Lecture 7 – Multiresolution Analysis

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Outline

- Definition of MRA in one dimension
- Finding the wavelet from the scaling function
- The Daubechies wavelets
- MRA in higher dimensions

Multiresolution Analysis

Definition

A *multiresolution analysis* on $\mathbb{R}$ is a sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}} \subseteq L^2(\mathbb{R})$ satisfying:

(a) For all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$.

(b) $\overline{\text{span}}\{V_j\}_{j\in\mathbb{Z}} = L^2(\mathbb{R})$. That is, the set $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.

(c) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(d) A function $f(x) \in V_0$ if and only if $D_{2^j} f(x) \in V_j$.

(e) There exists a function $\varphi(x)$, L^2 on $\mathbb{R}$, called the *scaling function* such that the collection $\{T_n\varphi(x)\}$ is an orthonormal basis for V_0.

- An MRA is completely determined by the scaling function $\varphi(x)$.
- Given φ with the property that $\{T_n\varphi(x)\}$ is an orthonormal system, define the subspace V_0 by $V_0 = \overline{\text{span}}\{T_n\varphi(x)\}$, and the subspaces V_j by $V_j = D_{2^j}V_0$, that is, $f \in V_j$ if and only if $D_{2^{-j}}f \in V_0$.
- Then verify that (a)–(e) hold for this sequence of subspaces.
- The following lemma holds.

Lemma

Given $\varphi \in L^2(\mathbb{R})$, the system $\{T_n\varphi(x)\}$ is an orthonormal system if and only if

$$\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\gamma+n)|^2 \equiv 1.$$

The Haar MRA

- If we let $\varphi(x) = \mathbf{1}_{[0,1]}(x)$, then the MRA so generated is called the *Haar MRA* and leads to the construction of the Haar wavelet.
- In this case, V_0 is the space of scale-0 dyadic step functions, and clearly $\{\varphi(x-n)\colon n \in \mathbb{Z}\}$ is an orthonormal basis for V_0.
- V_j is the space of scale-j dyadic step functions.

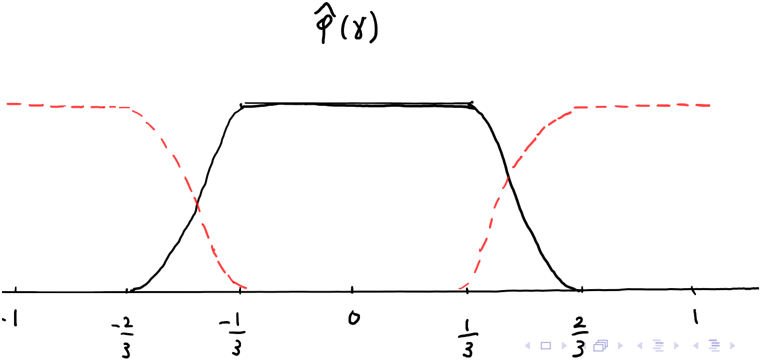
The Shannon MRA

- If we let $\varphi(x)$ be defined by $\widehat{\varphi}(\gamma) = \mathbf{1}_{[-1/2,1/2]}(\gamma)$, then the MRA so generated is called the *bandlimited MRA* and leads to the construction of the Bandlimited wavelet.
- By the Shannon Sampling Theorem,

$$\{\varphi(x-n)\colon n \in \mathbb{Z}\} = \left\{\frac{\sin \pi(x-n)}{\pi(x-n)}\colon n \in \mathbb{Z}\right\}$$

  is an orthonormal basis for V_0.
- The space V_j consists of those functions bandlimited to the interval $[-2^{j-1}, 2^{j-1}]$.

$$\widehat{\varphi}(\gamma) = \begin{cases} 0 & \text{if } |\gamma| \geq 2/3 \\ 1 & \text{if } |\gamma| \leq 1/3 \\ c(\gamma - 1/2) & \text{if } x \in (1/3, 2/3) \\ s(\gamma + 1/2) & \text{if } x \in (-2/3, -1/3) \end{cases}$$



- Because $\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\gamma+n)|^2 \equiv 1$, $\{\varphi(x-n) \colon n \in \mathbb{Z}\}$ is an orthornomal system.
- Define $V_0 = \overline{\text{span}}\{\varphi(x-n) \colon n \in \mathbb{Z}\}$.
- We can describe V_0 as follows.

$$V_0 = \left\{ f \in L^2(\mathbb{R}) \colon \widehat{f}(\gamma) = \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n\gamma}\, \widehat{\varphi}(\gamma) \colon (c_n) \in \ell^2(\mathbb{Z}) \right\}$$

Wavelets from MRA

Our goal will be to prove the following theorem.

