TOTAL TENSE ALGEBRAS AND SYMMETRIC SEMIASSOCIATIVE RELATION ALGEBRAS

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Abstract. *It is well known that the lattice Λ_{RA} of varieties of relation algebras has exactly three atoms. An unsolved problem, posed by B. Jónsson, is to determine the varieties of height two in Λ_{RA}.*

This paper solves the corresponding question for varieties generated by total tense algebras. More specifically, we show that there are exactly four finitely generated varieties and infinitely many nonfinitely generated varieties of height two. In the second half of the paper we show that total tense algebras are term equivalent to certain generalized relation algebras and extend our results to varieties of these algebras.

Introduction and definitions. Tense algebras are the algebraic counterpart of tense logic, which has been studied and used by modal logicians to reason about tenses in a language and about temporal processes. We first describe a typical tense algebra.

Let U be a set and R a binary relation on U. The relation R gives rise to two unary operations f and g on the set of all subsets of U, namely

$$\begin{aligned} f(X) &= \text{the image of } X \text{ under } R \\ &= \{u \in U : xRu \ \text{ for some } x \in X\} \text{ and} \end{aligned}$$

$$\begin{aligned} g(X) &= \text{the preimage of } X \text{ under } R \\ &= \{u \in U : uRx \ \text{ for some } x \in X\}. \end{aligned}$$

If U is interpreted as a set of events, and R is interpreted as a temporal relation between events in U, then $f(X)$ is the set of events in the future of some event in X, and $g(X)$ is the set of events in the past of some event in X. An example of a tense algebra can be obtained by adding the operations f and g to the Boolean algebra of all subsets of U, namely, the complex algebra

$$\mathbf{Cm}(U,R) = (\mathrm{Sb}(U), \cup, ^{-}, f, g).$$

It follows from a general result of Jónsson and Tarski [**51**] that every tense algebra can be embedded in one that arises in this concrete fashion.

The abstract definition of a tense algebra runs as follows. A tense algebra is an algebra of the form $\mathbf{A} = (A, +, ^{-}, f, g)$, where

(i) $(A, +, ^{-})$ is a Boolean algebra and

(ii) f and g are *conjugate operations*, i.e., for all $x, y \in A$

$$y \cdot f(x) = 0 \text{ iff } x \cdot g(y) = 0.$$

Let **A** be a tense algebra. Define $\cdot$ by $x \cdot y = \overline{\overline{x} + \overline{y}}$ for all x and y. By (i) we may define the distinguished elements 1 and 0 by $1 = x + \overline{x}$ and $0 = \overline{1}$.

Property (ii) is quite strong since it implies that f and g are *additive* ($f(x+y) = f(x) + f(y)$ and $g(x+y) = g(x) + g(y)$), *normal* ($f(0) = 0 = g(0)$), and *monotone* (if $x \leq y$ then $f(x) \leq f(y)$ and $g(x) \leq g(y)$). By Huntington [**33**], [**33a**] and Jónsson-Tarski [**51**], Theorem 1.15, both (i) and (ii) can be expressed by the equations listed below, so the class of all tense algebras is a variety.

$$
\begin{aligned}
x + (y + z) &= (x + y) + z \\
x + y &= y + x \\
x &= \overline{\overline{x} + \overline{y}} + \overline{\overline{x} + y} \\
f(0) &= 0 \\
g(0) &= 0 \\
f(x) \cdot y &\leq f(x \cdot g(y)) \\
g(y) \cdot x &\leq g(y \cdot f(x))
\end{aligned}
$$

Furthermore, the last four equations can be replaced by the following two:

$$
\begin{aligned}
f(x \cdot \overline{g(y)}) &\leq f(x) \cdot \overline{y} \\
g(y \cdot \overline{f(x)}) &\leq g(y) \cdot \overline{x}
\end{aligned}
$$

If **A** is complete and atomic then **A** is isomorphic to the complex algebra of its *atom structure* $\mathbf{At}(\mathbf{A}) = (U, R)$, where U is the set of atoms of **A** and $R = \{(u,v) \in U \times U : f(u) \geq v\}$. Finite tense algebras, in particular, are always isomorphic to the complex algebras of their atom structures.

A tense algebra is called *reflexive* if it satisfies the identity $x \leq f(x)$. An equivalent identity is $x \leq g(x)$. To see this, assume $x \leq f(x)$ for every x. Then $x \cdot \overline{g(x)} \leq f(x \cdot \overline{g(x)})$, so $x \cdot \overline{g(x)} = x \cdot \overline{g(x)} \cdot f(x \cdot \overline{g(x)})$. But $x \cdot \overline{g(x)} \cdot f(x \cdot \overline{g(x)}) \leq x \cdot f(x \cdot \overline{g(x)}) \leq x \cdot f(x) \cdot \overline{x} = 0$, hence $x \cdot \overline{g(x)} = 0$, i.e., $x \leq g(x)$. Similarly, if **A** satisfies $x \leq g(x)$, then it also satisfies $x \leq f(x)$. A tense algebra is called *total* if it satisfies the implication $x \neq 0 \Rightarrow f(x) + g(x) = 1$. Of course a binary relation R on a set U is reflexive if xRx for every $x \in U$, and total if xRy or yRx for all $x, y \in U$. Not surprisingly such relations give rise to correspondingly named tense algebras, that is, R is reflexive on U if and only if $\mathbf{Cm}(U, R)$ is a reflexive tense algebra, and R is total on U if and only if $\mathbf{Cm}(U, R)$ is a total tense algebra.

Every total tense algebra is a discriminator algebra, and hence is simple. The variety generated by all total tense algebras is finitely based. It is not hard to show that a basis is given by the equations above together with $x + f(f(x) + g(x)) + g(f(x) + g(x)) \leq f(x) + g(x)$. See Jipsen [**93**] for details.

Finally we note that if **A** is the complex algebra obtained from a structure (U, R), then every complete subalgebra of **A** corresponds to a partition of the set U with the property that the image of each block of the partition under f and g is a union of blocks. The blocks are the atoms of the subalgebra.

Total tense varieties of height 2. The lattice Λ_{TA} of all varieties of tense algebras has infinitely many atoms, but only one of these atoms is a variety of reflexive tense algebras, namely the variety generated by the tense algebra $\mathbf{T}_0 = \mathbf{Cm}(\{x\}, \{(x,x)\})$. This follows from the observation that every nontrivial reflexive tense algebra has a smallest subalgebra isomorphic to $\mathbf{T}_0$. We refer to this subalgebra as the *constant subalgebra*, since it contains only the constants 0 and 1.

