Nonrepresentable Sequential Algebras

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Abstract

The sequential calculus of von Karger and Hoare [18] is designed for reasoning about sequential phenomena, dynamic or temporal logic, and concurrent or reactive systems. Unlike the classical calculus of relations, it has no operation for forming the converse of a relation. Sequential algebras [15] are algebras that satisfy certain equations in the sequential calculus. One standard example of a sequential algebra is the set of relations included in a partial ordering. Nonstandard examples arise by relativizing relation algebras to elements that are antisymmetric, transitive, and reflexive. The incompleteness and non-finite-axiomatizability of the sequential calculus are examined here from a relation-algebraic point of view. New constructions of nonrepresentable relation algebras are used to prove that there is no finite axiomatization of the equational theory of antisymmetric dense locally linear sequential algebras. The constructions improve on previous examples in certain interesting respects and give yet another proof that the classical calculus of relations is not finitely axiomatizable.

Keywords: sequential algebras, relation algebras, completeness, axiomatizability, representability

DEFINITION 1.1 (von Karger [15])
Suppose $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, \triangleright, \triangleleft, 1'\rangle$. $\mathfrak{A}$ is a **sequential algebra** if

- The reduct $\langle A, +, \cdot, ^-, 0, 1\rangle$ is a Boolean algebra,
- the reduct $\langle A, ;, 1'\rangle$ is a monoid,

and, for all $p, q, r \in A$,

- $(p;q) \cdot r = 0$ iff $(p \triangleright r) \cdot q = 0$ iff $(r \triangleleft q) \cdot p = 0$,
- $p;(q \triangleleft r) \leq (p;q) \triangleleft r$,
- $1' \triangleleft p = p \triangleright 1'$.

If $\mathfrak{A}$ is a sequential algebra, then $\mathfrak{A}$ is given the name in the left column if it satisfies the identity in the right column:

symmetric	$1' \triangleleft 1 = 1$
antisymmetric	$1' \triangleleft 1 = 1'$
locally linear	$(p;q) \triangleleft r = p;(q \triangleleft r) + p \triangleleft (r \triangleleft q)$
dense	$\overline{1'} \leq \overline{1'};\overline{1'}$

Except for density, these concepts occur in [15].

DEFINITION 1.2
Suppose $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, \breve{}, 1' \rangle$. $\mathfrak{A}$ is a **relation algebra** if

- The reduct $\langle A, +, \cdot, ^-, 0, 1 \rangle$ is a Boolean algebra,
- The reduct $\langle A, ;, 1' \rangle$ is a monoid,

and, for all $p, q, r \in A$,

- $(p;q) \cdot r = 0$ iff $(\breve{p};r) \cdot q = 0$ iff $(r;\breve{q}) \cdot p = 0$.

For more background on relation algebras and sequential algebras, we refer the reader to [5], [10], [15], [18]. In, particular the definitions above can be restated in terms of equations, so both the class of all sequential algebras and the class of all relation algebras are varieties, denoted by RA and SeA respectively. Every relation algebra is (term-equivalent to) a sequential algebra if one defines binary operations $\triangleright$ and $\triangleleft$ by

$$p \triangleright q = \breve{p};q \qquad p \triangleleft q = p;\breve{q}. \tag{1.1}$$

Conversely, if the unary operation $\breve{}$ is defined by $\breve{p} = 1' \triangleleft p$ in a sequential algebra $\mathfrak{A}$, then $\mathfrak{A}$ is term-equivalent to a relation algebra iff $\mathfrak{A}$ satisfies the equations (1.1). In [8] it is also shown that a sequential algebra is (term-equivalent to) a relation algebra iff the equation $p;(q \triangleleft r) = (p;q) \triangleleft r$ holds. From an algebraic point of view, the two varieties differ in that RA is a discriminator variety while SeA is not (see *e.g.*, [4]). In many other respects sequential algebras are similar to relation algebras.

