0	d_1	d_2	d_3	$\ldots$	$\ldots$	c_3	c_2	c_1	c_0	b	a	0
d_0	d_0	d_0	b	b	b	b	$\ldots$	$\ldots$	b	b	b	0	0	0	0
d_1	d_1	d_1	b	b	b	b	$\ldots$	$\ldots$	b	b	0	0	0	0	0
d_2	d_2	d_2	b	b	b	b	$\ldots$	$\ldots$	b	0	0	0	0	0	0
d_3	d_3	d_3	b	b	b	b	$\ldots$	$\ldots$	0	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_3	c_3	c_3	b	b	$\mathbf{s}_2$	a	$\ldots$	$\ldots$	0	0	0	0	0	0	0
c_2	c_2	c_2	b	$\mathbf{s}_1$	a	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0
c_1	c_1	c_1	$\mathbf{s}_0$	a	0	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0
c_0	c_0	c_0	a	0	0	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0
b	b	b	0	0	0	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0
a	a	a	0	0	0	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0
0	0	0	0	0	0	0	$\ldots$	$\ldots$	0	0	0	0	0	0	0

Figure 4.

It is easy to check that this gives an associative operation since $xyz = 0$ whenever $e, 1 \notin \{x, y, z\}$. Now $1 = 0\backslash 0$, $d_0 = 1\backslash e$, $c_i = d_i\backslash 0$ and $d_{i+1} = c_i\backslash 0$, so the algebra is generated by 0. Distinct subsets S of ω produce nonisomorphic algebras, hence this construction gives uncountably many strictly simple algebras. However, since they are infinite, it does not yet follow that they generate distinct atoms. To complete the argument, one has to observe that any nontrivial subalgebra of an ultrapower of $\mathbf{C}_S$ contains a subalgebra isomorphic to $\mathbf{C}_S$ (since the generator 0 is definable by the universal formula $\phi(x) = \forall y(y < e \Rightarrow y^3 = x)$), and that for any distinct pair of such algebras one can find an equation that holds in one but not in the other.

Theorem 6.3. *There are uncountably many atoms in* $\mathbf{L}(\mathcal{RL})$ *that satisfy the identity* $x^3 = x^4$.

Not much is known about the global structure of $\mathbf{L}(\mathcal{RL})$. It is a dually algebraic distributive lattice, since $\mathcal{RL}$ is congruence distributive. The subvarieties of ℓ-groups and of Brouwerian algebras have been studied extensively, and for commutative integral residuated 0, 1-lattices, a recent monograph of Kowalski and Ono [KO] contains many important results.