The variety $\mathrm{Var}(\mathbf{T}_0)$ is in turn covered by infinitely many varieties of reflexive tense algebras. However we are interested in the varieties generated by total tense algebras, since they are later

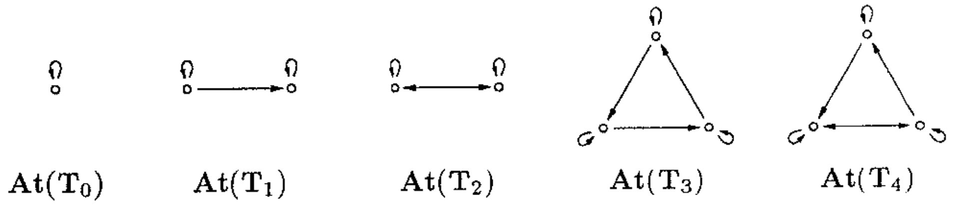


Figure 1. Necessary subalgebras of finite total tense algebras.

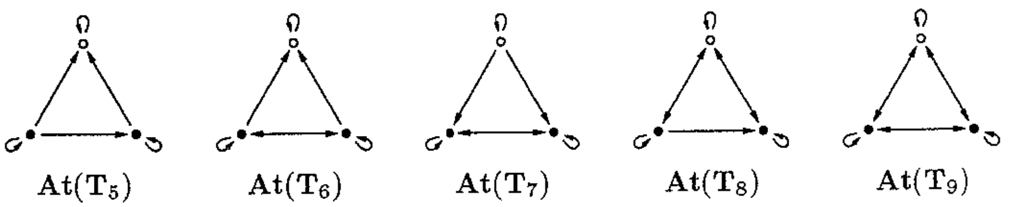


Figure 2. Remaining total tense algebras with 8 elements.

shown to correspond to varieties generated by simple subadditive semiassociative relation algebras. We now prove that only four of the varieties covering Var($\mathbf{T}_0$) are *finitely generated*, i. e. generated by finite total tense algebras. The atom structures of these four finite tense algebras are shown in Figure 1.

THEOREM 1 *If R is a total relation on a set U and $|U| \geq 4$, then* $\mathbf{Cm}(U, R)$ *has a proper nonconstant subalgebra.*

PROOF. For any $u, v \in U$, we say that "u points at v" just in case $(u, v) \in R$. The hypothesis that R is total can then be expressed by saying that for any $u, v \in U$, either u points at v or v points at u. Since this holds even when $u = v$, it follows that R is reflexive. Hence, for every $X \subseteq U$,

$$f(X) \supseteq X \text{ and } g(X) \supseteq X. \tag{1}$$

Suppose $u \in U$ and $u \notin f(X)$. Then no element of X points at u, equivalently, $g(\{u\}) \cap X = \emptyset$. Since R is total, u points at every element of X, hence $f(\{u\}) \supseteq X$. Summarizing, we have

$$\text{if } u \notin f(X) \text{ then } g(\{u\}) \cap X = \emptyset \text{ and } f(\{u\}) \supseteq X. \tag{2}$$

By similar reasoning we also have

$$\text{if } u \notin g(X) \text{ then } f(\{u\}) \cap X = \emptyset \text{ and } g(\{u\}) \supseteq X. \tag{3}$$

Choose any $X \subseteq U$ such that $|X| \geq 2$ and $X \neq U$. Consider the partition

$$X \text{ and } \{u\} \text{ for every } u \in U - X. \tag{4}$$

By the hypotheses on X and U, there are at least two sets in (4), and not all of them are singletons. Consequently, if the sets in (4) form the atoms of a subalgebra of $\mathbf{Cm}(U, R)$ then that subalgebra is both nonconstant and proper, hence we are done, so assume otherwise. Since $f(X)$ and $g(X)$ both contain X, by (1), they are clearly unions of sets in (4), so there must be some $u \in U - X$ such that either $f(\{u\})$ or $g(\{u\})$ is not a union of sets in (4). Since X is the only nonsingleton in (4), it follows that either X lies partly inside $f(\{u\})$ and partly outside $f(\{u\})$, or else X lies partly inside $g(\{u\})$ and partly outside $g(\{u\})$. In the former case, the second conclusion of (2) is false and the first conclusion of (3) is false, while in the latter case, the first conclusion of (2) is false and the second conclusion of (3) is false. In either case the hypotheses of (2) and (3) are false. Thus $u \in f(X) \cap g(X)$. Hence u points at some element $x \in X$, and some $y \in X$ points at u. We argue next that we may assume x and y are distinct. Suppose not. Since $|X| \geq 2$ we may choose $w \in X$ such that $w \neq x = y$. If w points at u, use w instead of y. If u points at w, use w instead of x. This shows that we may therefore assume

$$\text{if } X \subseteq U,\ |X| \geq 2, \text{ and } X \neq U, \text{ then there exist } u \in U - X \text{ and distinct } x, y \in X \text{ such that } u \text{ points at } x \text{ and } y \text{ points at } u. \quad (5)$$

Let $X, Y \subseteq U$. The pair (X, Y) will be called an $\mathbf{T}_2$-*pair* if the following conditions hold: $X \cap Y = \emptyset$, $f(X) \cap f(Y) \cap g(X) \cap g(Y) \supseteq X \cup Y$, and $|X \cup Y| \geq 3$. If (X, Y) is a $\mathbf{T}_2$-pair, then $X \neq \emptyset$ and $Y \neq \emptyset$, since otherwise $\emptyset \supseteq X \cup Y \neq \emptyset$ by the second condition, and either X or Y contains at least two elements by the third condition. The reason for the name is that if (X, Y) is a $\mathbf{T}_2$-pair then $\mathbf{Cm}(X \cup Y, R \cap ((X \cup Y) \times (X \cup Y)))$ is isomorphic to $\mathbf{T}_2$.

The collection of $\mathbf{T}_2$-pairs is closed under unions of chains, in the following sense. Let α be an ordinal and suppose (X_κ, Y_κ) is a $\mathbf{T}_2$-pair for every $\kappa < \alpha$. If $X_\kappa \subseteq X_\lambda$ and $Y_\kappa \subseteq Y_\lambda$ whenever $\kappa < \lambda < \alpha$, then $(\bigcup_{\kappa<\alpha} X_\kappa, \bigcup_{\kappa<\alpha} Y_\kappa,)$ is also a $\mathbf{T}_2$-pair.