Suppose $\mathfrak{A} = \langle A, +, \cdot, ^-, 0, 1, ;, \breve{}, 1' \rangle$ is a relation algebra and $u \in A$. We say that u is

reflexive	if $1' \leq u$,
transitive	if $u;u \leq u$,
symmetric	if $u = \breve{u}$,
antisymmetric	if $u \cdot \breve{u} \leq 1'$,
linear	if $1 = u + \breve{u}$,
locally linear	if $u;\breve{u} \cdot \breve{u};u \leq u + \breve{u}$,
dense	if $(u \cdot \overline{1'}) \leq (u \cdot \overline{1'});(u \cdot \overline{1'})$

Let

$$Rl_u\mathfrak{A} = \{x : u \geq x \in A\}.$$

Then $Rl_u\mathfrak{A}$ is closed under $+$ and $\cdot$, contains 0, and has maximum element u. Define relative complementation on $Rl_u\mathfrak{A}$ by $x^{-_u} = \overline{x} \cdot u$. Then

$$\left\langle Rl_u\mathfrak{A}, +, \cdot, ^{-_u}, 0, u \right\rangle$$

is again a Boolean algebra, the one obtained by relativizing to u. Suppose u is reflexive and transitive. Then $Rl_u\mathfrak{A}$ is also closed under ; since u is transitive, and $Rl_u\mathfrak{A}$ contains 1' since u is reflexive. Define two more binary operations $\triangleright$ and $\triangleleft$ on $Rl_u\mathfrak{A}$ by $x \triangleright y = \breve{x};y \cdot u$ and $x \triangleleft y = x;\breve{y} \cdot u$. This gives a relativized reduct algebra

$$\mathfrak{Seq}_u(\mathfrak{A}) = \left\langle Rl_u\mathfrak{A}, +, \cdot, ^{-_u}, 0, u, ;, \triangleright, \triangleleft, 1' \right\rangle.$$

THEOREM 1.3 (von Karger [15])
If $\mathfrak{A}$ is a relation algebra and u is a reflexive and transitive element of $\mathfrak{A}$, then $\mathfrak{Seq}_u(\mathfrak{A})$ is a sequential algebra.

In [15] it is shown that the relativization of a relation algebra with respect to an element that is locally linear or antisymmetric (as well as transitive and reflexive) results in a sequential algebra that is locally linear or antisymmetric, respectively. The same is true for density.

EXAMPLE 1.4
Let $\mathfrak{A}$ be the finite relation algebra whose atoms are 1', a, $\breve{a}$, b, $\breve{b}$, c, and $\breve{c}$, and whose relative multiplication is determined by the following table. (The "+" signs are omitted to save space, *e.g.*, $a;b = a + b$ and $a;\breve{a} = 1' + b + \breve{b} + c + \breve{c}$.)

;	1'	a	b	c	$\breve{a}$	$\breve{b}$	$\breve{c}$
1'	1'	a	b	c	$\breve{a}$	$\breve{b}$	$\breve{c}$
a	a	bc	ab	ac	$1'b\breve{b}c\breve{c}$	$a\breve{a}b\breve{b}$	$a\breve{a}c\breve{c}$
b	b	ab	ac	bc	$a\breve{a}b\breve{b}$	$1'a\breve{a}c\breve{c}$	$b\breve{b}c\breve{c}$
c	c	ac	bc	ab	$a\breve{a}c\breve{c}$	$b\breve{b}c\breve{c}$	$1'a\breve{a}b\breve{b}$
$\breve{a}$	$\breve{a}$	$1'b\breve{b}c\breve{c}$	$a\breve{a}b\breve{b}$	$a\breve{a}c\breve{c}$	$\breve{b}\breve{c}$	$\breve{a}\breve{b}$	$\breve{a}\breve{c}$
$\breve{b}$	$\breve{b}$	$a\breve{a}b\breve{b}$	$1'a\breve{a}c\breve{c}$	$b\breve{b}c\breve{c}$	$\breve{a}\breve{b}$	$\breve{a}\breve{c}$	$\breve{b}\breve{c}$
$\breve{c}$	$\breve{c}$	$a\breve{a}c\breve{c}$	$b\breve{b}c\breve{c}$	$1'a\breve{a}b\breve{b}$	$\breve{a}\breve{c}$	$\breve{b}\breve{c}$	$\breve{a}\breve{b}$

Let $u = 1' + a + b + c$. Then u is transitive and reflexive, so $\mathfrak{Seq}_u(\mathfrak{A})$ is a sequential algebra. Its atoms are 1', a, b, and c, and its non-Boolean binary operations are determined by the following tables.