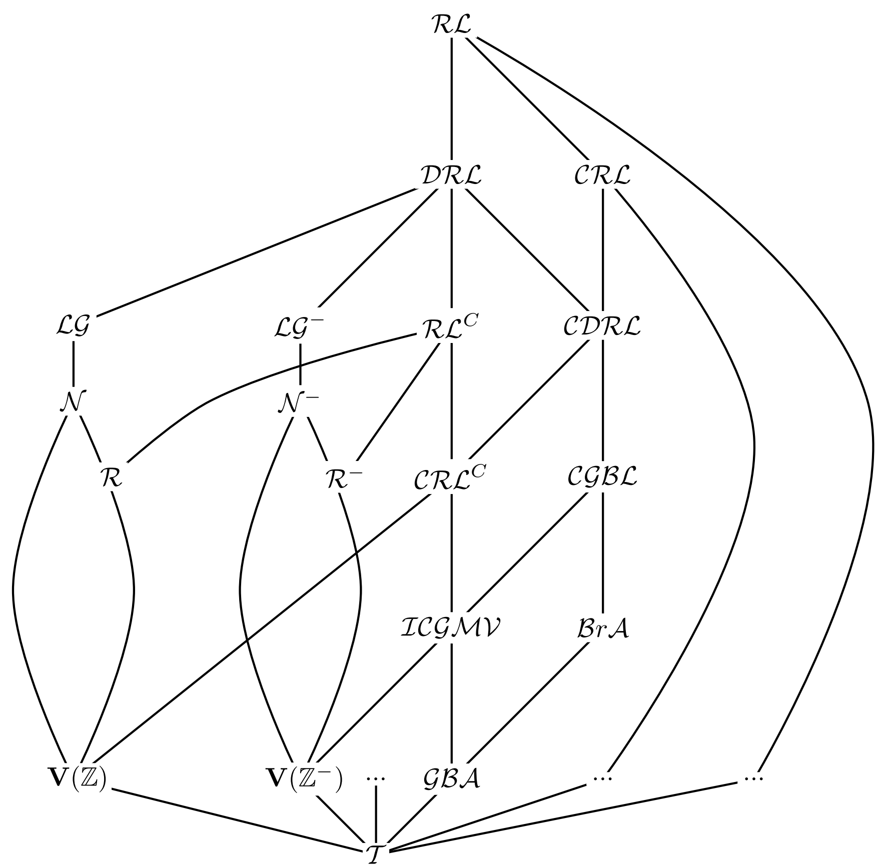


Figure 5: Some subvarieties of $\mathcal{RL}$

7 Decidability

In the first half of this section we present details of a result of Ono and Komori [OK85] which shows that the equational theory of residuated lattices is decidable and gives an effective algorithm based on a Gentzen system for the full Lambek calculus. Our approach is algebraic, and no familiarity with Gentzen systems or the Lambek calculus is assumed. In the second half we mention other results on the decidability of the equational and quasi-equational theory of various subvarieties of $\mathcal{RL}$. We note that this section is far from comprehensive, and that many related decidability results have been proved for the so-called *substructural logics* that correspond to subvarieties of $\mathcal{RL}$. The interested reader should consult the literature on relevance logic, full Lambek calculus and linear logic as a starting point.

An algebraic Gentzen system and decidability. Gentzen systems are usually defined for logics, and use pairs of sequences of formulas (so-called sequents) to specify the deduction rules of the logic. Since we are working within the algebraic theory of residuated lattices, we aim to present an algebraic version of Gentzen systems, in the hope that the reader will be persuaded by the effectiveness of the method, rather than burdened by syntactic differences between algebraic and logical notation. From an algebraic point of view, a Gentzen system is a finite set G of quasi-inequalities of the form $s_1 \le t_1 \;\&\ldots\&\; s_n \le t_n \Rightarrow s_0 \le t_0$, where s_i, t_i are terms. The notion of *Gentzen proof from* G is a restricted version of quasi-equational deduction that is usually simpler to work with than the standard equational deduction system of Birkhoff. In many cases it is decidable if a given inequality has a Gentzen proof, and a completeness theorem for a Gentzen system can be found to show that the concept of 'Gentzen proof from G' is in fact equivalent to 'quasi-equational proof from G'.

Let $\mathbf{T}$ be the term algebra on countably many variables $x_1, x_2, \ldots$ in the language of residuated lattices. Since the equational theory of monoids is decidable (in constant time) we may restrict our attention to deciding inequalities $s \le t$ up to associativity and multiplication by e. Effectively this means that we consider s, t as equivalence classes of terms in the quotient algebra $\mathbf{T}_M = \mathbf{T}/{\equiv_M}$ where $s \equiv_M s'$ iff the identity $s = s'$ is a consequence of the monoid identities.

We define $G_{\mathcal{RL}}$ to be the finite set of quasi-inequalities given in Table 1. Readers familiar with the traditional presentation of Gentzen rules may note that $\Rightarrow$ represents the horizontal line that is used in Gentzen rules, and that $s \le t$ is the equivalent of a sequent (assuming $\cdot$ is considered as comma). It is straightforward to check that each quasi-inequality of $G_{\mathcal{RL}}$ holds in $\mathcal{RL}$. For example, (\left) holds since if $x \le y$ and $uzv \le w$ then $ux(y\backslash z)v \le ux(x\backslash z)v \le uzv \le w$.

We now describe the notions of *proof-tree* and *Gentzen provable*. Recall that a *rooted tree* is a poset with a least element, called the root, and for each element, the set of all elements below it is linearly ordered. A *proof-tree* is a finite rooted tree in which each element is an inequality, and if $s_1 \le t_1, \ldots, s_n \le t_n$ are all the covers of the

$x \le x$	(refl)
$x \le z$ & $y \le w \Rightarrow xy \le zw$	($\cdot$ right)
$x \le y$ & $uzv \le w \Rightarrow ux(y\backslash z)v \le w$	($\backslash$ left)
$xy \le z \Rightarrow y \le x\backslash z$	($\backslash$ right)
$x \le y$ & $uzv \le w \Rightarrow u(z/y)xv \le w$	($/$ left)
$xy \le z \Rightarrow x \le z/y$	($/$ right)
$uxv \le w$ & $uyv \le w \Rightarrow u(x \vee y)v \le w$	($\vee$ left)
$x \le y \Rightarrow x \le y \vee z$	($\vee$ right)
$x \le z \Rightarrow x \le y \vee z$	($\vee$ right)
$uxv \le w \Rightarrow u(x \wedge y)v \le w$	($\wedge$ left)
$uyv \le w \Rightarrow u(x \wedge y)v \le w$	($\wedge$ left)
$x \le y$ & $x \le z \Rightarrow x \le y \wedge z$	($\wedge$ right)

Table 1: The algebraic Gentzen system $G_{\mathcal{RL}}$

element $s \le t$, then $n \in \{0, 1, 2\}$ and the quasi-inequality

$$s_1 \le t_1 \ \& \ \ldots \ \& \ s_n \le t_n \Rightarrow s \le t$$

is a substitution instance of a member of $G_{\mathcal{RL}}$. Hence each element has at most 2 covers, and an inequality has no covers if and only if it is an instance of (refl). An inequality is said to be *Gentzen provable* if there exists a proof-tree with this inequality as the root.[2] The *soundness* of $G_{\mathcal{RL}}$ is the observation that any Gentzen provable inequality holds in $\mathcal{RL}$ (since $\mathcal{RL} \models G_{\mathcal{RL}}$).