Suppose that there is a $\mathbf{T}_2$-pair in (U, R). By Zorn's Lemma there must be some maximal $\mathbf{T}_2$-pair (X, Y). If $X \cup Y = U$ then $\mathbf{Cm}(U, R)$ has a nonconstant proper subalgebra isomorphic to $\mathbf{T}_2$, as desired. On the other hand, if $X \cup Y \neq U$, then we claim that $\mathbf{Cm}(U, R)$ has a nonconstant proper subalgebra whose atoms are

$$X,\ Y, \text{ and } \{u\} \text{ for } u \in U - X - Y. \quad (6)$$

To prove this we need only check that the f-image and g-image of every set in (6) is a union of sets in (6). Since (X, Y) is a $\mathbf{T}_2$-pair, $f(X)$ contains both X and Y, but all others sets in (6) are singletons, so $f(X)$ is clearly a union of sets listed in (6). Similarly, $f(Y)$, $g(X)$, and $g(Y)$ are unions of sets in (6). Let $u \in U - X - Y$. It suffices to show that $f(\{u\})$ either contains or is disjoint from X, and either contains or is disjoint from Y. Since R is total and $X \neq \emptyset$, either $u \in f(X)$ or $u \in g(X)$. However, if $u \in f(X) \cap g(X)$, then $(X, Y \cup \{u\})$ is a $\mathbf{T}_2$-pair strictly larger than (X, Y), contradicting the maximality of (X, Y). Therefore either $u \in f(X) - g(X)$ or $u \in g(X) - f(X)$. If $u \in f(X) - g(X)$ then $f(\{u\}) \cap X = \emptyset$ and $g(\{u\}) \supseteq X$ by (3), while if $u \in g(X) - f(X)$, then $g(\{u\}) \cap X = \emptyset$ and $f(\{u\}) \supseteq X$ by (2). In any case both $f(\{u\})$ and $g(\{u\})$ either contain or are disjoint from X. By similar reasoning we also conclude that $f(\{u\})$ and $g(\{u\})$ either contain or are disjoint from Y.

So far we have shown that if there is a $\mathbf{T}_2$-pair in (U, R), then $\mathbf{Cm}(U, R)$ has a nonconstant proper subalgebra. We may therefore assume that

$$\text{there is no } \mathbf{T}_2\text{-pair in } (U, R). \quad (7)$$

This means, in particular, that there can be no "4-cycle" in R, that is, no four distinct elements $u_1, u_2, u_3, u_4 \in U$ such that $(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1) \in R$, for then $(\{u_1, u_3\}, \{u_2, u_4\})$ would be a $\mathbf{T}_2$-pair, contradicting (7).

Suppose there is a "3-cycle" in R, that is, three distinct elements $u_1, u_2, u_3 \in U$ such that u_1 points at u_2, u_2 points at u_3, and u_3 points at u_1. Let $X = \{u_1, u_2, u_3\}$. By (5) there exist $u \in U - X$ and distinct $x, y \in X$, such that u points at x and y points at u. Let z be the third element of X, distinct from x and y. Suppose y points at x. X forms a 3-cycle under R, so x points at z and z points at y (Figure 3(i)). It follows that there is a 4-cycle, from u to x to z to y to u. The existence of a 4-cycle contradicts (7), so we may assume x points at y. This implies that the 3-cycle in X goes from x to y to z to x. There are two final cases. If u points at z (Figure 3(ii)), then there is a 4-cycle from u to z to x to y to u, contradicting (7), but if z points at u (Figure 3(iii)), then there is a 4-cycle from z to u to x to y to z.

We may therefore assume that R contains no 3-cycle. Now suppose R contains a 2-cycle, that is, two distinct elements $u_1, u_2 \in U$ such that u_1 points at u_2 and u_2 points at u_1. By (5) there is some $u \in U - \{u_1, u_2\}$ which points at one of the elements of $\{u_1, u_2\}$ and is pointed at by the other. Either way we get a 3-cycle. We may consequently assume R has no 2-cycles, that is, R is antisymmetric.

So far we may assume R is reflexive, total, and antisymmetric. We may also assume R is transitive, since the only way it can fail to be transitive is to contain a 3-cycle. Hence R is a linear ordering of U. Choose any $u \in U$ which is not a minimum element under this ordering (where $(u, v) \in R$ is interpreted as "u is less than or equal to v"). Then $f(\{u\}) \neq U$ and it is easily checked that the sets $f(\{u\})$ and $U - f(\{u\})$ form the atoms of a nonconstant proper subalgebra of $\mathbf{Cm}(U, R)$ isomorphic to $\mathbf{T}_1$. □

COROLLARY 2 *In the lattice of varieties generated by total tense algebras,* $\mathrm{Var}(\mathbf{T}_i)$ ($i = 1, 2, 3, 4$) *are the only finitely generated covers of* $\mathrm{Var}(\mathbf{T}_0)$.

PROOF. Note that if $\mathbf{A}$ and $\mathbf{B}$ are two finite subdirectly irreducible algebras in a congruence distributive variety then it follows from Jónsson's Lemma that $\mathrm{Var}(\mathbf{A})$ is covered by $\mathrm{Var}(\mathbf{B})$ if and only if $\mathbf{A}$ is isomorphic to a maximal proper subalgebra of $\mathbf{B}$.

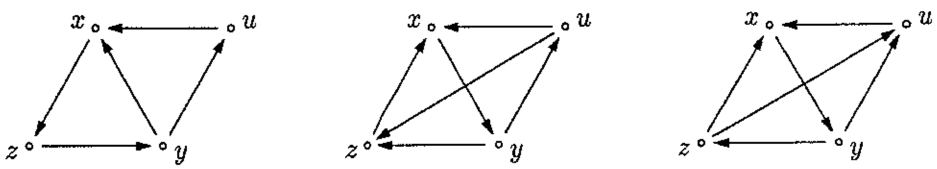
As mentioned in the introduction, a finite total tense algebra is isomorphic to the complex algebra of its atom structure, so the previous theorem implies that every finite total tense algebra with more than 8 elements has a proper nonconstant subalgebra (with at most 8 elements), and hence cannot generate a variety that covers $\mathrm{Var}(\mathbf{T}_0)$. Figures 1 and 2 show the atom structures of the ten nonisomorphic total tense algebras with at most 8 elements. Note that $\mathbf{T}_5$, $\mathbf{T}_6$ and $\mathbf{T}_7$ have a subalgebra isomorphic to $\mathbf{T}_1$, whereas $\mathbf{T}_8$, $\mathbf{T}_9$ have a subalgebra isomorpic to $\mathbf{T}_2$ (obtained by identifying the black vertices in the atom structures). On the other hand $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ have no proper subalgebras other than the constant one, hence they generate covers of $\mathrm{Var}(\mathbf{T}_0)$. □

A sequence of infinite total tense algebras. In this section we give countably many examples of nonfinitely generated varieties of total tense algebras that cover $\mathrm{Var}(\mathbf{T}_0)$. These examples are based on the veiled recession frame, used by W. Blok [80] to give similar examples of varieties of modal algebras.

