Lecture 9 – Coorbit Spaces

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Recap

- Starting from the notion of time-frequency representations/distributions of signals, two distinct branches of inquiry have emerged: (1) Gabor theory and (2) Wavelet theory.
- (1) Gabor theory had its origins in communication theory and quantum mechanics.
- The initial idea was to understand the information content in a signal in terms of fundamental units of information represented by rectangles of area one in time-frequency space.
- The goal was to represent a signal in terms of a tiling of the time-frequency plane in terms of such rectangles.
- The difficulties encountered in trying to realize that program led to the development of a mathematically rich field of time-frequency analysis.
- Included in this tapestry are deep structure theorems for Gabor systems and the general theory of frames.

Wavelet theory

- (2) Wavelet theory arose from the analysis of operators arising from differential equations and function spaces through which those operators can be understood.
- The effort to understand and characterize the fine oscillatory structure of these spaces led to simpler atomic decompositions and ultimately to smooth orthonormal bases that captured this structure.
- These bases are seen to also correspond to a tiling of the time-frequency plane in terms of rectangles of unit area.
- An elegant mathematical theory has developed that lends itself to efficient numerical algorithms and a rich array of applications in signal and image processing.

Motivation

Recall the definition of the Short Time Fourier Transform.

Definition

Given $g \in L^2(\mathbb{R}^d)$, we define the *short-time Fourier transform (STFT)* on $L^2(\mathbb{R}^d)$ by

$$V_g f(x,\gamma) = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i(t\cdot\gamma)}\,dt = \langle f, M_\gamma T_x g\rangle.$$

- The “coherent states” consist of applying the operators $T_a M_b$ with $(a,b) \in \mathbb{R} \times \mathbb{R}$ to a single window function function g.
- Is there a group structure underlying these transformations? Yes.

Definition (Heisenberg Group)

Let

$$\mathbb{H} = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$$

denote the Heisenberg group with group operation

$$(t_1, a_1, b_1) \cdot (t_2, a_2, b_2) = (t_1 t_2 e^{2\pi i b_1 a_2}, a_1 + a_2, b_1 + b_2).$$

Haar measure on this group is $dt\, da\, db$. Define the *Schrödinger representation* of $\mathbb{H}$ on $L^2(\mathbb{R})$ by

$$\pi(t, a, b) f(x) = t\, e^{2\pi i b(x-a)}\, f(x-a) = t\, T_a M_b f(x).$$

- $V_g f$ can now be thought of as a function on the group $\mathbb{H}$, so we write $V_g(t, a, b)$ instead of $Vg(a, b)$.
- The introduction of the extra component $t \in \mathbb{T}$ is immaterial to any of the preceding discussions.

Motivation

Recall the definition of the Continuous Wavelet Transform.

Definition

Given a function $g \in L^2(\mathbb{R})$, the *continuous wavelet transform* of a function $f \in L^2$ is defined by

$$W_g(f)(a,b) = \int_{-\infty}^{\infty} f(t)\, a^{1/2}\overline{g(at-b)}\, dt = \langle f, D_a T_b g\rangle_{L^2(\mathbb{R})}$$

for $a > 0$ and $b \in \mathbb{R}$.

- The "coherent states" consist of applying the operators $D_a T_b$ with $(a,b) \in \mathbb{R}_+ \times \mathbb{R}$ to a single wavelet function g.
- Is there a group structure underlying these transformations? Yes.

Definition (Affine Group)

Let

$$\mathbb{A} = \mathbb{R}_+ \times \mathbb{R}$$

denote the *affine group* with group operation

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_2 b_1 + b_2).$$

In this case, left-Haar measure on this group is $\dfrac{da}{a}\, db$.

- Define a representation, π of $\mathbb{A}$ on $L^2(\mathbb{R})$ by

$$\pi(a, b) f(x) = a^{1/2} f(ax - b).$$

- $W_g f$ is now thought of as a function on the group $\mathbb{A}$.

Motivation

- Co-orbit Theory (Feichtinger-Gröchenig, 1989) presents a unified framework for understanding the generating atomic decompositions in terms of Gabor or wavelet systems.
- The unifying principle is that each of these decompositions are in terms of Banach frames generated by a single vector under the action of a group of unitary transformations.
- The basic idea is that one can study Banach spaces which can in principle be very abstract by looking at a corresponding function space on a group, which can in principle be much more concrete.
- In particular, one can get atomic decompositions and frame expansions in these Banach spaces, which include Gabor expansions of modulation spaces, and wavelet expansions of Besov-Triebel-Lizorkin spaces.