We say that an inequality *matches* a quasi-inequality if it is a substitution instance of the conclusion of this quasi-inequality. Since we are considering terms equivalent modulo associativity for $\cdot$ and multiplication by e, we may re-associate freely and may match variables in a product by e. For example, the term x matches zw with z substituted by x, and w substituted by e (or vice versa). Note that for each of the members of $G_{\mathcal{RL}}$, the variables in the premises are a subset of the variables in the conclusion (this is referred to as the *subformula property* in logic). It follows that if an inequality matches a member of $G_{\mathcal{RL}}$, this determines exactly what inequalities must appear in the premises of the member. Hence the quasi-inequalities may be used as so-called rewrite rules in a search algorithm for a proof-tree of a given inequality.

For all members of $G_{\mathcal{RL}}$, the inequalities in the premises are structurally simpler than the inequalities in the conclusion. Hence the depth of a proof-tree is bounded by

[2] In the literature on Gentzen systems this corresponds to *cut-free provable* since the Gentzen system presented here does not mention the so-called *cut-rule* $x \le y$ & $uyv \le w \Rightarrow uxv \le w$.

the size (defined in a suitable way) of the inequality at the root. It follows that it is decidable whether a given sequent is Gentzen provable.

The following simple examples serve to illustrate the effectiveness of this decision procedure.

$$\begin{array}{ll} xy \le xy \Rightarrow xy \le xy \vee xz & \text{by } (\vee \text{ right}) \text{ and} \\ xz \le xz \Rightarrow xz \le xy \vee xz & \text{by } (\vee \text{ right}) \\ \Rightarrow x(y \vee z) \le xy \vee xz & \text{by } (\vee \text{ left}) \end{array}$$

$$\begin{array}{ll} y \le y \Rightarrow y \wedge z \le y & \text{by } (\wedge \text{ left}) \\ x \le x \;\&\; y \wedge z \le y \Rightarrow x(x\backslash(y \wedge z) \le y & \text{by } (\backslash \text{ left}) \\ \Rightarrow x\backslash(y \wedge z) \le x\backslash y & \text{by } (\backslash \text{ right}) \\ z \le z \Rightarrow y \wedge z \le z & \text{by } (\wedge \text{ left}) \\ x \le x \;\&\; y \wedge z \le z \Rightarrow x(x\backslash(y \wedge z) \le z & \text{by } (\backslash \text{ left}) \\ \Rightarrow x\backslash(y \wedge z) \le x\backslash z & \text{by } (\backslash \text{ right}) \\ \Rightarrow x\backslash(y \wedge z) \le x\backslash y \wedge x\backslash z & \text{by } (\wedge \text{ right}) \end{array}$$

An inequality such as $x \wedge (y \vee z) \le (x \wedge y) \vee (x \wedge z)$ is not Gentzen provable since no proof tree can be found: only ($\wedge$ left) and ($\vee$ right) match this inequality, and the premises of these quasi-inequalities only match ($\vee$ left) and ($\wedge$ right). But their premises, in turn, are not instances of $x \le x$ in all cases.

For lattice theorists it is also interesting to note that the quasi-inequalities for $\vee$ and $\wedge$ are essentially equivalent to Whitman's method for deciding if $s \le t$ holds in all lattices.

We now prove the result of Ono and Komori [OK85] which shows that an inequality $s \le t$ is Gentzen provable if and only if it holds in $\mathcal{RL}$. The semantical proof given here is based on a version in [OT99]. The forward implication is the soundness of the proof procedure, and follows from the observation that all the rules are valid (as quasi-inequalities) in $\mathcal{RL}$. The reverse implication is *completeness*, for which we need to prove that if $s \le t$ is not Gentzen provable, then there is a residuated lattice in which this identity fails.

We begin with a general lemma useful for constructing residuated lattices.

