Basic Logic algebras and lattice-ordered groups
as algebras of binary relations

Peter Jipsen
Chapman University, California

- Algebras of binary relations
- Embedding $\ell$-groups
- Embedding BL-algebras
- Finite representable generalized BL-algebras

Algebras of binary relations

Let us recall some standard results:

M. Stone: Every Boolean algebra is isomorphic to a subalgebra of all subsets of some set $U$, with $\cup$, $\cap$, $\neg$, $\emptyset$, $U$ as operations.

C. Holland: Every $\ell$-group is isomorphic to a subalgebra of all order-automorphisms of a chain, with pointwise order and $\circ, \neg^{-1}, id$ as operations.

Let $R, S$ be binary relations ($\subseteq U^2$)

relation composition:
$$R \circ S = \{(u, v) : \exists w (u, w) \in R \text{ and } (w, v) \in S\}$$

inverse: $R^{-1} = \{(v, u) : (u, v) \in R\}$ and

identity: $id_U = \{(u, u) : u \in U\}$

A representable relation algebra on $U$ is a set $A$ of relations that is closed under $\cup, \cap, \neg, \circ, \neg^{-1}, id_U$.

$RRA =$ class of all algebras isomorphic to representable relation algebras

Tarski: $RRA$ is a variety.

Monk ’64: $RRA$ is not finitely axiomatizable.

$\circ$ is like a multiplication

$\circ, \cap$ distribute over $\cup$ as in $\ell$-groups

Naive question: Can we embed $\ell$-groups into representable relation algebras?

Well, we don’t need complementation.

If $R \circ R^{-1} = id_U = R^{-1} \circ R$ then $R$ is a permutation.
So any ℓ-group element would have to map to a permutation.

But this is incompatible with preserving the order of the ℓ-group since distinct permutation are disjoint as relations.

Also, if \( R, S \subseteq \text{id}_U \) then \( R \circ S = R \cap S \).

But this is certainly no true in ℓ-groups.

So forget about \(-1, \text{id}_U\) and instead look at the “residuated lattice reducts” of relation algebras.

residuals: \( R \setminus S = \{(u, v) : R \circ \{(u, v)\} \subseteq S\} \) and \( R/S = \{(u, v) : \{(u, v)\} \circ S \subseteq R\} \)

**Definition:** A *residuated lattice of (binary) relations* is a set \( A \) of relations that is closed under \( \cup, \cap, \circ, \setminus, / \) and contains a relation \( 1 \) such that \( 1 \circ R = R \circ 1 = R \) for all \( R \in A \).

(Note that 1 is usually not the identity relation.)

RLR denotes the quasivariety of all residuated lattices of relations.

**Problem 1.** Is RLR a variety?

It is obvious that every residuated lattice of relations is a distributive residuated lattice.

**Problem 2.** Is the converse also true?

Andreka [1991] proved a general result that implies RLR is not finitely axiomatizable.

Since distributive residuated lattices form a finitely axiomatizable variety, the answer to Problem 2 would be no if RLR is a variety.

**Embedding ℓ-groups**

\( LG \) denotes the variety of *lattice-ordered groups* (residuated lattices that satisfy \( x(x\setminus 1) = 1 \), so \( x^{-1} = x\setminus 1 \)).

They are distributive residuated lattices.

**Question:** Is \( LG \subseteq RLR \)?

This is answered by the following result.

**Theorem.** Every ℓ-group is isomorphic to a residuated lattice of relations, hence \( LG \subseteq RLR \).

**Proof.** Let \( G = \langle \text{Aut}(\Omega), \lor, \land, \circ, \text{id}_\Omega, \setminus, / \rangle \) be the ℓ-group of order-automorphisms of a chain Ω.
Note that $\lor, \land$ are calculated pointwise.

By Holland’s embedding theorem, it suffices to embed $G$ into a residuated lattice of relations on $\Omega$.

For $g \in G$, let $R_g = \{(u, v) : u \leq g(v)\}$.

$R_g \cap R_h = R_{g \land h}$ since

$(u, v) \in R_g \cap R_h$
\[\iff u \leq g(v) \text{ and } u \leq h(v)\]
\[\iff u \leq \min\{g(v), h(v)\} = (g \land h)(v)\]
\[\iff (u, v) \in R_{g \land h}\]

$R_g \cup R_h = R_{g \lor h}$ is similar, using max.