Basic definitions

Definition

Let G be a locally compact group with left-invariant Haar measure $d\mu$, and $\mathcal{H}$ a Hilbert space.

(1) A *representation* π of G on $\mathcal{H}$ is a mapping $\pi\colon G \to \mathcal{L}(\mathcal{H})$ such that $\pi(x \cdot y) = \pi(x)\pi(y)$ for every $x, y \in G$.

(2) A vector $g \in \mathcal{H}$ is *admissible* if

$$\int_G |\langle g, \pi(x)g\rangle|^2 \, d\mu(x) < \infty.$$

(3) A vector $g \in \mathcal{H}$ is *cyclic* if $\overline{\text{span}}\{\pi(x)g\}_{x\in G} = \mathcal{H}$.

(4) π is *unitary* if the map $\pi(x)\colon \mathcal{H} \to \mathcal{H}$ is unitary for each $x \in G$.

(5) π is *irreducible* if every $g \in \mathcal{H} \setminus \{0\}$ is cyclic.

(6) π is *square-integrable* if π is irreducible and there exists an admissible $g \in \mathcal{H} \setminus \{0\}$.

The Voice Transform

Definition

Let G, $d\mu$, and $\mathcal{H}$ be as above. Assume that π a *unitary*, *square-integrable* group representation of G on $\mathcal{H}$. If $g \in \mathcal{H}$ is admissible, define the *voice transform* $\mathcal{V}_g$ on $\mathcal{H}$ by

$$\mathcal{V}_g(f)(x) = \langle f, \pi(x)g \rangle.$$

- $\mathcal{V}_g$ is a linear mapping from $\mathcal{H}$ into the collection of bounded continuous functions on G, and moreover

$$\|\mathcal{V}_g(f)\|_\infty \leq \|f\|_\mathcal{H} \|g\|_\mathcal{H}.$$

Admissibility

Theorem (Grossmann, Morlet, Paul, 1985)

There is a unique positive, self-adjoint, densely-defined operator A on $\mathcal{H}$ such that

(1) *$g \in dom(A)$ if and only if g is admissible,*

(2) $\int_G \mathcal{V}_{g_1}(f_1)(x)\,\overline{\mathcal{V}_{g_2}(f_2)(x)}\,d\mu(x) = \langle Ag_1, Ag_2\rangle\,\langle f_1, f_2\rangle$ *for $g_1,\ g_2$ admissible and $f_1,\ f_2 \in \mathcal{H}$.*

- The operator A is also referred to as the *Dufflo-Moore* operator.

Admissibility

- In the case of the Schrödinger representation of $\mathbb{H}$, the operator A is the identity. In this case, every $g \in L^2(\mathbb{R})$ is admissible.
- For the affine group $\mathbb{A}$, the operator A is given by

$$\widehat{Ag}(\gamma) = \frac{\widehat{g}(\gamma)}{|\gamma|^{1/2}}.$$

- In this case, g is admissible if and only if

$$\int_{-\infty}^{\infty} \frac{|\widehat{g}(\gamma)|^2}{|\gamma|}\, d\gamma < \infty.$$

- Note that if g is admissible and $\|Ag\| = 1$, then
$$\int_G |\mathcal{V}_g(f)(x)|^2 \, d\mu(x) = \|f\|_{\mathcal{H}}^2.$$
- Then $\mathcal{V}_g$ maps $\mathcal{H}$ isometrically onto a closed linear subspace $S \subseteq L^2(G)$.
- Since $\mathcal{V}_g f(x)$ is also bounded and continuous, the subspace S will consist of "nice" functions that can be used to study the Hilbert space $\mathcal{H}$ that may well consist of more "wild" objects.

Reproducing formula

- One particularly nice property of S is that it satisfies a reproducing formula.
- In order to be precise about this we need to make an additional assumption about π, namely that it is *integrable*.
- This means there exists $g \in \mathcal{H} \setminus \{0\}$ such that

$$\int_G |\langle g, \pi(x) g \rangle| \, d\mu(x) < \infty.$$

  In other words, $\mathcal{V}_g(g) \in L^1(G)$.
- This assumption on π will be important later as it will allow us to extend beyond the Hilbert space setting into more general Banach spaces.