;	1'	a	b	c
1'	1'	a	b	c
a	a	$b + c$	$a + b$	$a + c$
b	b	$a + b$	$a + c$	$b + c$
c	c	$a + c$	$b + c$	$a + b$

$\triangleleft$	1'	a	b	c
1'	1'	0	0	0
a	a	$1' + b + c$	$a + b$	$a + c$
b	b	$a + b$	$1' + a + c$	$b + c$
c	c	$a + c$	$b + c$	$1' + a + b$

$\triangleright$	1'	a	b	c
1'	1'	a	b	c
a	0	$1' + b + c$	$a + b$	$a + c$
b	0	$a + b$	$1' + a + c$	$b + c$
c	0	$a + c$	$b + c$	$1' + a + b$

Note that u is antisymmetric, dense, and linear, and that $\mathfrak{Seq}_u(\mathfrak{A})$ is therefore antisymmetric, dense, and locally linear.

Problem 1. Does every sequential algebra arise in this way, *i.e.*, is every sequential algebra $\mathfrak{S}$ isomorphic to $\mathfrak{Seq}_u(\mathfrak{A})$ for some relation algebra $\mathfrak{A}$ and some transitive and reflexive element u?

Problem 2. Is every locally linear sequential algebra $\mathfrak{S}$ isomorphic to $\mathfrak{Seq}_u(\mathfrak{A})$ for some relation algebra $\mathfrak{A}$ and some locally linear, transitive and reflexive element u?

If u is a preorder on X, then u is a transitive and reflexive element in the relation algebra $\mathfrak{Re}(X)$ of all binary relations on X and $\mathfrak{Seq}_u(\mathfrak{Re}(X))$ is the sequential algebra of all subrelations of u. We say that a sequential algebra is **representable** if it is isomorphic to a subalgebra of an algebra of the form $\mathfrak{Seq}_u(\mathfrak{Re}(X))$, where u is a preorder on some set X. If a sequential algebra happens to be a relation algebra, then this notion of representability agrees with the standard one for relation algebras.

Von Karger uses a different notion of representability, defined in terms of locally linear observation spaces. In [17], it is shown that these spaces can be extended to Brandt groupoids. Complex algebras of Brandt groupoids are representable relation algebras [7]. Hence representability defined via locally linear observation spaces implies representability in the sense used here. Our notion is more general since we do not assume local linearity. Under that assumption, however, the notions coincide.

For every sequential algebra $\mathfrak{A}$ define the unary operation c by $c(x) = x + 1;x + x;1 + x \rhd 1 + 1 \rhd x + x \lhd 1 + 1 \lhd x$ for every $x \in A$. Then c is self-conjugate: $x \cdot c(y) = 0$ iff $c(x) \cdot y = 0$. Every homomorphism h of a sequential algebra of $\mathfrak{A}$ determines an ideal $I = \{x \in A : h(x) = 0\}$. If $i \in I$ then $c(i) \in I$. On the other hand, if $c(z) = z \in A$ then the function $\langle x \cdot z : x \in A \rangle$ is a homomorphism of $\mathfrak{A}$ onto a sequential algebra whose universe is $Rl_z\mathfrak{A} = \{y : z \geq y \in A\}$ and whose operations are obtained from those of $\mathfrak{A}$ by relativizing to z.

Andréka and Németi pointed out in a letter to the first author that the following theorem can be proved, as is done here, by imitating the proof (due to Németi) of Theorem [3, 5.5.10], that cylindric-relativized set algebras form a variety.

Theorem 1.5
The class of representable sequential algebras is a variety.

Proof. It is obvious that the class of representable sequential algebras is closed under the formation of subalgebras, and not difficult to show that it is also closed under direct products. It therefore suffices to show that every homomorphic image of a representable sequential algebra is representable.

Assume that $\mathfrak{A} \subseteq \mathfrak{Seq}_u(\mathfrak{Re}(X))$, where u is a preorder on some set X, and that I is an ideal of $\mathfrak{A}$ determined by a homomorphism. We wish to embed $\mathfrak{A}/I$ into a representable sequential algebra. Let E be an ultrafilter on I such that $\{j : i \leq j \in I\} \in E$ for every $i \in I$. (E exists since the sets in question have the finite intersection property.) Let $\sim_E$ be the equivalence relation defined for $s, s' \in {}^IX$ by $s \sim_E s' \iff \{i \in I : s_i = s'_i\} \in E$ and define the equivalence class $s/\sim_E$ by $s/\sim_E = \{s' \in {}^IX : s \sim_E s'\}$. For every $x \in A$ let

$$\rho(x) = \{ \langle s/\sim_E, s'/\sim_E \rangle : s, s' \in {}^IX \wedge \{i \in I : \langle s_i, s'_i \rangle \in x\} \in E \}$$

It is easy to check that ρ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{Seq}_{\rho(u)}\left(\mathfrak{Re}\left({}^IX/\sim_E\right)\right)$.