$R_g \circ R_h = R_{g \circ h}$ since

$(u, v) \in R_g \circ R_h$
\[\iff \exists w \ [(u, w) \in R_g \text{ and } (w, v) \in R_h]\]
\[\iff \exists w \ [u \leq g(w) \text{ and } w \leq h(v)]\]
\[\iff u \leq g(h(v)) \quad (w = h(v) \text{ for } \iff)\]
\[\iff (u, v) \in R_{g \circ h}\]

$R_g \setminus R_h = R_{g \setminus h}$ since

$(u, v) \in R_g \setminus R_h$
\[\iff R_g \circ \{(u, v)\} \subseteq R_h\]
\[\iff \forall w \ [(w, u) \in R_g \implies (w, v) \in R_h]\]
\[\iff \forall w \ [w \leq g(u) \implies w \leq h(v)]\]
\[\iff g(u) \leq h(v)\]
\[\iff u \leq g^{-1}(h(v)) = (g \setminus h)(v)\]
\[\iff (u, v) \in R_{g \setminus h}\]

$R_g / R_h = R_{g / h}$ since

$(u, v) \in R_g / R_h$
\[\iff \{(u, v)\} \circ R_h \subseteq R_g\]
\[\iff \forall w \ [(v, w) \in R_h \implies (u, w) \in R_g]\]
\[\iff \forall w \ [v \leq h(w) \implies u \leq g(w)]\]
\[\iff \forall w \ [h^{-1}(v) \leq w \implies g^{-1}(u) \leq w]\]
\[\iff g^{-1}(u) \leq h^{-1}(v) \text{ for } \implies\]
Finally, \( R_{id} = \{(u, v) : u \leq v\} \) is an identity element since 
\( R_g \circ R_{id} = R_{g \circ id} = R_g = R_{id} \circ R_g \).
Therefore \( \{R_g : g \in G\} \) is a residuated lattice of relations that is isomorphic to \( G \). □

Embedding BL-algebras

**Theorem.** Every BL-algebra is isomorphic to some algebra of relations

**Proof.** The MV-algebra on \([0, 1]\) is isomorphic to \( \{M_r : r \in [0, 1]\} \) where \( M_r = \{(u, v) \in (0, 1]^2 : v \leq u - 1 + r\} \).

The Gödel algebra on \([0, 1]\) is isomorphic to \( \{G_r : r \in [0, 1]\} \) where 
\( G_r = \{(u, v) \in (0, 1]^2 : v \leq \min\{u, r\}\} \).

The product algebra on \([0, 1]\) is isomorphic to \( \{P_r : r \in [0, 1]\} \) where 
\( P_r = \{(u, v) \in (0, 1]^2 : v \leq r \cdot u\} \).

To complete the proof it suffices to show that \( \text{RLR} \) is closed under ordinal sums of integral members.

Suppose \( A, B \in \text{RLR}, \) with \( A \) integral, \( A \subseteq \mathcal{P}(U^2) \) and \( B \subseteq \mathcal{P}(V^2) \),
where \( U \) and \( V \) are disjoint.

Define \( C = A \cup \{R \cup 1^A : R \in B\} \). Then it is easy to check that 
\( C \cong A \oplus B \).

Note that integrality of \( A \) is required to ensure that \( 1^A \cup 1^B \) is an identity of \( C \). □

Finite representable generalized BL-algebras

**Generalized basic logic algebras,** or **GBL-algebras** for short, are residuated lattices that satisfy 
\( x \wedge y = ((x \wedge y) / y) y \) and \( x \wedge y = y(y \setminus (x \wedge y)) \).

The variety of GBL-algebras contains \( \text{LG} \), as well as the variety of **basic hoops** (defined by adding \( xy = yx \) and \( x \wedge y = (x/y)y \) to \( \text{RL} \)).

A residuated lattice is **integral** if the identity \( 1 \) is the top element.

This condition holds for basic hoops since \( x \wedge 1 = (x/1)1 = x \).

A GBL-algebra is called a **GBL-chain** if it is linearly ordered.

GBL-chains generate the variety of **representable GBL-algebras.**
**Pseudo BL-algebras** are bounded integral GBL-algebras expanded with a constant 0 denoting the least element, and that satisfy **prelinearity**: 
\[ x \backslash y \lor y \backslash x = 1 = x/y \lor y/x. \]

**BL-algebras** are commutative pseudo BL-algebras (in which case prelinearity can be derived from the basic hoop axioms). Hajek [1998] proved that all subdirectly irreducible BL-algebras are BL-chains.

