Lecture 8 – Wavelets in Functional Analysis

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Outline

- Modulation spaces
- The Feichtinger Algebra $S_0(\mathbb{R})$
- Pseudodifferential operators and Gabor frames
- Wavelets as unconditional bases for Banach spaces
- Wavelets and operators

Motivation

Suppose we are given a function $f(x)$.

- How can we measure the time-frequency concentration of f?
- Given g, α, $\beta > 0$, what do the Gabor coefficients

$$c_{k,n} = \langle f, T_{\alpha k} M_{\beta n} g \rangle$$

  tell us about the smoothness and decay properties of f?
- If we can write

$$f = \sum_{k,n} \langle f, T_{\alpha k} M_{\beta n} g \rangle \, T_{\alpha k} M_{\beta n} \gamma$$

  for some *analysis window* g and *synthesis window* γ, what sort of smoothness and decay properties can g and γ share?

Modulation Spaces

Definition (Short Time Fourier Transform)

Given $g \in L^2(\mathbb{R}^d)$, we define the *short-time Fourier transform (STFT)* on $L^2(\mathbb{R}^d)$ by

$$V_g f(x,\gamma) = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i(t\cdot\gamma)}\,dt = \langle f, M_\gamma T_x g\rangle.$$

Definition (Modulation Space)

Let $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$, and $1 \leq p, q \leq \infty$. The *modulation spaace* $M^{p,q}(\mathbb{R})$ consists of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |V_g(x,\omega)|^p\,dx \right)^{q/p} d\omega \right)^{1/q}$$

and the obvious changes being made when $p = \infty$ or $q = \infty$.

- $f \in M^{p,q}$ if and only if $V_g f$ is in a so-called *mixed-norm space*, where for all ω,

$$V_g(\cdot, \omega) \in L^p(\mathbb{R})$$

and

$$\|V_g(\cdot, \omega)\|_p \in L^q(\mathbb{R}).$$

- Intuitively, p measures the *decay* of f at infinity since

$$|V_g f(x, \omega)| \leq (|f| * |g|)(x)$$

so that if $f \in L^p$, $|V_g f(\cdot, \omega)| \in L^p$ for all ω.

- q measures the *smoothness* of f in the sense that

$$V_g f(x, \omega) = (\widehat{f \cdot T_x g})(\omega)$$

will on average be in L^q for each x.

- For $1 \leq p \leq \infty$, $M^{p,q}(\mathbb{R})$ is a Banach space.
- For $1 \leq p < \infty$, the dual space of $M^{p,q}(\mathbb{R})$ is identified with the modulation space $M^{p',q'}(\mathbb{R})$ where

$$1/p + 1/p' = 1/q + 1/q' = 1.$$

- $M^{p,q}(\mathbb{R})$ is independent of the window $g \in \mathcal{S}(\mathbb{R})$ in the sense that if another such window is used, the norms generated are equivalent.
- $M^{p,q}(\mathbb{R})$ is invariant under time and frequency shifts, and $f \in M^{p,q}(\mathbb{R})$ if and only if $\widehat{f} \in M^{q,p}(\mathbb{R})$.

Feichtinger Algebra

- The modulation space $M^{1,1}(\mathbb{R}) = S_0(\mathbb{R})$ is called the *Feichtinger Algebra* (Feichtinger, 1981).
- $S_0(\mathbb{R})$ is the smallest Banach space invariant under time frequency shifts and under the Fourier transform.
- This makes $S_0 = M^{1,1}$ and its dual $(S_0)^* = M^{\infty,\infty}$ ideal substitutes for the Schwartz functions $\mathcal{S}(\mathbb{R})$ and the tempered distributions $\mathcal{S}'(\mathbb{R})$ in many instances.
- $S_0(\mathbb{R})$ is a Banach algebra under pointwise multiplication and convolution.
- $S_0(\mathbb{R})$ is the largest Banach space on which the Poisson Summation Formula holds pointwise.