Next we construct a homomorphism h_x for every $x \in A \sim I$. Let $x \in A \sim I$. Then $x \not\leq i$ for every $i \in I$, so there are $s(x), s'(x) \in {}^IX$ such that

$$\langle s(x)_i, s'(x)_i \rangle \in x \cdot \bar{i} \quad \text{for every } i \in I. \tag{1.2}$$

Let $r(x) = \{\langle s(x)/\sim_E, s'(x)/\sim_E \rangle\}$, $z = \bigcup\{c^n(r(x)) : n \in \omega\}$, and set $h_x(y) = \rho(y) \cap z$ for every $y \in A$. Then $c(z) = z \subseteq \rho(u)$, so h_x is a homomorphism of $\mathfrak{A}$ into $\mathfrak{Seq}_z\left(\mathfrak{Re}\left({}^IX/\sim_E\right)\right)$. By (1.2) we have $\{i \in I : \langle s(x)_i, s'(x)_i \rangle \in x\} = I \in E$, so

$r(x) \subseteq h_x(x)$, hence $h_x(x) \neq \emptyset$. Next we show that if $j \in I$ then $h_x(j) = \emptyset$. Suppose $j \in I$. Since $h_x(j) = \bigcup\{\rho(j) \cap c^n(r(x)) : n \in \omega\}$, it suffices to prove $\rho(j) \cap c^n(r(x)) = \emptyset$ for every $n \in \omega$. We get $c^n(j) \in I$ from $j \in I$, so $\{i \in I : c^n(j) \leq i\} \in E$ by the choice of E. However, from (1.2) we get

$$\{i \in I : c^n(j) \leq i\} \subseteq \{i \in I : \langle s(x)_i, s'(x)_i\rangle \notin c^n(j)\}$$

so $\langle s(x)/{\sim_E}, s'(x)/{\sim_E}\rangle \notin \rho(c^n(j))$, i.e., $r(x) \cap \rho(c^n(j)) = \emptyset$. Since ρ is a homomorphism and c is self-conjugate, this gives $c^n(r(x)) \cap \rho(j) = \emptyset$, as desired. For every $x \in A \sim I$, let $\mathfrak{A}_x$ be the image of $\mathfrak{A}$ under the homomorphism h_x. Each $\mathfrak{A}_x$ is a representable sequential algebra, so the product $\mathfrak{B} = \prod_{x \in A \sim I} \mathfrak{A}_x$ is also representable. Since the ideal of h_x contains I, there is a homomorphism $g_x : \mathfrak{A}/I \to \mathfrak{A}_x$ such that $g_x(y/I) = h_x(y)$ for every $y \in A$. In particular, $g_x(x/I) = h_x(x) \neq \emptyset$. Define a function f from $\mathfrak{A}/I$ into $\mathfrak{B}$ by $f(y/I) = \langle g_x(y/I) : x \in A \sim I\rangle$ for every $y \in A$. Then f is a homomorphism and f is one-to-one since $g_x(x/I) \neq \emptyset$ whenever $x \in A \sim I$. ∎

As one referee observed, every nonrepresentable relation algebra is term-equivalent to a nonrepresentable sequential algebra. Hence the existence of nonrepresentable sequential algebras is not surprising. Furthermore, since the variety of representable relation algebras is not finitely axiomatizable, it follows that the same is true for the variety of representable sequential algebras. However, this argument does not apply to the subvariety of antisymmetric sequential algebras. Indeed, if a relation algebra $\mathfrak{A}$ is term-equivalent to an antisymmetric sequential algebra then $\mathfrak{A}$ must be Boolean, since $1' = 1' \triangleleft 1 = 1' ; \breve{1} = 1$, and all Boolean relation algebras are representable. To obtain nonrepresentable antisymmetric sequential algebras we relativize nonrepresentable relation algebras to antisymmetric elements.

THEOREM 1.6
The equation

$$z \cdot (x \cdot p;q);(y \cdot r;s) \leq p;[q;r \cdot (p \triangleright z \cdot q;y) \triangleleft s \cdot p \triangleright (x;r \cdot z \triangleleft s)];s \qquad (1.3)$$

holds in every representable sequential algebra but fails in 1774 of the 3677 finite indecomposable antisymmetric sequential algebras that have exactly 4 atoms. For example, equation (1.3) fails in the 16-element sequential algebra $\mathfrak{A}$ (defined in example 1.4) when $p = a$, $q = a$, $r = a$, $s = b$, $x = c$, $y = b$, and $z = c$.