Also, by definition, BL-algebras are commutative and integral.

The lattice-reduct of any GBL-algebra is distributive (see e.g. J. and Tsinakis [2002])

But in general, GBL-algebras are neither integral nor commutative nor representable, (consider any nonrepresentable ℓ-group).

Note that if a GBL-algebra has a top element \( \top \), then \( \top = 1 \):

From \( x \land y = y(y \backslash (x \land y)) \), we deduce

\[ 1 = 1 \land \top = \top(\top \backslash (1 \land \top)) = \top(\top \backslash 1) \]

hence \( \top = \top 1 = \top \top = \top(\top \backslash 1) = 1 \).

(More generally this shows 1 is a maximal idempotent in any GBL-algebra.)

Therefore any finite GBL-algebra is integral.

We now prove that all finite GBL-chains are in fact commutative and hence basic hoops.

This result also holds for pseudo BL-chains (since any finite GBL-chain can be expanded to a pseudo BL-chain by adding a constant 0 to denote the least element)

**Problem 3.** Are all finite GBL-algebras commutative?

The GBL identities are equivalent to the following property that is also called **divisibility**: 
\[ x \leq y \iff (\exists z (x = zy) \land \exists z (x = yz)). \]

As usual, the symbol \(≺\) denotes the covering relation, and \(\preceq\) denotes the covering-or-equal relation.

As usual, the symbol \(≺\) denotes the covering relation, and \(\preceq\) denotes the covering-or-equal relation.
Lemma 1. In any integral GBL-chain, if $a \prec b$ then for all $c$ we have $ac \preceq bc$ and $ca \preceq cb$.

Proof. We show the contrapositive. Suppose $a, b, c, x$ are elements in an integral GBL-chain such that $ac < x < bc$.

By integrality $x \leq c$, hence by divisibility $\exists z$ such that $x = zc$.

Now $zc < bc \implies b \not\geq z$, so the linear order implies $z < b$.

Similarly, $ac < zc$ implies $a < z$.

Hence $a$ is not covered by $b$.

The argument for $ca < x < cb$ is similar. $\square$

Let $L$ and $M$ be integral residuated lattices with no elements in common.

Suppose further that the identity element of $L$ is join-irreducible.

The ordinal sum of $L$ and $M$ is an integral residuated lattice defined on the set $(L \setminus \{1\}) \cup M$ as follows:

· restricted to $L$ and $M$ agrees with the original product on $L$ and $M$ respectively, and for $x \in L \setminus \{e\}$ and $y \in M$,

$$x \cdot y = x = y \cdot x.$$  

The order on the ordinal sum also agrees with the original order on $L$ and $M$, and all elements of $L \setminus \{e\}$ are below all elements of $M$.

Let $a$ be an idempotent element $(aa = a)$ of an integral residuated lattice $L$.

Then it is easy to check that $\uparrow a = \{x \in L : x \leq a\}$ is a subalgebra of $L$.

Similarly, $\downarrow a = \{x \in L : x \leq a\}$ is closed under $\cdot$ and the lattice operations, and

· is residuated by the operations

$$x \downarrow y = x \setminus y \land a \quad \text{and} \quad x /\downarrow y = x / y \land a.$$  

The next result shows that for finite GBL-chains $a$ acts as the identity element on $\downarrow a$.

Lemma 2. If a finite GBL-chain $L$ contains an idempotent $a \neq 0,1$ then $\downarrow a$ is a residuated lattice with identity $1^\downarrow = a$, and $L$ decomposes as the ordinal sum of $\uparrow a$ and $\downarrow a$. 

Proof. To conclude that \( \downarrow a \) is a residuated lattice, it suffices to show that \( ax = x = xa \) for all \( x \leq a \).

This follows from the preceding lemma since \( L \) is a finite chain, \( aa = a, a0 = 0 \), and the map \( x \mapsto ax \) preserves \( \preceq \).

To see that \( L \) decomposes as an ordinal sum, note that if \( x \leq a \) and \( y \geq a \) then \( yx = x \), since \( 1x = x \), \( ax = x \), and \( \cdot \) is order preserving.