A structure (U, R) is called an *n-recession frame* if $U = \{a_1, \ldots, a_n\}$ and

$$R = \{(a_i, a_j) : i \geq j\} \cup \{(a_i, a_{i+1}) : 1 \leq i < n\}.$$

If we instead consider sets $U = \{a_1, a_2, a_3, \ldots\}$ or $U = \{\ldots, a_{-1}, a_0, a_1, a_2, \ldots\}$ with the natural extension of R then we obtain the *ω-recession frame* and the *ℤ-recession frame* respectively. A *veiled* recession frame is the subalgebra of $\mathbf{Cm}(U, R)$ generated by the singletons (or finite subsets)



(i) (ii) (iii)

Figure 3.

of Sb(U). This subalgebra is denoted by $\mathbf{Cm}_f(U, R)$. A diagram of the $\mathbb{Z}$-recession frame is given in Figure 4(i). The downward pointing bold arrow represents the linear order part of R, and is intended to indicate that, in addition to the arrows shown, there are also arrows from every element to all elements below it (as well as to itself). Observe that $\mathbf{Cm}(U, R)$ is a total tense algebra.

We now define a sequence of structures (U_n, R_n) that are essentially $\mathbb{Z}$ copies of the n-recession frame combined like a $\mathbb{Z}$-recession frame (Figure 4(ii)). Let $U_n = \{a_{i,j} : i \in \mathbb{Z}, 1 \leq j \leq n\}$ and define

$$\begin{aligned} R_n = &\{(a_{i,j}, a_{k,l}) : i > k \text{ or } (i = k \text{ and } j \geq l)\} \\ &\cup \{(a_{i,1}, a_{i+1,j}) : i \in \mathbb{Z}, 1 \leq j \leq n\} \\ &\cup \{(a_{i,j}, a_{i,j+1}) : i \in \mathbb{Z}, 1 \leq j < n\}. \end{aligned}$$

For $n = 1$ the structure is (isomorphic to) the $\mathbb{Z}$-recession frame, and it is easy to check that, for fixed n and i, the set $V_i = \{a_{i,j} : 1 \leq j \leq n\}$ together with the relation $R_n \cap (V_i \times V_i)$ forms an n-recession frame whose complex algebra is generated by $\{a_{i,1}\}$. The algebras $\mathbf{B}_n$ are defined as $\mathbf{Cm}_f(U_n, R_n)$.

THEOREM 3 *For each* $n \in \omega$ *the varieties* $\mathrm{Var}(\mathbf{B}_n)$ *are distinct and cover* $\mathrm{Var}(\mathbf{T}_0)$.

PROOF. We first show that $\mathbf{B}_n$ has no nonconstant subalgebras. Note that an element of B_n is either a finite subset of U_n or a union of a finite subset together with $\bigcup_{k=i}^{\omega} V_i$ for some $k \in \omega$ or a complement of either such sets (since this collection of sets is a subalgebra of $\mathbf{Cm}(U_n, R_n)$). Let $x \in B_n$ and suppose $0 < x < 1$. Replacing x by its complement if necessary, we may assume that $f(x) < 1$. From the definition of R_n it follows that $\underline{f^k(x) < 1}$, and it is easy to see that $f^3(x) \cdot \overline{f^2(x)}$ is one of the V_k for some k. Consequently $f^{3+i}(x) \cdot \overline{f^{2+i}(x)} = V_{k+i}$ for each $i \in \omega$. To generate the sets V_{k-i}, we choose the smallest m such that $V_{k-i} \subseteq g^m(V_k)$, define $y = \overline{g^m(V_k)}$, then $f^2(y) \cdot \overline{(f(y)} = V_{k-i}$. Finally $g^2(V_i) \cdot \overline{g(V_i)} = \{a_{i-2,1}\}$, and now we can use the operations f and g relativized to V_{i-2} (i.e. $\hat{f}(x) = f(x) \cdot V_{i-2}$ and $\hat{g}(x) = g(x) \cdot V_{i-2}$) to generate each singleton subset of V_{i-2}. So we conclude that $\mathbf{B}_n$ is generated by each element x different from $\emptyset$ or U_n.

This however does not yet imply that $\mathrm{Var}(\mathbf{B}_n)$ covers $\mathrm{Var}(\mathbf{T}_0)$. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathrm{Var}(\mathbf{B}_n)$ that is not isomorphic to $\mathbf{T}_0$. We need to show that $\mathrm{Var}(\mathbf{B}_n) \subseteq \mathrm{Var}(\mathbf{A})$. Since tense algebras are congruence distributive we have $\mathbf{A} \in \mathbb{HSP}_U(\mathbf{B}_n)$, and since B_n is a discriminator algebra ($x \neq 0$ implies $f(x) + g(x) = 1$), we conclude that all members of $\mathbb{SP}_U(\mathbf{B}_n)$ are simple and that (an isomorphic copy of) $\mathbf{A}$ is among them. So we may assume that $\mathbf{A}$ is a subalgebra of $\mathbf{B}_n^I/F$ for some index set I and some nonprincipal ultrafilter F over I. Consider an

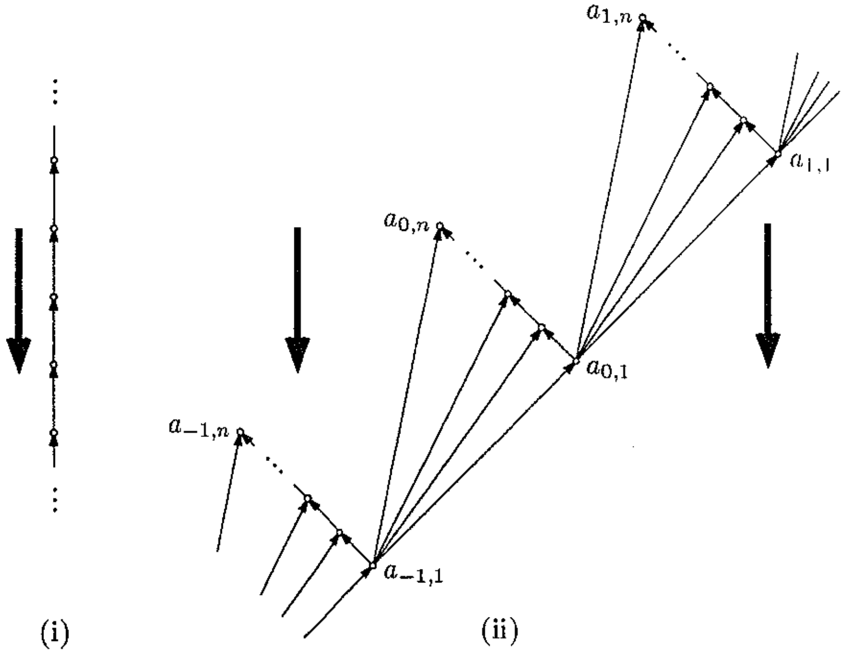


Figure 4. Infinite total tense algebras with no proper nonconstant subalgebras.

element $x \in A$ such that $0 < x < 1$. Again replacing x by its complement if necessary, we assume that $f(x) < 1$. Let $J = \{j \in I : f(x_j) < 1\}$. Then $J \in F$ and $f(x_j)$ generates all of B_n (in the jth coordinate) uniformly for each $j \in J$. But this implies that $f(x)$ generates a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}_n$, hence $\mathrm{Var}(\mathbf{B}_n) \subseteq \mathrm{Var}(\mathbf{A})$.