PROOF. To check that (1.3) holds in all representable sequential algebras is a simple matter of assuming that an ordered pair $\langle v, w\rangle$ is an element of the left hand side of (1.3), which implies the existence of three other elements, and then showing that $\langle v, w\rangle$ is always an element of the right hand side.

The number of nonisomorphic indecomposable sequential algebras were found by computer, and the result shows that there are many small nonrepresentable antisymmetric sequential algebras. However, it can be shown that all sequential algebras with at most 8 elements are representable.

To see that (1.3) fails in $\mathfrak{A}$, we note that under the given assignment, the left hand side evaluates to c, while the right hand side gives $a;[(b+c) \cdot (a+b) \cdot (a+c)];b = 0$. ∎

Next we construct a sequence of relation algebras that have nonrepresentable antisymmetric sequential algebras as relative subalgebras and are also locally linear and dense. Any nonprincipal ultraproduct of these algebras will however be representable. This result implies that the equational theory of the variety of antisymmetric dense locally linear representable sequential algebras is not finitely axiomatizable.

EXAMPLE 1.7
For each $n \geq 3$ let $\mathfrak{A}_n$ be the relation algebra that has $2n+1$ atoms 1', a_1, $\breve{a}_1$, ..., a_n, $\breve{a}_n$, and whose relative multiplication is such that

$$\begin{aligned} 1' ; a_i &= a_i = a_i ; 1', \\ a_i ; a_i &= \overline{a_i} \cdot (a_1 + \ldots + a_n), \\ a_i ; a_j &= a_1 + \ldots + a_n \quad \text{wherenever } 1 \leq i \neq j \leq n. \end{aligned}$$

These conditions determine relative multiplication completely. For example, we can deduce that

$$\begin{aligned} \breve{a}_i ; \breve{a}_i &= \overline{\breve{a}_i} \cdot (\breve{a}_1 + \ldots + \breve{a}_n) \\ \breve{a}_i ; \breve{a}_j &= \breve{a}_1 + \ldots + \breve{a}_n \quad \text{if } i \neq j, \\ a_i ; \breve{a}_i &= \breve{a}_i ; a_i = \overline{a_i + \breve{a}_i}, \\ a_i ; \breve{a}_j &= \breve{a}_i ; a_j = 0' \quad \text{if } i \neq j. \end{aligned}$$

The element $u_n = 1' + a_1 + \ldots + a_n$ is a reflexive transitive antisymmetric linear dense element of $\mathfrak{A}_n$. Therefore $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$ is a sequential algebra that is antisymmetric, dense, locally linear, and has $n+1$ atoms.

For example, considering the case $n = 3$, we find that the table for relative multiplication in $\mathfrak{Seq}_{u_3}(\mathfrak{A}_3)$ is

;	1'	a_1	a_2	a_3
1'	1'	a_1	a_2	a_3
a_1	a_1	$a_2 + a_3$	$a_1 + a_2 + a_3$	$a_1 + a_2 + a_3$
a_2	a_2	$a_1 + a_2 + a_3$	$a_1 + a_3$	$a_1 + a_2 + a_3$
a_3	a_3	$a_1 + a_2 + a_3$	$a_1 + a_2 + a_3$	$a_1 + a_2$

Every diversity atom in $\mathfrak{A}_n$ has a "color", a number from 1 to n, and an "orientation", either "up" or "down". The orientation of a_i is "up" and its color is i. The orientation of $\breve{a}_i$ is "down" and its color is i. 1' has no color and no orientation. Suppose a, b, c are diversity atoms. Then the only circumstances under which we do *not* have $a ; b \geq c$ are

- the orientations of a, b, c violate $u_3 ; u_3 = u_3$ (*e.g.*, a up, b up, c down)
- a, b, c all have the same color
- the identity law is violated (*e.g.*, $a = 1'$ but $b \neq c$).

For example, $a_i ; a_j \not\geq \breve{a}_k$ because $u_3 ; u_3 = u_3$, $a_i ; a_i \not\geq a_i$ because three atoms of the same color are involved, and $a_i ; 1' \not\geq a_j$ when $i \neq j$ by the identity law.

THEOREM 1.8
For $3 \leq n$, $\mathfrak{A}_n$ is a nonrepresentable relation algebra.