Theorem

If $\{V_j\}$ is an MRA, then there exists a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k}\}$ is an orthonormal wavelet basis for $L^2(\mathbb{R})$.

Outline of proof.

- For each j we define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e.
$$V_{j+1} = V_j \oplus W_j.$$
- Find a function $\psi(x)$ with the property that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for the space W_0.
- Then $\{D_{2^j}T_k\psi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j.
- Finally we observe that
$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$$
so that $\{D_{2^j}T_k\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

The dilation equation

- There exists $\{h(k)\} \in \ell^2$ such that

$$\varphi(x) = \sum_k h(k)\, 2^{1/2} \varphi(2x - k).$$

  This equation is referred to as the *two-scale dilation equation* and the sequence $\{h(k)\}$ is referred to as the *scaling sequence* or *scaling filter*.

- That φ satisfies such an equation is a simple consequence of the fact that

$$\varphi \in V_0 \subseteq V_1$$

  and that

$$\{2^{1/2} \varphi(2x - k) \colon k \in \mathbb{Z}\}$$

  is an orthonormal basis for V_1.

We may write

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2),$$

where

$$m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k)\, e^{2\pi i k\gamma}$$

is called the *auxiliary function*.

$$\begin{aligned}
\widehat{\varphi}(\gamma) &= \sum_k h(k)\,(D_2 T_k \varphi)^\wedge(\gamma) \\
&= \sum_k h(k)\,(D_{1/2} M_k \widehat{\varphi})(\gamma) \\
&= \left(\sum_k h(k)\, 2^{-1/2}\, e^{2\pi i n(\gamma/2)} \right) \widehat{\varphi}(\gamma/2) \\
&= m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2).
\end{aligned}$$

Haar case.

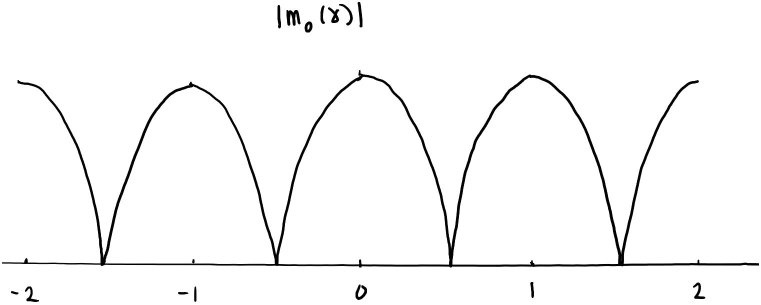
- With $\varphi(x) = \mathbf{1}_{[0,1]}(x)$,

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) = \frac{1}{\sqrt{2}}(2^{1/2}\varphi(2x) + 2^{1/2}\varphi(2x-1)).$$

- Therefore,

$$h(k) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 0,\ 1 \\ 0 & \text{otherwise} \end{cases}$$

$$m_0(\gamma) = \frac{1}{2}(1 + e^{2\pi i\gamma}) = e^{\pi i\gamma}\cos(\pi\gamma)$$



Shannon case

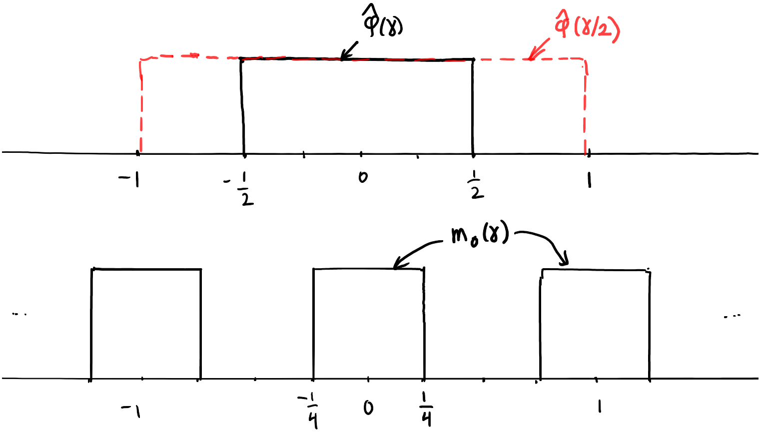
$$\widehat{\varphi}(\gamma) = \mathbf{1}_{[-1/2,1/2]}(\gamma),$$