S_0 and Banach frames

Definition

Given $g \in L^2(\mathbb{R})$ and $\alpha,\ \beta > 0$, the Gabor frame operator $S_{g,g}$ is defined by

$$S_{g,g}f = \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f,\, T_{\alpha k} M_{\beta n} g\rangle\, T_{\alpha k} M_{\beta n} g.$$

We denote the collection $\{T_{\alpha k} M_{\beta n} g \colon k, n \in \mathbb{Z}\}$ by $\mathcal{G}(g, \alpha, \beta)$. Recall that $S_{g,g}$ is an isomorphism of $L^2(\mathbb{R})$ if and only if the collection $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$.

Choosing g from $M^{1,1}(\mathbb{R})$ turns out to be the right choice of window class for Gabor frames.

$S_{g,g}$ on S_0

Theorem

If $g \in S_0(\mathbb{R})$, then the following are equivalent.

(1) *$S_{g,g}$ is invertible on $M^{1,1}(\mathbb{R})$.*

(2) *$S_{g,g}$ is invertible on all of the modulation spaces $M^{p,q}(\mathbb{R})$, $1 \leq p \leq \infty$.*

In this case, $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$ and the dual window $\gamma^\circ \in M^{1,1}(\mathbb{R})$ as well.

- This theorem allows us to move toward the notion of a *Banach frame* for $M^{p,q}$
- The idea is to characterize membership of $f \in M^{p,q}$ by some condition on the Gabor coefficients

$$\{\langle f, T_{\alpha k} M_{\beta n} g \rangle\}$$

where the window $g \in M^{1,1}$.

Banach frames

Definition (Gröchenig, 1991)

A sequence $\{e_n \colon n \in \mathbb{N}\}$ in a Banach space B is called a *Banach frame* if there exists an associated sequence space $B_d(\mathbb{N})$, a constant $C > 0$, and a continuous operator $R \colon B_d \to B$ such that for all $f \in B$,

(1) $\frac{1}{C}\|f\|_B \leq \|\langle f, e_n\rangle\|_{B_d} \leq C\|f\|_B$, and

(2) $R(\langle f, e_n\rangle) = f$.

Definition (Discrete mixed-norm spaces)

For $1 \leq p \leq \infty$, define $\ell^{p,q}$ to be the space of sequences $a = (a_{k,n})_{k,n\in\mathbb{Z}}$, for which

$$\|a\|_{\ell^{p,q}} = \left(\sum_{n\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|a_{k,n}|^p\right)^{q/p}\right)^{1/q}$$

is finite

Theorem

Assume that $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R})$ with $\alpha\beta \in \mathbb{Q}$ and $g \in M^{1,1}$ Then there exists $C > 0$ such that for all $1 \leq p \leq \infty$, and $f \in M^{p,q}$,

$$\frac{1}{C}\|f\|_{M^{p,q}} \leq \left(\sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\langle f, T_{\alpha k} M_{\beta n} g\rangle|^p \right)^{q/p} \right)^{1/q} \leq \|f\|_{M^{p,q}}.$$

Moreover, there exists $\gamma \in M^{1,1}$ such that $f \in M^{p,q}$ can be recovered by

$$f = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, T_{\alpha k} M_{\beta n} g\rangle \, T_{\alpha k} M_{\beta n} \gamma$$

where the series converges unconditionally in $M^{p,q}$ if $1 \leq p,\, q < \infty$ and weakly if p or $q = \infty$.

Pseudodifferential operators

Definition

Let σ be a function or distribution on $\mathbb{R}^{2d}$. Then the operator

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\widehat{f}(\omega)\, e^{2\pi i(x\cdot\omega)}\, d\omega$$

is the *pseudodifferential operator* with *symbol* σ.

- Pseudodifferential operators arose in the mid-1960s and were formally described by Kohn and Nirenberg, 1965.
- The notion arose earlier in a different context in the study of time-varying communication channels by Zadeh, 1950.