Lemma

Suppose that $g \in \mathcal{H}$ satisfies $\mathcal{V}_g(g) \in L^1(G)$ and $\|Ag\| = 1$. Then for $f \in \mathcal{H}$,

$$\mathcal{V}_g(f) * \mathcal{V}_g(g) = \int_G \mathcal{V}_g(f)(x)\, \mathcal{V}_g(g)(x^{-1}y)\, d\mu(x) = \mathcal{V}_g(f).$$

From the orthogonality relations,

$$\begin{aligned}
&\int_G \langle f, \pi(x)g\rangle \, \langle g, \pi(x^{-1}y)g\rangle \, d\mu(x) \\
= \quad & \int_G \langle f, \pi(x)g\rangle \, \overline{\langle \pi(y)g, \pi(x)g\rangle} \, d\mu(x) \\
= \quad & \langle Ag, Ag\rangle \, \langle f, \pi(y)g\rangle
\end{aligned}$$

Discretization

- $S = \mathrm{range}\,(V_g)$ is a closed subspace of $L^2(G)$ and the above lemma identifies S as a *reproducing kernel Hilbert space*.
- Typically such RKHS are associated with *sampling theorems* based on the intuition that such spaces consist of smooth functions.
- How can such sampling theorems be obtained in general?
- The idea is to approximate the convolution integral (the identity) by a sum (like a Riemann sum) and arrive at a *discrete* representation of functions in $L^2(G)$ and $\mathcal{H}$.

Discrete sets in G

Definition

Let $X = \{x_i \colon i \in I\} \subseteq G$ be a countable family in G.

(1) For a neighborhood U of the identity in G, X is *U-dense* if

$$\bigcup_{i \in I} x_i U = G.$$

(2) X is *relatively separated* if for any relatively compact set $W \subseteq G$ with non-empty interior,

$$\sup_{i \in I} \#\{k \in I \colon x_k W \cap x_i W \neq \emptyset\} < \infty.$$

(3) X is said to be *well-spread* if it is both U-dense for some U and relatively separated.

BUPUs

Definition

Let U be a compact neighborhood of the identity in G, a family of functions $\{\psi_i \colon i \in I\} \subseteq C_0(G)$ is a *bounded uniform partition of unity (BUPU)* provided that

(1) $0 \leq \psi_i \leq 1$ for all $i \in I$.

(2) There is a well-spread family $\{x_i \colon i \in I\} \subseteq G$ such that

$$\operatorname{supp} \psi_i \subseteq x_i U, \ \forall i \in I.$$

(3) $\displaystyle\sum_{i \in I} \psi_i(x) \equiv 1.$

Approximating convolution

Returning now to our reproducing formula

$$\mathcal{V}_g(f) * \mathcal{V}_g(g) = \int_G \mathcal{V}_g(f)(x)\,\mathcal{V}_g(g)(x^{-1}y)\,d\mu(x) = \mathcal{V}_g(f)$$

we can write for some BUPU $\{\psi_i\}$

$$\begin{aligned} F * \mathcal{V}_g(g)(x) &= \int_G F(x)\,\mathcal{V}_g(g)(x^{-1}y)\,d\mu(x) \\ &= \sum_{i\in I} \int_{x_i U} F(x)\,\psi_i(x)\,\mathcal{V}_g(g)(x^{-1}y)\,d\mu(x) \end{aligned}$$

If U is small enough and since $V_g(g)$ is at least continuous,

$$V_g(g)(x^{-1}y) \approx V_g(g)(x_i^{-1}y)$$

on x_iU.

$$\begin{aligned} & \sum_{i\in I} \int_{x_iU} F(x)\,\psi_i(x)\,\mathcal{V}_g(g)(x^{-1}y)\,d\mu(x) \\ \approx\ & \sum_{i\in I} \left(\int_{x_iU} F(x)\,\psi_i(x)\,d\mu(x) \right) V_g(g)(x_i^{-1}y) \\ =\ & \sum_{i\in I} \langle F, \psi_i \rangle\, V_g(g)(x_i^{-1}y) \end{aligned}$$

A frame for S

- Define T_Ψ on $S = \mathrm{range}\,(\mathcal{V}_g)$ by

$$T_\Psi F(y) = \sum_{i \in I} \langle F, \psi_i \rangle\, \mathcal{V}_g(g)(x_i^{-1}y).$$