PROOF. Assume, to the contrary, that $\mathfrak{A}_n$ is representable. Since $\mathfrak{A}_n$ is also simple, there is a set X and an isomorphism ρ that maps $\mathfrak{A}_n$ onto a subalgebra of $\mathfrak{Re}(X)$. From the properties of u it follows that $\rho(u_n)$ is a dense linear ordering without endpoints. Consequently X is infinite. We will use a special case of Ramsey's Theorem to derive a contradiction, thus showing that $\mathfrak{A}_n$ cannot be representable. To apply Ramsey's Theorem, we show that the 2-element subsets of X can be partitioned into n disjoint sets $T_1, \ldots, T_n$. For each $i = 1, \ldots, n$ let

$$T_i = \{\{x, x'\} : \{\langle x, x'\rangle, \langle x', x\rangle\} \subseteq \rho(a_i + \breve{a}_i)\}$$

If $1 \le i < j \le n$ then the relations $\rho(a_i + \breve{a}_i)$ and $\rho(a_j + \breve{a}_j)$ are disjoint, so T_i and T_j are also disjoint. The relation $\rho(a_i + \breve{a}_i)$ is symmetric, so for any 2-element subset $\{x, x'\}$ of X, the symmetric binary relation $\{\langle x, x'\rangle, \langle x', x\rangle\}$ is either included in $\rho(a_i + \breve{a}_i)$ or else is disjoint from $\rho(a_i + \breve{a}_i)$. Thus the sets T_i form a partition of the 2-element subsets of X. Since X is infinite, Ramsey's Theorem implies that for some $i \in \{1, \ldots, n\}$ there is an infinite subset Y of X that is "homogeneous for T_i", i.e., all 2-element subsets of Y are in T_i. We only need to know that Y has three elements to get a contradiction. Choose $y, y', y'' \in Y$. Then $\{y, y'\}$, $\{y', y''\}$, and $\{y, y''\}$ are in T_i, so $\{\langle y, y'\rangle, \langle y', y''\rangle, \langle y, y''\rangle\} \subseteq \rho(a_i + \breve{a}_i)$, hence $\langle y, y''\rangle \in \rho((a_i + \breve{a}_i);(a_i + \breve{a}_i) \cdot (a_i + \breve{a}_i))$, but this is contradictory since

$$\begin{aligned}
&(a_i + \breve{a}_i);(a_i + \breve{a}_i) \cdot (a_i + \breve{a}_i) \\
&= (a_i;a_i + a_i;\breve{a}_i + \breve{a}_i;a_i + \breve{a}_i;\breve{a}_i) \cdot (a_i + \breve{a}_i) \\
&= (\overline{a_i} \cdot (a_1 + \ldots + a_n) + \overline{a_i + \breve{a}_i} + \overline{\breve{a}_i} \cdot (\breve{a}_1 + \ldots + \breve{a}_n)) \cdot (a_i + \breve{a}_i) \\
&= 0
\end{aligned}$$

∎

THEOREM 1.9
For $3 \le n$, $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$ is a nonrepresentable antisymmetric dense locally linear sequential algebra.

PROOF. We imitate the proof that $\mathfrak{A}_n$ is a nonrepresentable relation algebra. Assume $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$ has a representation ρ. Since $\rho(u_n)$ is transitive, dense, and antisymmetric, the field of $\rho(u_n)$ must be infinite. By Ramsey's Theorem, that field has an infinite subset that is homogeneous for some color i, hence $\rho(a_i;a_i \cdot a_i) \neq \emptyset$. But $a_i;a_i \cdot a_i = 0$, so $\rho(a_i;a_i \cdot a_i) = \emptyset$, a contradiction. ∎

Every finite nonrepresentable relation algebra $\mathfrak{A}$ can be used to construct an equation that is satisfied by all representable relation algebras but fails in $\mathfrak{A}$. Such an equation can be obtained by translating into equational form the universal first-order sentence asserting that there is no subalgebra isomorphic to $\mathfrak{A}$. In special cases simpler equations can be found. For example, the following equation holds in all representable relation and sequential algebras but fails in both $\mathfrak{A}_3$ and $\mathfrak{Seq}_{u_3}(\mathfrak{A}_3)$ when $u = w = y = a_1$ and $v = x = z = a_2$.

$$u \triangleright v \cdot w;x \cdot y \triangleleft z \le (u \triangleright [u;y \cdot v;z \cdot (u;w \cdot v \triangleleft x);(w \triangleright y \cdot x;z)]) \triangleleft z \tag{1.4}$$

Equation (1.4) is closely related to equation (1.3) and originates with Lyndon [9].