Since total tense algebras are discriminator algebras, the varieties $\mathrm{Var}(\mathbf{B}_n)$ will be distinct if we find a universal sentence ϕ_n that holds in $\mathbf{B}_n$ and fails in $\mathbf{B}_m$ for $m < n$. Let $a = f^3(x) \cdot \overline{f^2(x)}$, $b = f^4(x) \cdot \overline{f^3(x)}$ and $c = g(b) \cdot a = \hat{g}(b)$. Then the sentence

$$\phi_n : \quad 0 < f(x) < 1 \Rightarrow a = \hat{g}^n(c)$$

has the required property. □

Application to symmetric semiassociative relation algebras. We now show that tense algebras are term-equivalent to reducts of certain semiassociative relation algebras. This allows us to extend the previous results to varieties of semiassociative relation algebras.

A *nonassociative relation algebra* is of the form $\mathbf{A} = (A, +, ^-, ;, \breve{}, 1')$, where $(A, +, ^-)$ is a Boolean algebra, $\breve{}$ is a unary operation, 1' is an identity element with respect to the binary operation ; and for all $x, y, z \in A$

$$x \cdot y;z = 0 \quad \text{iff} \quad y \cdot x;\breve{z} = 0 \quad \text{iff} \quad z \cdot \breve{y};x = 0. \tag{8}$$

(Throughout we use the convention that ; has precedence over $\cdot$ which in turn takes precedence over $+$). The above equivalence can be expressed by equations, so the class of all nonassociative relation algebras is a variety (Maddux [**82**], Corollary 1.5, or [**91**], Theorem 2). A nonassociative relation algebra is *symmetric* if it satisfies the equation $\breve{x} = x$. Every nonassociative relation algebra satisfies $(x;y)\breve{} = \breve{y};\breve{x}$ (Maddux [82], Theorem1.13(13)), so every symmetric nonassociative relation algebra is *commutative*, that is, the operation ; is commutative. A nonassociative relation algebra is *reflexive* if it satisfies $x \leq x;x$.

If we omit the requirement of an identity element for a symmetric nonassociative relation algebra, then we obtain reducts $(A, +, ^{-}, ;)$ which we will refer to as *sr-algebras* (symmetric residuated algebras). Thus $(A, +, ^{-}, ;)$ is a sr-algebra just in case $(A, +, ^{-})$ is a Boolean algebra and for all $x, y, z \in A$

$$x \cdot y;z = 0 \quad \text{iff} \quad y \cdot x;z = 0 \quad \text{iff} \quad z \cdot y;x = 0. \tag{9}$$

Sr-algebras are also commutative. Listed below are some other properties of nonassociative relation algebras and sr-algebras that we will use later. The first two equations assert that ; is normal, the next two that it distributes over +, and the last two properties assert that ; is monotone in each variable.

$$\begin{aligned} x;0 = 0;x &= 0 \\ x;(y+z) &= x;y + x;z \\ (x+y);z &= x;z + y;z \\ \text{if } x \leq y &\text{ then } x;z \leq y;z \\ \text{if } x \leq y &\text{ then } z;x \leq z;y \end{aligned}$$

The following two results establish a term-equivalence between the variety of reflexive tense algebras and a subvariety of sr-algebras.

THEOREM 4 *Let* $\mathbf{A} = (A, +, ^{-}, f, g)$ *be a tense algebra and define a binary operation* ; *on* A *by*

$$x;y = f(x \cdot y) + x \cdot g(y) + y \cdot g(x).$$

Then $(A, +, ^{-}, ;)$ *is a sr-algebra that satisfies the equation* $x;(\overline{x} \cdot y) \leq x + y$*. If* $\mathbf{A}$ *is reflexive then* $(A, +, ^{-}, ;)$ *is reflexive, and the operations* f *and* g *can be recovered from* ; *by the term functions* $f(x) = x;x$ *and* $g(x) = x + x;\overline{x}$*.*

PROOF. Note that $z \cdot x;y = 0$ if and only if $z \cdot f(x \cdot y) + z \cdot x \cdot g(y) + z \cdot y \cdot g(x) = 0$, which is equivalent to

$$x \cdot y \cdot g(z) = 0 \quad \text{and} \quad z \cdot x \cdot g(y) = 0 \quad \text{and} \quad z \cdot y \cdot g(x) = 0.$$

This last statement is unchanged by permuting the variables x, y, and z, so the statements $z \cdot x;y = 0$, $x \cdot z;y = 0$ and $y \cdot x;z = 0$ are equivalent. To check that $(A, +, ^{-}, ;)$ satisfies the equation $x;(\overline{x} \cdot y) \leq x + y$, we compute

$$x;(\overline{x} \cdot y) = f(x \cdot \overline{x} \cdot y) + x \cdot g(\overline{x} \cdot y) + \overline{x} \cdot y \cdot g(x) \leq 0 + x + \overline{x} \cdot y = x + y.$$

Assume that $\mathbf{A}$ is reflexive. Then $x \leq f(x)$ and $x \leq g(x)$, so

$$\begin{aligned} x;x &= f(x \cdot x) + x \cdot g(x) + x \cdot g(x) = f(x) + x \cdot g(x) = f(x) + x = f(x). \\ x + x;\overline{x} &= x + f(x \cdot \overline{x}) + x \cdot g(\overline{x}) + \overline{x} \cdot g(x) \\ &= x + f(0) + \overline{x} \cdot g(x) \\ &= x \cdot g(x) + \overline{x} \cdot g(x) \\ &= g(x), \end{aligned}$$

and $(A, +, ^{-}, ;)$ is reflexive since $x \leq f(x) = x;x$. □

Since the equation $x;(\overline{x}\cdot y)\leq x+y$ is satisfied by all sr-algebras obtained in this way, we can only expect a converse for such algebras. A sr-algebra that satisfies this equation is called *subadditive* because for disjoint x and y the relative product $x;y$ is below $x+y$. (Indeed, from $x\cdot y=0$ and subadditivity we get $x;y=x;(\overline{x}\cdot y)+x;(x\cdot y)=x;(\overline{x}\cdot y)+x;0=x;(\overline{x}\cdot y)\leq x+y$.)