- For U small, $T_\Psi \approx Id$ so is a bounded isomorphism of S.
- We can write for $F \in S$,

$$F(y) = \sum_{i \in I} \langle T_\Psi^{-1} F, \psi_i \rangle\, \mathcal{V}_g(g)(x_i^{-1}y).$$

- It can be shown directly that for some A, $B > 0$, and all $F \in S$,

$$A\|F\|_{L^2(G)} \leq \|(\langle T_\Psi^{-1} F, \psi_i \rangle)\|_{\ell^2} \leq B\|F\|_{L^2(G)}.$$

- In other words, $\{\mathcal{V}_g(g)(x_i^{-1}y)\}$ is a frame for S.

A frame for $\mathcal{H}$

If $f \in \mathcal{H}$, we can write

$$\begin{aligned}
\langle f, \pi(y)g\rangle &= \mathcal{V}_g(f)(y) \\
&= T_\Psi(T_\Psi^{-1}\mathcal{V}_g(f))(y) \\
&= \sum_{i\in I} \langle T_\Psi^{-1}\mathcal{V}_g(f), \psi_i\rangle\, \mathcal{V}_g(g)(x_i^{-1}y) \\
&= \sum_{i\in I} \langle T_\Psi^{-1}\mathcal{V}_g(f), \psi_i\rangle\, \langle \pi(x_i)g, \pi(y)g\rangle \\
&= \left\langle \sum_{i\in I} \langle T_\Psi^{-1}\mathcal{V}_g(f)\psi_i, \rangle \pi(x_i)g, \pi(y)g \right\rangle
\end{aligned}$$

- Hence

$$f = \sum_i \lambda_i(f)\pi(x_i)g \text{ where } \lambda_i(f) = \langle T_\Psi^{-1} V_g(f), \psi_i \rangle$$

- Because we have a frame for $L^2(G)$, there are constants $A_0,\ B_0 > 0$ such that

$$A_0\|f\|_{\mathcal{H}} \leq \|(\lambda_i(f))\|_2 \leq B_0\|f\|_{\mathcal{H}}$$

  for all $f \in \mathcal{H}$.
- In other words, $\{\pi(x_i)g \colon i \in I\}$ is a frame for $\mathcal{H}$.

Banach spaces

- How can we go outside the Hilbert space setting to more general Banach spaces?
- The key is our assumption that π is integrable, that is, that there exists $g \in \mathcal{H} \setminus \{0\}$ such that

$$\int_G |\langle g, \pi(x)g \rangle| \, d\mu(x) < \infty.$$

  In other words, $\mathcal{V}_g(g) \in L^1(G)$.
- Since always $\mathcal{V}_g(g) \in L^\infty(G)$, if g satisfies the above then g is admissible.
- Define $\mathcal{H}_0 = \{g \in \mathcal{H} \colon \mathcal{V}_g(g) \in L^1(G)\}$, and note that if $g \in \mathcal{H}_0$ then the natural domain for the operator $\mathcal{V}_g$ is $(\mathcal{H}_0)'$, the dual space of $\mathcal{H}_0$.

Heisenberg group

- Recall that the voice transform $\mathcal{V}_g$ generated by the Schrödinger representation of $\mathbb{H}$ on $L^2(\mathbb{R})$ is

$$\mathcal{V}_g f(t,a,b) = \langle f, \pi(t,a,b)g\rangle = t\,\langle f, T_a M_b g\rangle = t\,V_g(f)(a,b)$$

  where here V_g is the usual short-time Fourier transform.
- Then π is clearly integrable since for any $g \in S_0$, $t\,V_g(g) \in L^1(\mathbb{H})$.
- Hence the natural domain for $\mathcal{V}_g$ with $g \in S_0$ is the dual Feichtinger algebra $S_0' = M^{\infty,\infty}$.

Affine group

- The representation π of the affine group $\mathbb{A}$ on $L^2(\mathbb{R})$ is also integrable.
- It turns out that the space of g for which $\mathcal{V}_g(g) \in L^1(\mathbb{A})$ is the so-called *minimal Besov space* $B_1^{0,1}$ defined to be those distributions in $\mathcal{S}_0'$ such that

$$\|f\| = \int_0^\infty \|\varphi_t * f\|_1 \, \frac{dt}{t} < \infty.$$

Co-orbit spaces

- Let Y be a Banach space of functions on G with the property of *solidity*, i.e., if $f \in Y$ and g satisfies $|g(x)| \leq |f(x)|$ for all $x \in G$ then $g \in Y$ and $\|g\|_Y \leq \|f\|_Y$.