solving

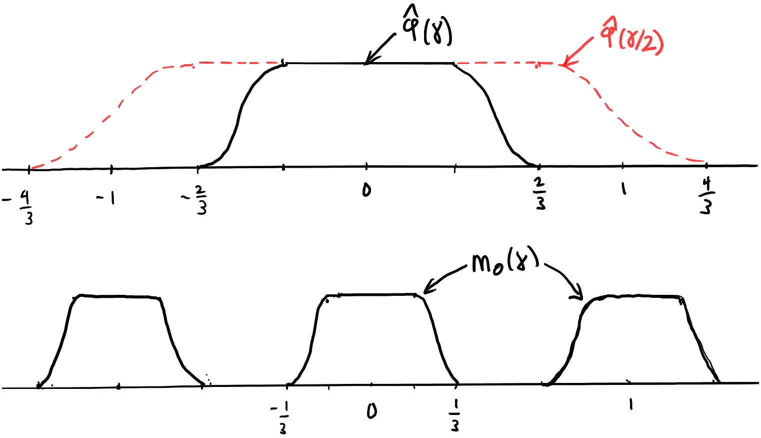
$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\widehat{\varphi}(\gamma/2)$$

yields

$$m_0(\gamma) = \mathbf{1}_{[-1/4,1/4]}(\gamma) \text{ on } [-1/2, 1/2]$$



Meyer case


$\hat{\varphi}(\gamma)$
$\hat{\varphi}(\gamma/2)$
$-\frac{4}{3}$
-1
$-\frac{2}{3}$
0
$\frac{2}{3}$
1
$\frac{4}{3}$
$m_0(\gamma)$
$-\frac{1}{3}$
0
$\frac{1}{3}$
1

The auxiliary function

Lemma

If $\{T_n\varphi(x)\}$ is an orthonormal system and if $\varphi(x)$ satisfies the two-scale dilation equation with scaling filter $\{h(k)\}$. Then the auxiliary function $m_0(\gamma)$ satisfies

$$|m_0(\gamma)|^2 + |m_0(\gamma + 1/2)|^2 \equiv 1.$$

Proof:

$$\begin{aligned}
1 &= \sum_n |\widehat{\varphi}(\gamma+n)|^2 = \sum_n \left| m_0\left(\frac{\gamma+n}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\gamma+n}{2}\right)\right|^2 \\
&= \sum_k \left| m_0\left(\frac{\gamma+2k}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\gamma+2k}{2}\right)\right|^2 \\
&\quad + \left| m_0\left(\frac{\gamma+2k+1}{2}\right)\right|^2 \left|\widehat{\varphi}\left(\frac{\gamma+2k+1}{2}\right)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k |m_0(\gamma/2+k)|^2 \, |\widehat{\varphi}(\gamma/2+k)|^2 \\
&\qquad + \sum_k |m_0(\gamma/2+1/2+k)|^2 \, |\widehat{\varphi}(\gamma/2+1/2+k)|^2 \\
&= |m_0(\gamma/2)|^2 \sum_k |\widehat{\varphi}(\gamma/2+k)|^2 \\
&\qquad + |m_0(\gamma/2+1/2)|^2 \sum_k |\widehat{\varphi}(\gamma/2+1/2+k)|^2 \\
&= |m_0(\gamma/2)|^2 + |m_0(\gamma/2+1/2)|^2.
\end{aligned}
$$

The wavelet recipe.

- We seek a function ψ such that $\{\psi(x-k)\colon k \in \mathbb{Z}\}$ is an orthonormal basis for W_0.
- Since $W_0 \subseteq V_1$,

$$\psi(x) = \sum_k g(k)\, 2^{1/2} \varphi(2x-k)$$

  or equivalently

$$\widehat{\psi}(\gamma) = m_1(\gamma/2)\, \widehat{\varphi}(\gamma/2)$$

  where

$$m_1(\gamma) = \frac{1}{\sqrt{2}} \sum_k g(k)\, e^{2\pi i k \gamma}.$$

- How does the function m_1 relate to m_0?

Given a function $f \in V_1$, we write

$$f = f_0 + g_0$$

where $f_0 \in V_0$ and $g_0 \in W_0$.

By our assumptions

$$\begin{aligned} f(x) &= \sum_k a(k)\, 2^{1/2} \varphi(2x - k), \\ f_0(x) &= \sum_k b(k)\, \varphi(x - k), \\ g_0(x) &= \sum_k c(k)\, \psi(x - k). \end{aligned}$$

Taking Fourier transforms gives

$$\begin{aligned}\widehat{f}(\gamma) &= A\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right),\\ \widehat{f_0}(\gamma) &= B(\gamma)\,\widehat{\varphi}(\gamma),\\ \widehat{g_0}(\gamma) &= C(\gamma)\,\widehat{\psi}(\gamma)\end{aligned}$$

where $A(\gamma)$, $B(\gamma)$, and $C(\gamma)$ all have period 1.