THEOREM 5 *Let* $\mathbf{A}=(A,+,^{-},;)$ *be a subadditive sr-algebra and define*

$$f(x)=x+x;x \qquad \text{and} \qquad g(x)=x+x;\overline{x}.$$

Then $(A,+,^{-},f,g)$ *is a reflexive tense algebra. If* $\mathbf{A}$ *is reflexive then* ; *can be recovered from* f *and* g *by* $x;y=f(x\cdot y)+x\cdot g(y)+y\cdot g(x)$.

PROOF. We show that $y\cdot f(x)=0$ iff $x\cdot g(y)=0$. It follows that $(A,+,^{-},f,g)$ is a tense algebra. Furthermore, $(A,+,^{-},f,g)$ is reflexive by the definitions of f and g. Suppose $y\cdot f(x)=y\cdot(x+x;x)=0$. Then $y\cdot x=0$ and $y\cdot x;x=0$, hence, by distributivity, monotonicity, and subadditivity,

$$y\cdot x;\overline{y}=y\cdot x;(x\cdot\overline{y}+\overline{x}\cdot\overline{y})=y\cdot x;(x\cdot\overline{y})+y\cdot x;(\overline{x}\cdot\overline{y})\leq y\cdot x;x+y\cdot(x+\overline{y})=0.$$

This shows that $x\cdot y;\overline{y}=0$ by (9) and therefore $x\cdot g(y)=x\cdot(y+y;\overline{y})=0$. For the converse, suppose $x\cdot g(y)=0$. Then $x\cdot y=0$ and $x\cdot y;\overline{y}=0$. The latter equation implies $x;y=y;x\leq y$ by commutativity and (9). It follows that $x\cdot x;y\leq x\cdot y=0$, hence also $y\cdot x;x=0$. Consequently, $y\cdot f(x)=y\cdot(x+x;x)=0$. So far we know that $(A,+,^{-},f,g)$ is a reflexive tense algebra.

Next we show that

$$f(x\cdot y)+x\cdot g(y)+y\cdot g(x)=x;y+x\cdot y. \tag{10}$$

First we expand the left side of (10).

$$\begin{aligned} f(x\cdot y)+x\cdot g(y)+y\cdot g(x)&=x\cdot y+(x\cdot y);(x\cdot y)+x\cdot(y+y;\overline{y})+y\cdot(x+x;\overline{x})\\ &=x\cdot y+(x\cdot y);(x\cdot y)+x\cdot y;\overline{y}+y\cdot x;\overline{x}.\end{aligned} \tag{11}$$

Next, by distributivity, monotonicity, and subadditivity,

$$x\cdot y;\overline{y}=x\cdot y;(\overline{y}\cdot x)+x\cdot y;(\overline{y}\cdot\overline{x})\leq y;x+x\cdot(y+\overline{x})=x;y+x\cdot y.$$

and, similarly,

$$y\cdot x;\overline{x}\leq x;y+x\cdot y,$$

so

$$x\cdot y;\overline{y}+y\cdot x;\overline{x}\leq x;y+x\cdot y.$$

Combining the previous equation with (11) and the observation that $(x\cdot y);(x\cdot y)\leq x;y$, we get half of (10), namely,

$$f(x\cdot y)+x\cdot g(y)+y\cdot g(x)\leq x;y+x\cdot y.$$

For the inclusion in the other direction, we first expand $x;y$.

$$\begin{aligned} x;y&=(x\cdot y+x\cdot\overline{y});(x\cdot y+\overline{x}\cdot y)\\ &=(x\cdot y);(x\cdot y)+(x\cdot\overline{y});(x\cdot y)+(x\cdot y);(\overline{x}\cdot y)+(x\cdot\overline{y});(\overline{x}\cdot y).\end{aligned}$$

Next we show the last three terms in the expansion of $x;y$ are included in $x \cdot y;\overline{y} + y \cdot x;\overline{x}$. From monotonicity we have

$$(x \cdot \overline{y});(x \cdot y) \leq \overline{y};y$$

and, by subadditivity,

$$(x \cdot \overline{y});(x \cdot y) \leq x \cdot \overline{y} + x \cdot y = x$$

so

$$(x \cdot \overline{y});(x \cdot y) \leq x \cdot \overline{y};y.$$

In a similar way we also get

$$(x \cdot y);(\overline{x} \cdot y) \leq y \cdot x;\overline{x}.$$

By monotonicity,

$$(x \cdot \overline{y});(\overline{x} \cdot y) \leq x;\overline{x} \cdot \overline{y};y$$

and, by subadditivity,

$$(x \cdot \overline{y});(\overline{x} \cdot y) \leq x \cdot \overline{y} + \overline{x} \cdot y \leq x + y$$

so

$$\begin{aligned}(x \cdot \overline{y});(\overline{x} \cdot y) &\leq (x + y) \cdot x;\overline{x} \cdot \overline{y};y \\ &= x \cdot x;\overline{x} \cdot \overline{y};y + y \cdot x;\overline{x} \cdot \overline{y};y \\ &\leq x \cdot \overline{y};y + y \cdot x;\overline{x}.\end{aligned}$$

We may now conclude that

$$x;y \leq (x \cdot y);(x \cdot y) + x \cdot y;\overline{y} + y \cdot x;\overline{x}.$$

From this last equation and (11) we get the second half of (10), so (10) holds.

Now if **A** is reflexive, then $x \cdot y \leq (x \cdot y);(x \cdot y) \leq x;y$, hence $x;y + x \cdot y = x;y$, so by (10),

$$f(x \cdot y) + x \cdot g(y) + y \cdot g(x) = x;y,$$

as desired. □

COROLLARY 6 *The variety of reflexive tense algebras is term-equivalent to the variety of reflexive subadditive sr-algebras.*

PROOF. Let $\mathbf{A} = (A, +, ^-, f, g)$ be a reflexive tense algebra. Define ; by

$$x;y = f(x \cdot y) + x \cdot g(y) + y \cdot g(x).$$

By Theorem 4, $(A, +, ^-, ;)$ is a reflexive subadditive sr-algebra in which the operations f, g are definable by $f(x) = x;x$ and $g(x) = x + x;\overline{x}$.

For the converse, let $\mathbf{A} = (A, +, ^-, ;)$ be a reflexive subadditive sr-algebra. Define f and g by

$$f(x) = x;x \qquad \text{and} \qquad g(x) = x + x;\overline{x}.$$

Reflexivity implies $f(x) = x + x;x$, so by Theorem 5, $(A, +, ^-, f, g)$ is a reflexive tense algebra in which ; can be recovered from f and g by $x;y = f(x \cdot y) + x \cdot g(y) + y \cdot g(x)$. □

To establish a connection between subadditive sr-algebras and total tense algebras we require the *semiassociative* identity

$$(x;1);1 = x;1$$

introduced in Maddux [78].