Definition (Co-orbit space)

Given a solid Banach function space Y, and $g \in \mathcal{H}_0$, we define the *co-orbit space* $Co(Y)$ by

$$Co(Y) = \{f \in (\mathcal{H}_0)' : \mathcal{V}_g(f) \in Y\}$$

with norm given by $\|f\|_{Co(Y)} = \|\mathcal{V}_g f\|_Y$. $Co(Y)$ is a Banach space under this norm.

- $Co(Y)$ is independent of the choice of $g \in \mathcal{H}_0$ with equivalent norms being generated by different g.

Co-orbit spaces on $\mathbb{H}$

- For our space Y we choose the *mixed-norm space* $L^{p,q}(\mathbb{H})$ given by

$$L^{p,q}(\mathbb{H}) = \Big\{F(t,a,b) \colon \|F\|_{L^{p,q}} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{T}}|F(t,a,b)\,dt|^p\,da\right)^{q/p} db\right)^{1/q} < \infty\Big\}.$$

- In this case the co-orbit space $Co(L^{p,q})$ is the modulation space $M^{p,q}$, i.e.

$$Co(L^{p,q}) = \{f \in (S_0)'(\mathbb{R}) \colon t\,\mathcal{V}_g(f)(a,b) \in L^{p,q}\}.$$

- $Co(L^1)$ recovers the Feichtinger algebra S_0.

Affine group

- In this case we again take for Y the mixed-norm spaces and in this case,

$$
\begin{aligned}
L^{p,q}(\mathbb{A}) &= \{F(a,b)\colon \|F\|_{L^{p,q}} \\
&= \left(\int_0^\infty \left(\int_{-\infty}^\infty |F(a,b)|^q \, db \right)^{p/q} \frac{da}{a} \right)^{1/p} < \infty \Bigg\}.
\end{aligned}
$$

- In this case, the co-orbit space $Co(L^{p,q})$ is the Besov space $\dot{B}_p^{0,q}$.

Reproducing formula

Lemma

Suppose that $g \in \mathcal{H}_0$, with $\|Ag\| = 1$, and $f \in (\mathcal{H}_0)'$. Then

$$V_g(f) * V_g(g) = \int_G V_g(f)(x)\, V_g(g)(x^{-1}y)\, d\mu(x) = V_g(f).$$

- Our goal is to define Banach frames for spaces Y and $Co(Y)$.
- The idea is to discretize the reproducing formula as before utilizing BUPUs.
- In order to have a Banach frame we must specify a sequence space associated to Y and $Co(Y)$.

Definition

Given a well-spread family $X = \{x_i \colon i \in I\} \subseteq G$ and a solid, translation-invariant Banach space Y of functions on G, we define the *sequence space* $Y_d(X)$ by

$$Y_d(X) = \{(\lambda_i)_{i \in I} \colon \sum_{i \in I} \lambda_i \mathbf{1}_{x_i W} \in Y\}$$

where W is a compact subset of G with non-empty interior. The norm on $Y_d(X)$ is given by

$$\|(\lambda_i)\|_{Y_d} = \Big\| \sum_{i \in I} \lambda_i \mathbf{1}_{x_i W} \Big\|_Y .$$

- $Y_d(X)$ does not depend on W as different W will generate equivalent norms on $Y_d(X)$.
- $Y_d(X)$ also does not necessarily depend on X. For example, if $Y = L^p(G)$, then $Y_d(X) \approx \ell^p(I)$ for any well-spread family X.

A Banach frame for S

- Define T_Ψ on $S = \mathrm{range}\,(V_g)$ (a closed subspace of Y) by

$$T_\Psi F(y) = \sum_{i \in I} \langle F, \psi_i \rangle \, \mathcal{V}_g(g)(x_i^{-1} y).$$

- For U small, $T_\Psi \approx Id$ is a bounded isomorphism of S.
- We can write for $F \in S$,

$$F(y) = \sum_{i \in I} \langle T_\Psi^{-1} F, \psi_i \rangle \, T_\Psi^{-1} \mathcal{V}_g(g)(x_i^{-1} y).$$

- For some A, $B > 0$, and all $F \in S$,

$$A \|F\|_Y \leq \|(\langle T_\Psi^{-1} F, \psi_i \rangle)\|_{Y_d} \leq B \|F\|_Y.$$

- In other words, $\{\mathcal{V}_g(g)(x_i^{-1} y)\}$ is a Banach frame for S.