Similarly \( xy = x \). \( \square \)

Since ordinal sums of commutative integral GBL-algebras are commutative, it now only remains to show that every finite integral GBL-algebras without idempotents (other than \( 0 \) and \( 1 \)) is commutative.

Lemma. Let \( A = \{a_0, a_1, \ldots, a_n\} \) be the elements of a finite GBL-algebra, with \( a_0 = 1, a_n = 0 \) and \( a_i \succ a_{i+1} \) for \( i < n \).

Suppose that \( A \) has no idempotents other than \( 1, 0 \), and that for some fixed \( m \leq n \) and all \( i + j < m \) we have \( a_i \cdot a_j = a_{i+j} \).

Then \( a_{m-k} \cdot a_k = a_m \) for all \( k \leq m \).

Lemma 3. Let \( A = \{a_0, a_1, \ldots, a_n\} \) be the elements of a finite GBL-algebra, with \( a_0 = 1, a_n = 0 \) and \( a_i \succ a_{i+1} \) for \( i < n \).

Suppose that \( A \) has no idempotents other than \( 1, 0 \), and that for some fixed \( m \leq n \) and all \( i + j < m \) we have \( a_i \cdot a_j = a_{i+j} \).

Then \( a_{m-k} \cdot a_k = a_m \) for all \( k \leq m \).

Proof. Assume the stated conditions hold, and let \( k \leq m \).

If \( k = 0 \), then the conclusion follows immediately.

For \( k = 1 \), Lemma 1 implies that \( a_{m-1} \cdot a_1 \) is either \( a_{m-1} \) or \( a_m \), since \( a_{m-1} \cdot a_0 = a_{m-1} \).

We claim that the first case is impossible since it implies that \( a_{m-1} \) is idempotent.

This follows from the observation that if \( a_{m-1} \cdot a_1 = a_{m-1} \) then \( a_{m-1} \cdot a_1 \cdot a_1 \cdots a_1 = a_{m-1} \), and the product of \( m - 1 \) copies of \( a_1 \) is \( a_{m-1} \) by assumption.

Therefore \( a_{m-1} \cdot a_1 = a_m \). (End of basis step)

Now suppose that \( a_{m-(k-1)} \cdot a_{k-1} = a_m \).
Then $a_{m-k} \cdot a_k = a_{m-k} \cdot a_1 \cdot a_{k-1} = a_{m-k+1} \cdot a_{k-1} = a_m$.

So the desired result follows by induction on $k$. \[\square\]

The $n$-element Wajsberg chain is a basic hoop with elements $a_0 \succ a_1 \succ \cdots \succ a_{n-1}$ such that $a_i \cdot a_j = a_{\min(i+j,n-1)}$, hence commutative.

**Theorem.** Every finite GBL-chain is commutative (hence a basic hoop).

**Proof.** Suppose $A$ is a GBL-chain that has elements $a_0 \succ a_1 \succ \cdots \succ a_n = 0$.

Any finite GBL-algebra is integral, hence $a_0 = 1$.

If $n = 1$, then $A$ is the 2-element Wajsberg chain ($= BA$).

Now suppose $n > 1$. If $a_i$ is idempotent for some $0 < i < n$, then $A$ decomposes by Lemma 2 into the ordinal sum of two smaller GBL-chains.

So we may assume that $A$ has no idempotents other that $1, 0$.

Therefore by Lemma 1, $a_1 \cdot a_1 = a_2$.

If $n = 2$, then $A$ is the 3-element Wajsberg chain, and

if $n > 2$, then the assumptions of Lemma 3 are satisfied with $m = 3$.

Using this lemma as the inductive step we see that $A$ has the structure of the $n+1$-element Wajsberg chain. \[\square\]

So the finite GBL-chains are just ordinal sums of Wajsberg chains.

This makes it easy to count the number nonisomorphic (G)BL-chains with $n$ elements.

We just have to choose which of the $n - 2$ elements between 1 and 0 are idempotents.

For each of the $2^{n-2}$ different choices we obtain a nonisomorphic (G)BL-chain.

**Corollary.** For $n > 1$ there are $2^{n-2}$ GBL-chains with $n$ elements.

Since there are noncommutative representable integral GBL-algebras, we also have the following result.

**Corollary.** The variety of representable GBL-algebras and the variety of integral representable GBL-algebras are not generated by their finite members (i.e. they do not have the finite model property).
References