We now turn to the task of showing that the ultraproduct of the algebras $\mathfrak{A}_n$ for $3 \le n$ is representable. For any relation algebra $\mathfrak{A}$ and any $n > 3$, let $B_n\mathfrak{A}$ be the set of n-by-n matrices of atoms of $\mathfrak{A}$ that satisfy the following conditions:

(B_0) $m_{ii} \leq 1'$ for all $i < n$,

(B_1) $\breve{m}_{ij} = m_{ji}$ for all $i, j < n$,

(B_2) $m_{ij} \leq m_{ik} ; m_{kj}$ for all $i, j, k < n$.

The elements of $B_n\mathfrak{A}$ are called $(n-1)$-dimensional **simplices** (The geometric dimension of a simplex is $n-1$, one less than its size as a matrix.) For $i \leq n$, two simplices m and m' in $B_n\mathfrak{A}$ **agree up to** i (in symbols $m \equiv_{(i)} m'$) if the only differences between them are confined to their ith rows and ith columns, more precisely, $m_{kl} = m'_{kl}$ whenever $k, l < n$ and $k, l \neq i$. A subset M of $B_n\mathfrak{A}$ is called an n-**dimensional relational basis** for $\mathfrak{A}$ if

(R_0) for every atom a of $\mathfrak{A}$ there is a simplex $m \in M$ such that $m_{01} = a$, and

(R_1) if $m \in M$, $k \neq i, j < n$ and a, b are atoms of $\mathfrak{A}$ for which $m_{ij} \leq a;b$ then there is a simplex $m' \in M$ such that $m'_{ik} = a$, $m'_{kj} = b$ and $m \equiv_{(k)} m'$.

RA_n is defined to be the class of all relation algebras that are subalgebras of some complete and atomic relation algebra which has an n-dimensional relational basis. It is proved in [11] that $\mathsf{RA} = \mathsf{RA}_4 \supseteq \mathsf{RA}_5 \supseteq \mathsf{RA}_6 \supseteq \cdots$ is a decreasing sequence of varieties, and the intersection of all of them is the variety of representable relation algebras. (It is proved in [13] that the sequence is strictly decreasing.) The next result shows that, although the algebras $\mathfrak{A}_n$ of example 1.7 are nonrepresentable, they do become "more representable" for larger n.

THEOREM 1.10
For $3 \leq n$, $B_{n+1}\mathfrak{A}_n$ is an $(n+1)$-dimensional relational basis for $\mathfrak{A}_n$, hence $\mathfrak{A}_n \in \mathsf{RA}_{n+1}$.

PROOF. Condition (R_0) is easy to satisfy when $M = B_{n+1}\mathfrak{A}_n$: given an atom a, define m by $m_{ij} = a$ if $i = 0 < j \leq n$, $m_{ij} = \breve{a}$ if $j = 0 < i \leq n$, and $m_{ij} = 1'$ otherwise.

To verify condition (R_1), let $m \in B_{n+1}\mathfrak{A}_n$, fix $k \neq i, j \leq n$ and choose atoms a, b such that $m_{ij} \leq a;b$. We need to define m'_{lk} for $l \neq i, j, k$ (the remaining entries for m' follow from (B_0), (B_1) and $m \equiv_{(k)} m'$). If $a = 1'$ we take $m'_{lk} = m_{li}$, and if $b = 1'$ we take $m'_{lk} = m_{lj}$. The interesting case $a \neq 1' \neq b$ is handled as follows:

For $p, q \leq n$ with $p, q \neq k$, define $p \preceq q$ iff $m_{pq} \leq u_n$ $(= 1' + a_1 + \cdots + a_n)$, and define $p \approx q$ iff $p \preceq q$ and $q \preceq p$ (iff $m_{pq} \leq 1'$). Since u_n is a reflexive, transitive, linear element of $\mathfrak{A}_n$, $\preceq$ is a linear preorder and hence $\approx$ is an equivalence relation. To satisfy (B_2) we first assign the same color to m'_{pk}, m'_{qk} iff $p \approx q$. (This is possible since there are at most $n-2$ assignments to be made, and $n-2$ colors available, not counting the colors of a, b.)