A Banach frame for $Co(Y)$

If $f \in Co(Y)$, we can write

$$\begin{aligned}
\langle f, \pi(y)g \rangle &= V_g(f)(y) \\
&= T_\Psi(T_\Psi^{-1} V_g(f))(y) \\
&= \sum_{i \in I} \langle T_\Psi^{-1} \mathcal{V}_g(f), \psi_i \rangle \, \mathcal{V}_g(g)(x_i^{-1} y) \\
&= \sum_{i \in I} \langle T_\Psi^{-1} \mathcal{V}_g(f), \psi_i \rangle \, \langle \pi(x_i)g, \pi(y)g \rangle \\
&= \left\langle \sum_{i \in I} \langle T_\Psi^{-1} \mathcal{V}_g(f), \psi_i \rangle \, \pi(x_i)g, \pi(y)g \right\rangle.
\end{aligned}$$

- Hence

$$f = \sum_i \lambda_i(f)\pi(x_i)g \text{ where } \lambda_i(f) = \langle T_\Psi^{-1} V_g(f), \psi_i \rangle$$

- Because we have a Banach frame for Y, there are constants $A_0, B_0 > 0$ such that

$$A_0\|f\|_{Co(Y)} \leq \|(\lambda_i(f))\|_{Y_d} \leq B_0\|f\|_{Co(Y)}$$

  for all $f \in Co(Y)$.
- In other words, $\{\pi(x_i)g \colon i \in I\}$ is a Banach frame for $Co(Y)$.

Shearlets

Definition

The *Shearlet group* $\mathbb{S}$ is given by

$$\mathbb{S} = \mathbb{R} \setminus \{0\} \times \mathbb{R} \times \mathbb{R}^2.$$

Define

$$A_a = \begin{pmatrix} a & 0 \\ 0 & sgn(a)\sqrt{|a|} \end{pmatrix} \text{ and } S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

and let $T_t f(x) = f(x-t)$ and $D_M f(x) = |\det(M)|^{-1/2} f(M^{-1}x)$ for M an invertible 2×2 matrix. Then $\mathbb{S}$ becomes a group under the operation

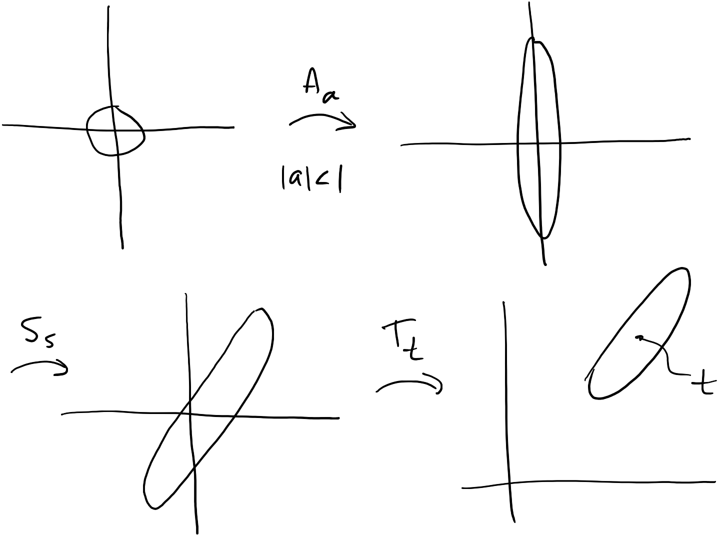
$$(a, s, t) \cdot (a', s', t') = (aa', s + s'\sqrt{|a|}, t + S_s A_a t').$$

Also $d\mu_{\mathbb{S}} = \dfrac{da}{|a|^3}\, ds\, dt$ defined Haar measure on $\mathbb{S}$.

- We define a representation π on $L^2(\mathbb{R}^2)$ by

$$\pi(a, s, t)\psi(x) = T_t D_{S_s A_a}\psi(x).$$

- Under these assuptions, the full co-orbit theory of Frichtinger and Gröchenig is applicable.


A_a
$|a|<1$
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t