Motivation from PDE

- Consider the N-th order differential operator with nonconstant coefficients given by

$$Af(x) = \sum_{|\alpha| \leq N} a_\alpha(x)\, D^\alpha f(x).$$

- By Fourier inversion,

$$D^\alpha f(x) = \int_{\mathbb{R}^d} (2\pi i\omega)^\alpha \, \widehat{f}(\omega)\, e^{2\pi i (x\cdot\omega)}\, d\omega.$$

- $Af(x) = \displaystyle\int_{\mathbb{R}^d} \Bigg(\sum_{|\alpha| \leq N} a_\alpha(x)\, (2\pi i\omega)^\alpha \Bigg) \widehat{f}(\omega)\, e^{2\pi i (x\cdot\omega)}\, d\omega$

  which is the pseudodifferential operator with symbol

$$\sigma(x,\omega) = \sum_{|\alpha| \leq N} a_\alpha(x)\, (2\pi i\omega)^\alpha$$

Motivation from Communications

- The standard model for a time-invariant communication channel is convolution.

$$Hf(x) = \int_{\mathbb{R}} h(t)\, f(x-t)\, dt.$$

- The *impulse response h* completely characterizes the channel and does not change with time.
- In mobile communications, the impulse response can change with time, so the general model is

$$Hf(x) = \int_{\mathbb{R}} h(x,t)\, f(x-t)\, dt.$$

- Letting $\sigma(x,\omega) = \int_{\mathbb{R}} h(x,t)\, e^{-2\pi i\omega t}\, dt$, then by Fourier inversion
$$h(x,t) = \int_{\mathbb{R}} \sigma(x,\omega)\, e^{2\pi i\omega t}\, d\omega.$$
- Substituting gives

$$\begin{aligned} Hf(x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(x,\omega)\, e^{2\pi i\omega t}\, f(x-t)\, d\omega\, dt \\ &= \int_{\mathbb{R}} \sigma(x,\omega)\, e^{2\pi i\omega x} \int_{\mathbb{R}} f(t)\, e^{-2\pi i\omega t}\, dt\, d\omega \\ &= \int_{\mathbb{R}} \sigma(x,\omega)\, e^{2\pi i\omega x}\, \widehat{f}(\omega)\, d\omega. \end{aligned}$$

- Letting $\eta(t,\nu) = \int_{\mathbb{R}} h(x,t)\, e^{-2\pi i\nu x}\, dx$, then by Fourier inversion
$$h(x,t) = \int_{\mathbb{R}} \eta(t,\nu)\, e^{2\pi i\nu x}\, d\nu.$$
- Substituting gives

$$\begin{aligned} Hf(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \eta(t,\nu)\, e^{2\pi i\nu x}\, f(x-t)\, d\nu\, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \eta(t,\nu)\, M_\nu\, T_t f(x)\, dt\, d\nu \end{aligned}$$

so that H is realized as a superposition of time delays and Doppler shifts.

- $\eta(t,\nu)$ is called the *spreading function* of the operator H and measures how much a delta impulse is “spread” in time and how a pure tone is “spread” in frequency.
- Note that

$$\begin{aligned}\eta(t,\nu) &= \int_{\mathbb{R}} h(x,t)\, e^{-2\pi i\nu x}\, dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\omega)\, e^{2\pi i\omega t}\, e^{-2\pi i\nu x}\, d\omega\, dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,\omega)\, e^{-2\pi i(\nu x-\omega t)}\, d\omega\, dx\end{aligned}$$

so that the spreading function is the *symplectic Fourier transform* of the symbol.

- All of this suggests that Gabor analysis is a natural setting for studying the properties of pseudodifferential operators.
- An illustration of this is the following generalization of the Calderón-Vaillancourt theorem on the L^2-boundedness of pseudodifferential operators.

Theorem (Calderón-Vaillancourt, 1971)

Given a smooth symbol σ with bounded derivatives up to order $2d + 1$, the pseudodifferential operator with symbol σ is bounded on $L^2(\mathbb{R}^d)$.

Theorem (Gröchenig and Heil, 1999)

If $\sigma \in M^{\infty,1}(\mathbb{R}^d)$, then the pseudodifferential operator with symbol σ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$, with uniform bound

$$\|K_\sigma\|_{op} \le C\|\sigma\|_{M^{\infty,1}}.$$

In particular, K_σ is bounded on $L^2(\mathbb{R}^d) = M^{2,2}(\mathbb{R}^d)$.

- Since the space of smooth symbols with bounded derivatives up to order $2d + 1$ is embedded in $M^{\infty,1}$, this result is a generalization of the C-V Theorem.
- Note that the result assumes no smoothness on σ.