Lemma 7.1. *Suppose $M = \langle M, \cdot, e^{\mathbf{M}} \rangle$ is a monoid and $\mathcal{L}$ is a set of subsets of $\mathbf{M}$ such that*

(P$_1$) *$\mathcal{L}$ is closed under arbitrary intersections and*

(P$_2$) *for all $X \subseteq M$, $Y \in \mathcal{L}$ we have $X\backslash Y$ and $Y/X \in \mathcal{L}$, where*

$$X\backslash Y = \{z \in M : X\{z\} \subseteq Y\}, \quad Y/X = \{z \in M : \{z\}X \subseteq Y\} \quad \textit{and}$$

$$XY = \{xy : x \in X,\ y \in Y\}.$$

Then $\mathbf{L} = (\mathcal{L}, \vee, \wedge, \cdot^{\mathbf{L}}, \backslash, /, e^{\mathbf{L}})$ *is a residuated lattice, with*

$$X^C = \bigcap\{Z \in \mathbf{L} : X \subseteq Z\}, \quad \textit{the closure of } X, \textit{ and}$$

$$X \vee Y = (X \cup Y)^C \qquad X \wedge Y = X \cap Y \qquad X \cdot^{\mathbf{L}} Y = (XY)^C \quad e^{\mathbf{L}} = \{e^{\mathbf{M}}\}^C.$$

Proof. $\mathbf{L}$ is a lattice (in fact a complete lattice) since it is the collection of closed sets of a closure operation. By definition of $\backslash$ we have $Z \subseteq X\backslash Y$ if and only if $XZ \subseteq Y$, and for $Y \in \mathbf{L}$ this is equivalent to $X \cdot^{\mathbf{L}} Z = (XZ)^C \subseteq Y$. Similarly, $Z \subseteq Y/X$ is equivalent to $Z \cdot^{\mathbf{L}} X \subseteq Y$. It remains to show that $\cdot^{\mathbf{L}}$ is associative and e is an identity.

For all $X, Y \subseteq M$, $XY \subseteq (XY)^C$ implies $Y \subseteq X\backslash(XY)^C$, hence

$$XY^C \subseteq X(X\backslash(XY)^C)^C = X(X\backslash(X \cdot^{\mathbf{L}} Y)) \subseteq X \cdot^{\mathbf{L}} Y$$

where the middle equality makes use of the fact that $X\backslash(XY)^C$ is closed by (P_2). Similarly $X^C Y \subseteq X \cdot^{\mathbf{L}} Y$, hence $X^C Y^C \subseteq X^C \cdot^{\mathbf{L}} Y = (X^C Y)^C \subseteq (X \cdot^{\mathbf{L}} Y)^C = X \cdot^{\mathbf{L}} Y$. Since we also have $XY \subseteq X^C Y^C$, it follows that $(XY)^C = (X^C Y^C)^C$. Now

$$\begin{aligned}(X \cdot^{\mathbf{L}} Y) \cdot^{\mathbf{L}} Z &= ((XY)^C Z)^C = ((XY)^{CC} Z^C)^C = ((XY)Z)^C \\ &= (X(YZ))^C = (X^C (YZ)^C)^C = X \cdot^{\mathbf{L}} (Y \cdot^{\mathbf{L}} Z),\end{aligned}$$

and $e^{\mathbf{L}} \cdot^{\mathbf{L}} X = (\{1\}^C X)^C = (\{1\}^{CC} X^C)^C = (\{1\}X)^C = X^C = X$ for $X \in \mathbf{L}$. ■

The notion of a subterm of a term p is defined in the standard way: p is a subterm of p, and if $s \diamond t$ is a subterm of p for $\diamond \in \{\vee, \wedge, \cdot, \backslash, /\}$ then s, t are subterms of p. Let $S(p) \subseteq \mathbf{T}_M$ be the set of subterms of p and let $M(p)$ be the submonoid of $\mathbf{T}_M$ generated by $S(p)$. For $q, q' \in M(p)$, $u \in S(p)$, define

$$[q, q', r] = \{s \in M(p) : qsq' \leq r \text{ is Gentzen provable}\}.$$

Further let

$$\begin{aligned}&\mathcal{L}'(p) = \{[q, q', r] : q, q' \in M(p),\ r \in S(p)\} \quad \text{and} \\ &\mathcal{L}(p) = \Big\{\bigcap \mathcal{K} : \mathcal{K} \subseteq \mathcal{L}'(p)\Big\}.\end{aligned}$$

In the subsequent proofs we will frequently make use of the following observation:

($*$) For any $X \subseteq M(p)$, $s \in S(p)$,
$s \in X^C$ iff for all $q, q' \in M(p)$ and $r \in S(p)$, $X \subseteq [q, q', r]$ implies $s \in [q, q', r]$.