THEOREM 7 *The variety generated by all total tense algebras is term-equivalent to the variety of reflexive subadditive semiassociative sr-algebras.*

PROOF. Let $(A, +, ^-, f, g)$ be an algebra in the variety generated by all total tense algebras, so that $(A, +, ^-, f, g)$ is a tense algebra which satisfies the identity

$$x + f(f(x) + g(x)) + g(f(x) + g(x)) \leq f(x) + g(x). \tag{12}$$

Let $h(x) = f(x) + g(x)$ for every $x \in A$. Then (12) is equivalent to

$$x + h(h(x)) \leq h(x). \tag{13}$$

A consequence of (13) is that $(A, +, ^-, f, g)$ satisfies $x \leq h(x)$. From the latter identity we can prove that $(A, +, ^-, f, g)$ is reflexive. To see this, let $z = x \cdot \overline{f(x)}$. Then

$$z = x \cdot \overline{f(x)} \leq h(x \cdot \overline{f(x)}) = f(x \cdot \overline{f(x)}) + g(x \cdot \overline{f(x)}) \leq f(x) + \overline{x} = \overline{z}.$$

From $z \leq \overline{z}$ it follows that $z = 0$, hence $x \leq f(x)$. Define the binary operation ; on A by

$$x;y = f(x \cdot y) + x \cdot g(y) + y \cdot g(x). \tag{14}$$

By Theorem 4, $(A, +, ^-, ;)$ is a reflexive subadditive sr-algebra with $f(x) = x;x$ and $g(x) = x + x;\overline{x}$. We will show $(A, +, ^-, ;)$ also satisfies the semiassociative identity. From (14) and the definition of h we have

$$x;1 = f(x \cdot 1) + x \cdot g(1) + 1 \cdot g(x) = h(x) + x \cdot g(1).$$

Since $(A, +, ^-, f, g)$ is a tense algebra, the operations f and g are monotone and distribute over $+$. It follows that the same properties apply to h, so we can derive one half of the semiassociative identity from a consequence of (13), namely $h(h(x)) \leq h(x)$, as follows:

$$\begin{aligned}(x;1);1 &= h(h(x) + x \cdot g(1)) + (h(x) + x \cdot g(1)) \cdot g(1) \\ &= h(h(x)) + h(x \cdot g(1)) + h(x) \cdot g(1) + x \cdot g(1) \\ &\leq h(x) + x \cdot g(1) \\ &= x;1\end{aligned}$$

For the other direction, note first that $x \leq h(x)$ by (13), hence $x \leq h(x) + x \cdot g(1) = x;1$ for every x in A. Replacing x by $x;1$ yields $x;1 \leq (x;1);1$.

Now suppose that $(A, +, ^-, ;)$ is a reflexive subadditive semiassociative sr-algebra. Define f and g by

$$f(x) = x;x \qquad \text{and} \qquad g(x) = x + x;\overline{x}.$$

Reflexivity implies $f(x) = x + x;x$, so $(A, +, ^-, f, g)$ is a reflexive tense algebra and $x;y = f(x \cdot y) + x \cdot g(y) + y \cdot g(x)$ by Theorem 5. We use the semiassociative identity to derive (13). First note that $h(x) = f(x) + g(x) = x;x + x + x;\overline{x} = x + x;(x + \overline{x}) = x + x;1$. Then

$$\begin{aligned} x + h(h(x)) &= x + h(x + x;1) \\ &= x + (x + x;1) + (x + x;1);1 \\ &= x + x;1 + (x;1);1 \\ &= x + x;1 \\ &= h(x) \end{aligned}$$

Since $(A, +, ^-, f, g)$ is a reflexive tense algebra satisfying (13), which is equivalent to (12), it is in the variety generated by all total tense algebras. □

Before we extend our results about total tense algebras to subadditive symmetric semiassociative relation algebras, we recall what is known about the bottom of the lattice Λ_{SA} of semiassociative relation algebra varieties. Jónsson and Tarski [**52**] proved that there are exactly three atoms, generated by the algebras $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$, in the lattice Λ_{RA} of varieties of relation algebras. Later Andréka, Jónsson and Németi [**91**] pointed out that the proof given there requires only the semiassociative identity, so the same result holds in Λ_{SA}. Furthermore, $\mathrm{Var}(\mathbf{A}_1)$ has no join-irreducible covers and $\mathrm{Var}(\mathbf{A}_2)$ has exactly one join-irreducible cover, generated by the relation algebra of all binary relations on a 2-element set.

THEOREM 8 *Let* $\mathbf{A} = (A, +, ^-, ;, \breve{}, 1')$ *be a finite simple subadditive symmetric semiassociative relation algebra with more than 4 atoms. Then* $\mathbf{A}$ *has a proper nonconstant subalgebra.*

PROOF. A subalgebra of $\mathbf{A}$ is "proper", of course, if it is not the same as $\mathbf{A}$, but the meaning of "nonconstant" is slightly different in this case, because semiassociative relation algebras have a distinguished identity element 1'. Indeed, every simple finite semiassociative relation algebra has a subalgebra whose only elements are 0, 1, 1', and $\overline{1'}$, and it is this subalgebra that we refer to as the "constant subalgebra" of $\mathbf{A}$.

A nonassociative relation algebra is said to be *integral* if $x;y \neq 0$ implies either $x \neq 0$ or $y \neq 0$. If $\mathbf{A}$ is not integral, then $\mathbf{A}$ has a proper nonconstant subalgebra by Maddux [**90**], Theorem 3. We may therefore assume that $\mathbf{A}$ is integral. By Maddux [**90**], Theorem 4, this is equivalent to assuming that 1' is an atom.

Next we show that we may also assume $\mathbf{A}$ is reflexive. Let $\mathbf{A}' = (A, +, ^-, ;', \breve{}, 1')$ where, for all $x, y \in A'$,

$$x;'y = x;y + x \cdot y.$$

We claim that $\mathbf{A}'$ is a reflexive finite integral symmetric semiassociative relation algebra with exactly the same subalgebras as $\mathbf{A}$. Obviously $\mathbf{A}'$ satisfies $\breve{x} = x$ since $\mathbf{A}$ does so. Hence to prove that (8) holds for $\mathbf{A}'$ we need only show (9) holds for $\mathbf{A}'$, namely,

$$x \cdot y;'z = 0 \quad \text{iff} \quad y \cdot x;'z = 0 \quad \text{iff} \quad z \cdot y;'x = 0.$$