We still have to assign orientations. If a, $\breve{b}$ are both up (down), let m'_{lk} be up (down), respectively. If a, b are both up, then $i \preceq j$ since $m_{ij} \leq a;b \leq u_n;u_n = u_n$. Let m'_{lk} be up whenever $l \preceq i$, and down otherwise. This inserts k as the $\preceq$-successor of i in the linear preorder $\preceq$. The case if a, b are both down is handled dually. In all cases m' is a simplex since $\preceq$ (defined on $\{0, 1, \ldots, n\}$) is a linear preorder and for any $p, q, r \leq n$ the atoms m'_{pq}, m'_{qr}, m'_{pr} are never all the same color. ∎

THEOREM 1.11
If E is a nonprincipal ultrafilter on the index set $I = \{3, 4, 5, \ldots\}$, then

- the ultraproduct of the relation algebras $\mathfrak{A}_n$ by E is a representable relation algebra, and

- the ultraproduct of the sequential algebras $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$ by E is a antisymmetric dense locally linear representable sequential algebra.

PROOF. Let $\sim_E$ be the equivalence relation on the direct product algebra $\mathfrak{B} = \prod_{n\in I}\mathfrak{A}_n$ defined for arbitrary $a, b \in B$ by $a \sim_E b$ iff $\{i \in I : a_i = b_i\} \in E$. Then $\sim_E$ is a congruence relation on the algebra $\mathfrak{B}$, and the quotient algebra $\mathfrak{C} = \mathfrak{B}/{\sim_E}$ is the ultraproduct of the relation algebras $\mathfrak{A}_n$. Let $\tilde{\mathfrak{B}}$ be the product of the sequential algebras $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$ for $n \in I$, and let $\approx_E$ be the restriction of $\sim_E$ to $\tilde{\mathfrak{B}} \subseteq \mathfrak{B}$. Then $\approx_E$ is the E-ultrafilter congruence on $\tilde{\mathfrak{B}}$. Define

$$u_\omega = \langle u_n : n \in I\rangle /{\sim_E} = \{a \in \mathfrak{B} : a \sim_E \langle u_n : n \in I\rangle\}.$$

It is straightforward to check that the ultraproduct $\tilde{\mathfrak{C}} = \tilde{\mathfrak{B}}/{\approx_E}$ is isomorphic to $\mathfrak{Seq}_{u_\omega}(\mathfrak{C})$ via the map $a/{\approx_E} \mapsto a/{\sim_E}$.

Consider an arbitrary $k \geq 3$. If $n \geq k-1$ then $\mathsf{RA}_{n+1} \subseteq \mathsf{RA}_k$, and $\mathfrak{A}_n \in \mathsf{RA}_{n+1}$ for every $n \geq 3$, so $\{n : \mathfrak{A}_n \in \mathsf{RA}_k\} \supseteq \{k-1, k, k+1, k+2, \ldots\}$. E is nonprincipal, hence $\{n : \mathfrak{A}_n \in \mathsf{RA}_k\} \in E$. Every RA_k is an elementary class by [11], so $\mathfrak{C} \in \mathsf{RA}_k$. This shows that $\mathfrak{C} \in \bigcap_{3\leq k}\mathsf{RA}_k$. But $\bigcap_{3\leq k}\mathsf{RA}_k$ is the variety of representable relation algebras, so $\mathfrak{C}$ is representable. $\tilde{\mathfrak{C}}$ is also simple because simplicity has a first-order characterization for relation algebras and each $\mathfrak{A}_n$ is simple. It follows that $\mathfrak{C}$ is isomorphic to a subalgebra of an algebra $\mathfrak{Re}(X)$ for some set X. Now $\tilde{\mathfrak{C}}$ is a antisymmetric dense locally linear representable sequential algebra since it is isomorphic to a relativization of $\mathfrak{C}$ by the reflexive transitive antisymmetric dense linear element u_ω. ∎

COROLLARY 1.12
The variety of antisymmetric dense locally linear representable sequential algebras is not finitely axiomatizable.

PROOF. This follows from the existence of antisymmetric dense locally linear representable ultraproducts of the nonrepresentable antisymmetric dense locally linear sequential algebras $\mathfrak{Seq}_{u_n}(\mathfrak{A}_n)$. The previous theorem says there are many. ∎

Of course, theorem 1.11 also shows that the variety of representable relation algebras is not finitely axiomatizable. This has already been proved and generalized in several ways, by Monk [14], Andréka [1], [2], Jónsson [6], and Maddux [12]. The construction in example 1.7 is simpler than previous ones in certain respects, including the sizes of the algebras and the proofs of their properties. The corollary was first proved by Andréka and von Karger [16] for the concept of representability via linear categories (*i.e.*, antisymmetric locally linear observation spaces), using sequential algebras $\mathfrak{A}_n$ with $3n! + n + 1$ atoms instead of $n + 1$.

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