Lemma 7.2. $\mathbf{L}(p) = \langle \mathcal{L}(p), \vee, \wedge, \cdot, e, \backslash, / \rangle$ *is a residuated lattice, with the operations defined as in the preceding lemma.*

Proof. (P_1) holds by construction. To prove (P_2), let $X \subseteq M(p)$ and $Y \in \mathcal{L}(p)$. Now $s \in X\backslash Y$ if and only if $X\{s\} \subseteq Y$ if and only if for all $t \in X$, $ts \in Y = Y^C$. By $(*)$ this is equivalent to showing that $Y \subseteq [q, q', r]$ implies $ts \in [q, q', r]$. This last containment holds if and only if $qtsq' \leq r$ is provable if and only if $s \in [qt, q', r]$. Hence

$$s \in X\backslash Y \text{ iff } s \in \bigcap \Big\{ [qt, q', r] : t \in X \text{ and } Y \subseteq [q, q', r] \Big\},$$

which implies that $X\backslash Y \in \mathcal{L}(p)$, and Y/X is similar. ∎

The following result is the central part of the completeness argument. As usual, an assignment $h : \{x_1, x_2, \ldots\} \to \mathbf{L}$ extends to a homomorphism from the (quotient) term algebra $\mathbf{T}_M$ to $\mathbf{L}$, with $h(e)$ defined as $e^{\mathbf{L}}$. We now fix h to be the assignment $h(x_i) = [x_i]$, where the notation $[r]$ is shorthand for $[e, e, r]$.

Lemma 7.3. *Let $\mathbf{L}(p)$ and h be defined as above. For any subterm t of p we have $t \in h(t) \subseteq [t]$. In particular, if $e \in h(t)$ then the inequality $e \leq t$ is Gentzen provable.*

Proof. By induction on the structure of the subterm. If it is a variable of p, say x, then $h(x) = [x]$ by definition, and $x \in [x]$ since $x \leq x$ is Gentzen provable by (refl). Suppose s, t are subterms of p, and $s \in h(s) \subseteq [s]$, $t \in h(t) \subseteq [t]$.

$s \vee t \in h(s \vee t) \subseteq [s \vee t]$: Note that $h(s \vee t) = (h(s) \cup h(t))^C$. Let $q \in h(s) \cup h(t)$. If $q \in h(s)$, then $q \in [s]$, so $q \leq s$ is Gentzen provable. By ($\vee$ right) it follows that $q \leq s \vee t$ is Gentzen provable, hence $q \in [s \vee t]$ and therefore $h(s) \subseteq [s \vee t]$. Similarly $h(t) \subseteq [s \vee t]$, and since $[s \vee t]$ is closed, $h(s \vee t) \subseteq [s \vee t]$.

To see that $s \vee t \in h(s \vee t)$, we use observation $(*)$: Suppose $h(s) \cup h(t) \subseteq [q, q', r]$ where $q, q' \in M(p)$, $r \in S(p)$. Then $qsq' \leq r$ and $qtq' \leq r$ are Gentzen provable (since $s \in h(s)$ and $t \in h(t)$). Therefore $q(s \vee t)q' \leq r$ is Gentzen provable by ($\vee$ left) and so $s \vee t \in [q, q', r]$. By $(*)$ we conclude that $s \vee t \in (h(s) \cup h(t))^C = h(s \vee t)$.

$s \wedge t \in h(s \wedge t) \subseteq [s \wedge t]$: Let $q \in h(s \wedge t) = h(s) \cap h(t)$. Then $q \in [s] \cap [t]$, hence $q \leq s$ and $q \leq t$ are Gentzen provable. So now $q \leq s \wedge t$ is Gentzen provable by ($\wedge$ right), which shows that $q \in [s \wedge t]$.

Suppose $h(s) \subseteq [q, q', r]$. Then $qsq' \leq r$ is Gentzen provable, and by ($\wedge$ left) $q(s \wedge t)q' \leq r$ is Gentzen provable. Therefore $s \wedge t \in [q, q', r]$, and it follows from $(*)$ that $s \wedge t \in h(s)^C = h(s)$. Similarly $s \wedge t \in h(t)$, hence $s \wedge t \in h(s \wedge t)$.

$st \in h(st) \subseteq [st]$: Since $h(st) = (h(s)h(t))^C$, we have $st \in h(st)$. Now consider $r \in h(s)h(t)$. Then $r = qq'$, where $q \in h(s) \subseteq [s]$ and $q' \in h(t) \subseteq [t]$. Therefore $q \leq s$ and $q' \leq t$ are Gentzen provable, hence by ($\cdot$ right) $qq' \leq st$ is Gentzen provable, and so $r \in [st]$. It follows that $h(s)h(t) \subseteq [st]$, and since $[st]$ is closed, $h(st) \subseteq [st]$.