Suppose $x \cdot y;'z = 0$. Then $0 = x \cdot (y;z + y \cdot z) = x \cdot y;z + x \cdot y \cdot z$, so $0 = x \cdot y;z$ and $0 = x \cdot y \cdot z$. Using (8) and the symmetry of $\mathbf{A}$, we get $0 = y \cdot x;z$, so $0 = y \cdot x;z + x \cdot y \cdot z = y \cdot (x;z + x \cdot z) = y \cdot x;'z$. The other equivalences can be proved similarly. Next,

$$x;'1' = x;1' + x \cdot 1' = x + x \cdot 1' = x,$$

and, similarly, $1';'x = x$, so $1'$ is an identity for $;'$. To show the semiassociative identity holds for $\mathbf{A}'$, use the assumption that it holds for $\mathbf{A}$ to get

$$(x;'1);'1 = (x;1 + x \cdot 1);'1 = (x;1+x);1 + (x;1+x)\cdot 1 = (x;1);1 + x;1 + x = x;1 + x = x;'1.$$

So far we have shown that $\mathbf{A}'$ is a symmetric semiassociative relation algebra. Obviously $\mathbf{A}'$ is finite. $\mathbf{A}$ and $\mathbf{A}'$ have the same Boolean algebra as reduct, so $1'$ is an atom in both, hence $\mathbf{A}'$ is integral. $\mathbf{A}'$ is reflexive since $x \leq x;x + x = x;x + x\cdot x = x;'x$. What remains is to show $\mathbf{A}$ and $\mathbf{A}'$ have the same subalgebras. The operations of $\mathbf{A}'$ are term-definable from those of $\mathbf{A}$, so every subalgebra of $\mathbf{A}$ is a subalgebra of $\mathbf{A}'$. For the converse, assume $\mathbf{B}$ is a subalgebra of $\mathbf{A}'$. We wish to show the universe B of $\mathbf{B}$ is closed under the operations of $\mathbf{A}$. The only operation of $\mathbf{A}$ which is not also an operation of $\mathbf{A}'$ is $;$. Hence we need only show that B is closed under $;$. Let $x, y \in B$. First expand $x;y$ as follows.

$$x;y = (x\cdot y);(x\cdot y) + (x\cdot \overline{y});(x\cdot y) + (x\cdot y);(\overline{x}\cdot y) + (x\cdot\overline{y});(\overline{x}\cdot y).$$

Note that $;$ and $;'$ coincide on disjoint arguments, that is, if $u\cdot v = 0$, then $u;'v = u;v + u\cdot v = u;v + 0 = u;v$. We can apply this observation to the last three terms in the expansion of $x;y$, obtaining

$$x;y = (x\cdot y);(x\cdot y) + (x\cdot \overline{y});'(x\cdot y) + (x\cdot y);'(\overline{x}\cdot y) + (x\cdot\overline{y});'(\overline{x}\cdot y).$$

Thus we need only show $(x\cdot y);(x\cdot y) \in B$. Let $z = x\cdot y$. We claim that either $z \leq z;z$, in which case $z;z = z;'z \in B$, or else $z\cdot z;z = 0$, in which case $z;z = z;'z\cdot\overline{z} \in B$. Suppose both cases fail. Then $u \neq 0 \neq v$, where $u = z\cdot\overline{z;z}$ and $v = z\cdot z;z$. Since $\mathbf{A}$ is integral and subadditive, we have $0 \neq u;v \leq u + v$. But $u;v \leq z;z$ and $u \leq \overline{z;z}$, so $0 \neq u;v \leq (u\cdot\overline{z;z} + v)\cdot z;z = v\cdot z;z \leq v$. Thus $0 \neq u;v\cdot v$. By (8) and symmetry, this yields $0 \neq v;v\cdot u$. But $v \leq z$ and $u \leq \overline{z;z}$, so $v;v\cdot u \leq z;z\cdot\overline{z;z} = 0$, a contradiction. This completes the proof of our claim that $\mathbf{A}'$ is a reflexive finite integral symmetric semiassociative relation algebra and $\mathbf{A}'$ has the same subalgebras as $\mathbf{A}$, so we might as well assume that $\mathbf{A}$ is reflexive.

Let $\mathbf{A}' = (A', +, ^{-'}, ;')$ be the algebra whose universe A' is defined by $A' = \{x \in A : x \leq \overline{1'}\}$, where, for all $x, y \in A'$,

$$x^{-'} = \overline{x}\cdot\overline{1'},$$
$$x;'y = x;y\cdot\overline{1'}.$$

We claim that $\mathbf{A}'$ is a sr-algebra. First of all, $(A', +, ^{-'})$ is a Boolean algebra. Since $\mathbf{A}$ is symmetric, (9) holds. When (9) is applied to $\mathbf{A}'$, that is, with $;$ replaced by $;'$, it is equivalent to

$$x\cdot y;z\cdot\overline{1'} = 0 \quad\text{iff}\quad y\cdot x;z\cdot\overline{1'} = 0 \quad\text{iff}\quad z\cdot y;x\cdot\overline{1'} = 0.$$

But for $x, y, z \in A'$ we have $x = x\cdot\overline{1'}$, $y = y\cdot\overline{1'}$, and $z = z\cdot\overline{1'}$, so the latter condition actually coincides with (9). Thus $\mathbf{A}'$ is a sr-algebra.

By Theorem 5, $\mathbf{A}'$ is term-equivalent to a reflexive tense algebra $\mathbf{A}'' = (A', +, ^{-}, f, g)$, and since $\mathbf{A}$ was assumed to be simple and symmetric, $\mathbf{A}'$ satisfies the implication $x \neq 0 \Rightarrow x;'1_{A'} = 1_{A'}$, whence $\mathbf{A}''$ is total. Note that $\mathbf{A}'$ has more than 3 atoms, so we can apply Theorem 1 to the atom structure of $\mathbf{A}''$ and conclude that it has a proper nonconstant subalgebra. Of course the same conclusion holds for $\mathbf{A}'$, since it is term-equivalent to $\mathbf{A}''$. Let B' be the universe of this subalgebra of $\mathbf{A}'$, and define $B = B' \cup \{x + 1' \in A : x \in B\}$. Then the equations

$$x;y = x;'y + x;y\cdot 1',$$
$$(x+1');y = x;'y + y + x;y\cdot 1',$$
$$(x+1');(y+1') = x;'y + x + y + 1',$$

together with the assumption that 1' is an atom, show that $(B, +, ^-, ;, 1')$ is a proper nonconstant subalgebra of $\mathbf{A}$. □

Since the preceding theorem did not assume reflexivity, there are more than four finitely generated varieties covering $\mathrm{Var}(\mathbf{A}_3)$. The exact number is found counting the number of nonisomorphic structures that can be obtained by deleting loops (x, x) from the atom structures of $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$. Nonfinitely generated covers of $\mathrm{Var}(\mathbf{A}_3)$ can be constructed from the total tense algebras $\mathbf{B}_n$ of Theoren 3 via the term-equivalence, similar to the proof above.

COROLLARY 9 *In the lattice of varieties of subadditive symmetric semiassociative relation algebras there are exactly 19 finitely generated varieties, and infinitely many nonfinitely generated varieties, that cover* $\mathrm{Var}(\mathbf{A}_3)$.

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