$s\backslash t \in h(s\backslash t) \subseteq [s\backslash t]$: Here $h(s\backslash t) = h(s)\backslash h(t) = \{q \in M(p) : h(s)\{q\} \subseteq h(t)\}$. Thus $q \in h(s\backslash t)$ implies $sq \in h(t) \subseteq [t]$, since we are assuming $s \in h(s)$. This means $sq \leq t$ is Gentzen provable, so by ($\backslash$ right) $q \in [s\backslash t]$. Therefore $h(s\backslash t) \subseteq [s\backslash t]$.

Suppose $h(t) \subseteq [q, q', r]$. Then $t \in h(t)$ implies $qtq' \leq r$ is Gentzen provable. For any $s' \in h(s) \subseteq [s]$ we have that $s' \leq s$ is Gentzen provable, so from ($\backslash$ left) we get that $s'(s\backslash t) \in [q, q', r]$. By $(*)$ it follows that $s'(s\backslash t) \in h(t)$ whenever $s' \in h(s)$, hence $h(s)\{s\backslash t\} \subseteq h(t)$. This implies $s\backslash t \in h(s)\backslash h(t) = h(s\backslash t)$.

The case for $s/t \in h(s/t) \subseteq [s/t]$ is similar.

Since we are assuming that h has been extended to a homomorphism from $\mathbf{T}_M$ to $\mathbf{L}$, we have $h(e) = e^{\mathbf{L}} = \{e\}^C$. Suppose $\{e\} \subseteq [q, q', r]$, then $qq' \leq r$ is Gentzen provable, and $e \in [q, q', r]$. Hence $(*)$ implies $e \in h(e)$. Finally, $h(e) \subseteq [e]$ holds since $\{e\} \subseteq [e]$.

The second statement is a simple consequence: if $e \in h(t)$ then $e \in [t]$ which means $e \leq t$ is Gentzen provable. ∎

Theorem 7.4. *For any $\mathcal{RL}$-term p the following statements are equivalent:*

(i) $\mathcal{RL} \models e \leq p$

(ii) $\mathbf{L}(p) \models e \leq p$

(iii) $e \leq p$ *is Gentzen provable.*

Proof. (i) implies (ii) by Lemma 7.2. Assuming (ii) holds, we have $h(e) \subseteq h(p)$. Since $e \in \{e\}^C = h(e)$, (iii) follows by Lemma 7.3. Finally, (iii) implies (i) by a standard soundness argument using the observation that $\mathcal{RL} \models G_{\mathcal{RL}}$. ∎

Since it was observed earlier that condition (iii) is decidable, and since any equation can be reduced to this form, the equational theory of $\mathcal{RL}$ is decidable. Okada and Terui [OT99] go on to prove that $\mathcal{RL}$ is generated by its finite members, and they also consider several subvarieties and expansions of $\mathcal{RL}$. For example, to decide inequalities for residuated $0, 1$-lattices, one simply adds the two inequalities $x0y \leq z$ and $x \leq 1$ to $G_{\mathcal{RL}}$. In fact their results are formulated for what amounts to commutative residuated $0, 1$-lattices, and the non-commutative case is only mentioned briefly at the end. To obtain a Gentzen system for $\mathcal{CRL}$, it suffices to add the so-called *exchange rule*

$$uxyv \leq w \Rightarrow uyxv \leq w \text{ corresponding to commutativity } xy = yx$$

or equivalently one replaces $\mathbf{T}_M$ by its commutative quotient algebra $\mathbf{T}_{CM}$. Other well known Gentzen quasi-inequalities are the *weakening rule*

$$uv \leq w \Rightarrow uxv \leq w \text{ corresponding to integrality } x \leq e$$

and the *contraction rule*

$$uxxv \leq w \Rightarrow uxv \leq w \text{ corresponding to } x \leq xx.$$

Adding any combination of these quasi-inequalities to $G_{\mathcal{RL}}$ gives a decision procedure for the corresponding subvariety of $\mathcal{RL}$ (as shown in [OK85], [OT99]). Note that in the presence of integrality, the contraction rule is equivalent to *idempotence* $x = xx$ and implies the identity $xy = x \wedge y$ that defines the subvariety $\mathcal{B}r\mathcal{A}$ of Brouwerian algebras (since $xy \leq x \wedge y = (x \wedge y)(x \wedge y) \leq xy$).

The results above show that Gentzen systems are a versatile approach to proving decidability and can be adapted to cover other subvarieties of $\mathcal{RL}$.

The Finite Model Property and the Finite Embedding Property. A class $\mathcal{K}$ of algebras has the *finite model property* (FMP) if every identity that fails in some member of $\mathcal{K}$ also fails in some finite member of $\mathcal{K}$. The *strong finite model property* (SFMP) is defined in the same way, except for quasi-identities instead of identities. Since every identity is a quasi-identity, SFMP implies FMP. If we denote the finite members of $\mathcal{K}$ by $\mathcal{K}_F$, then FMP is equivalent to $\mathbf{V}(\mathcal{K}) = \mathbf{V}(\mathcal{K}_F)$, and SFMP is equivalent to $\mathbf{Q}(\mathcal{K}) = \mathbf{Q}(\mathcal{K}_F)$, where $\mathbf{Q}(\mathcal{K})$ is the quasivariety generated by $\mathcal{K}$. For a finitely based variety, FMP implies that the equational theory is decidable, and similarly for a finitely based quasivariety, SFMP implies that the quasi-equational theory is decidable. In both cases the argument is that there is an algorithm to enumerate all (quasi-)identities that are consequences of the finite basis, and there is an algorithm to enumerate all finite members of the given (quasi-)variety. Since we are assuming (S)FMP, any specific (quasi-)identity must either appear on the first list or fail in one of the algebras on the second list. Unlike a Gentzen system, the algorithm just outlined usually cannot be applied in practice, but the (S)FMP has proven valuable in establishing theoretical decidability results.

If a class $\mathcal{K}$ of structures is closed under finite products, then any universal formula is equivalent to a finite number of quasi-identities and negated equations. If, in addition, $\mathcal{K}$ contains one-element models, negated equations are not satisfied, hence $\mathcal{K}$ has a decidable universal theory if and only if it has a decidable quasi-equational theory. In particular, this argument shows that for varieties and quasivarieties of residuated lattices the decidability of the universal theory and the quasi-equational theory coincide.

Let $\mathbf{A}$ be an algebra, and B any subset of A. The partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ is obtained by restricting the fundamental operations of $\mathbf{A}$ to the set B. A class $\mathcal{K}$ of algebras has the *finite embedding property* (FEP) if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded in a finite member of $\mathcal{K}$. In [Fe92] it is shown that for quasivarieties, FEP and SFMP are equivalent (see also [BvA1]).

In recent work of Blok and van Alten [BvA2], algebraic methods are used to show that $\mathcal{IRL}$, $\mathcal{ICRL}$, and $\mathcal{B}r\mathcal{A}$ have the FEP, hence these varieties have decidable universal theories. Their very general results also apply to nonassociative residuated lattices and to various subreducts that are obtained when the lattice operations are omitted. Kowalski and Ono [KO] show that the variety of integral commutative residuated $0, 1$-lattices is generated by its finite *simple* members, and the subvarieties defined by $x^{n+1} = x^n$ have the FEP.

Further Results. One should compare the decidability of residuated lattices with two important results about ℓ-groups.

Theorem 7.5. [HM79] *The variety of ℓ-groups has a decidable equational theory.*

Theorem 7.6. [GG83] *The (local) word problem for ℓ-groups is undecidable; i.e., there exists a finitely presented ℓ-group (with one relator) for which there is no algorithm that decides if two words are representatives of the same element.*

Corollary 7.7. *The quasi-equational theory of ℓ-groups is undecidable.*

This is also referred to as the undecidability of the global (or uniform) word problem.

Theorem 7.8. (i) [Hi66] *The universal theory of abelian ℓ-groups is decidable.*

(ii) [We86] *The universal theory of abelian ℓ-groups is co-NP-complete.*

(iii) [Gu67] *The first-order theory of abelian ℓ-groups is hereditarily undecidable.* (See also [Bu85].)

Theorem 7.9. *The quasi-equational theory of residuated lattices is undecidable. The same holds for any subvariety that contains all powersets of finite monoids.*

Proof. Consider the class $\mathcal{K}$ of all monoid reducts of $\mathcal{RL}$. Note that a quasi-identity that uses only $\cdot$, holds in all residuated lattices iff it holds in $\mathcal{K}$ iff it holds in all subalgebras of $\mathcal{K}$.

Let $\mathcal{SG}$ be the variety of all semigroups. Any semigroup S can be embedded in some member of $\mathcal{K}$ follows. Embed S in a monoid $S_e = S \cup \{e\}$ and construct the powerset $\mathcal{P}(S_e)$, which is a complete residuated lattice with multiplication of complexes as product. The collection of singletons is closed under this operation and isomorphic to S_e.

On the other hand, $\mathcal{K} \subseteq \mathcal{SG}$, hence $\mathcal{SG}$ coincides with the class of all subalgebras of $\mathcal{K}$. But the quasi-equational theory of semigroups is undecidable, hence the same is true for residuated lattices.

For the second part of the theorem, we use the result of [GL84] that the quasi-equational theory of semigroups is recursively inseparable from the quasi-equational theory of finite semigroups. ∎

Since $\mathcal{P}(S_e)$ is distributive, it follows that the variety of distributive residuated lattices has an undecidable quasi-equational theory. This was proved earlier by N. Galatos [Ga] using the machinery of von Neumann n-frames. His result is somewhat stronger and also applies to many subvarieties not covered by the above argument.

Theorem 7.10. [Ga] *The (local) word problem (and hence the quasi-equational theory) for any variety between $\mathcal{DRL}$ and $\mathcal{CDRL}$ is undecidable.*

Variety	Name	Eq. Theory	Word prob.	Univ. Th.
Residuated Lattices	$\mathcal{RL}$	FMP[OT99]		Und. 7.9
Commutative $\mathcal{RL}$	$\mathcal{CRL}$	FMP[OT99]		
Distributive $\mathcal{RL}$	$\mathcal{DRL}$		Und.[Ga]	Und. 7.9
$\mathcal{CRL} \cap \mathcal{DRL}$	$\mathcal{CDRL}$		Und.[Ga]	Undecidable
Modular $\mathcal{RL}$	$\mathcal{MRL}$	Und. 7.11	Undecidable	Undecidable
$\mathbf{V}$(Resid. Chains)	$\mathcal{RL}^C$			
Commutative $\mathcal{RL}^C$	$\mathcal{CRL}^C$			
Integral $\mathcal{RL}$	$\mathcal{IRL}$	FMP[OT99]	Decidable	FEP [BvA2]
Commutative $\mathcal{IRL}$	$\mathcal{CIRL}$	FMP[OT99]	Decidable	FEP [BvA1]
Cancellative $\mathcal{RL}$	$\mathcal{C}an\mathcal{RL}$			
Comm. $\mathcal{C}an\mathcal{RL}$	$\mathcal{C}an\mathcal{CRL}$			
ℓ-groups	$\mathcal{LG}$	Dec.[HM79]	Und.[GG83]	Undecidable
Abelian ℓ-groups	$\mathbf{V}(\mathbb{Z})$	Decidable	Decidable	Dec.[Hi66]
Generalized BL Algs	$\mathcal{GBL}$			
Generalized MV Algs	$\mathcal{GMV}$			
Brouwerian Algs	$\mathcal{B}r\mathcal{A}$	Decidable	Decidable	FEP[MT44]
Gen. Boolean Algs	$\mathcal{GBA}$	Decidable	Decidable	FEP

Table 2: (Un)decidability of some subvarieties of $\mathcal{RL}$

The next result shows that there are indeed varieties of residuated lattices with undecidable equational theory.

Theorem 7.11. *The variety of modular residuated lattices has an undecidable equational theory.*

Proof. Let $\mathbf{M}$ be a modular lattice and define $A = M \cup \{0, e, 1\}$ where $0 < e < x < 1$ for all $x \in M$. Then $\mathbf{A}$ is still modular (in fact in the same variety as M).

We define $\cdot$ on $\mathbf{A}$ by $x0 = 0 = 0x$, $xe = x = ex$ and $xy = 1$ if $x, y \neq 0, e$. It is easy to check that $\cdot$ is associative and residuated, hence the $\vee, \wedge$-equational theory of modular residuated lattices coincides with the equational theory of modular lattices. Since modular lattices have an undecidable equational theory ([Fr80]), the same is true for modular residuated lattices. ∎

Table 2 summarizes what is currently known about the decidability of various subvarieties of $\mathcal{RL}$.

8 Open Problems

Problem 8.1. *Is every lattice a subreduct of a* commutative *cancellative residuated lattice (see Theorem 5.3)?*

Problem 8.2. *Are there commutative, cancellative, distributive residuated lattices that are not in $\mathcal{C}an\mathcal{RL}^C$? In the noncommutative case the 2-generated free ℓ-group is an example.*

Problem 8.3. *Does $\mathcal{C}an\mathcal{RL}^C$ have a decidable equational theory?*

Problem 8.4. *Is there a Weinberg-type description of free algebras in $\mathcal{C}an\mathcal{RL}^C$? See e.g. Powell and Tsinakis* [PT89].

Problem 8.5. *Theorem 6.2 proves that the only two atoms in $\mathbf{L}(\mathcal{RL})$ that are cancellative and commutative are $\mathcal{V}(\mathbb{Z}^-)$ and $\mathcal{V}(\mathbb{Z})$. Are there any other cancellative atoms in $\mathbf{L}(\mathcal{RL})$? It follows from Theorem 6.1 that if this is the case then they are generated by nonintegral residuated lattices.*

Problem 8.6. *Are there uncountably many atoms in $\mathbf{L}(\mathcal{RL})$ that satisfy the commutative identity or the identity $x^2 = x^3$?*

Problem 8.7. *Kowalski and Ono* [KO00] *have shown that there are no nontrivial splitting varieties in the lattice of subvarieties of commutative integral residuated* $0, 1$-*lattices. What is the situation for $\mathbf{L}(\mathcal{RL})$ or some of the ideals determined by subvarieties of $\mathcal{RL}$?*

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