$$\begin{aligned}A\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) &= B(\gamma)\,\widehat{\varphi}(\gamma) + C(\gamma)\,\widehat{\psi}(\gamma)\\ &= B(\gamma)\,m_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) + C(\gamma)\,m_1\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)\end{aligned}$$

$$\begin{bmatrix} m_0\left(\frac{\gamma}{2}\right) & m_1\left(\frac{\gamma}{2}\right) \\ m_0\left(\frac{\gamma+1}{2}\right) & m_1\left(\frac{\gamma+1}{2}\right) \end{bmatrix} \begin{bmatrix} B(\gamma) \\ C(\gamma) \end{bmatrix} = \begin{bmatrix} A\left(\frac{\gamma}{2}\right) \\ A\left(\frac{\gamma+1}{2}\right) \end{bmatrix}$$

Lemma

If $m_1(\gamma) = e^{2\pi i(\gamma+1/2)}\,\overline{m_0(\gamma+1/2)}$ *then the matrix*

$$\begin{bmatrix} m_0\left(\frac{\gamma}{2}\right) & m_1\left(\frac{\gamma}{2}\right) \\ m_0\left(\frac{\gamma+1}{2}\right) & m_1\left(\frac{\gamma+1}{2}\right) \end{bmatrix}$$

is unitary. Moreover, if

$$m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k)\, e^{2\pi i k\gamma} \text{ and } m_1(\gamma) = \frac{1}{\sqrt{2}} \sum_k g(k)\, e^{2\pi i k\gamma}$$

then

$$g(k) = (-1)^k\, \overline{h(1-k)}.$$

Theorem

Let $\{V_j\}$ *be an MRA with scaling function* $\varphi(x)$ *and scaling filter* $h(k)$*. Define the* wavelet $\psi(x)$ *by*

$$\psi(x) = \sum_k (-1)^k \, \overline{h(1-k)} \, 2^{1/2} \, \varphi(2x - k).$$

Then

$$\{\psi_{j,k}(x) \colon j,k \in \mathbb{Z}\} = \{2^{j/2}\psi(2^j x - k) \colon j,k \in \mathbb{Z}\}$$

is a wavelet orthonormal basis on $\mathbb{R}$*.*

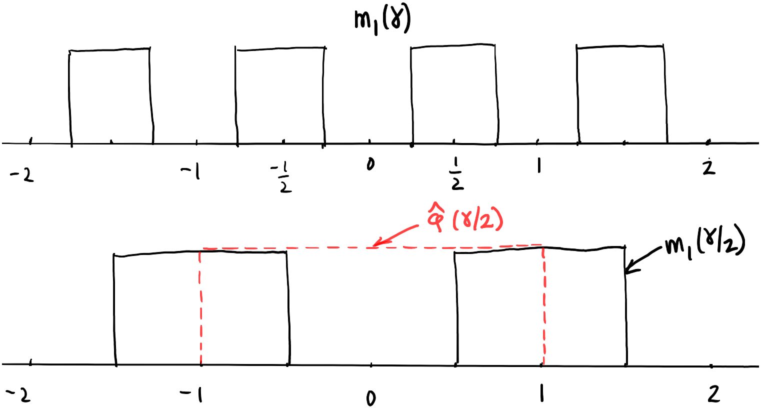
Haar wavelet

$$h(k) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad g(k) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } k = 0 \\ -\frac{1}{\sqrt{2}} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

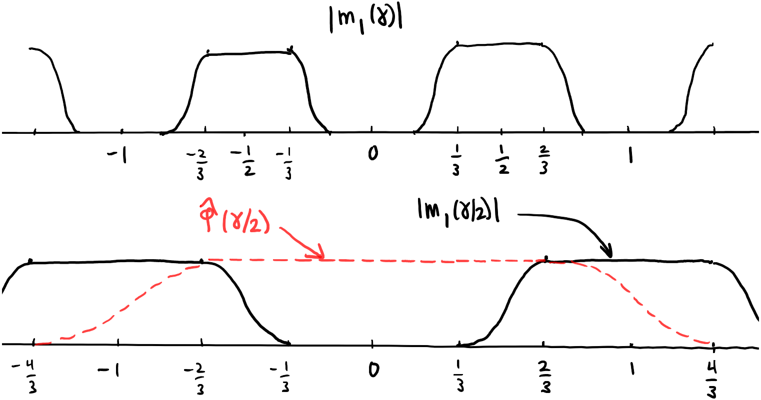
$$\psi(x) = \frac{1}{\sqrt{2}}(2^{1/2}\varphi(2x) - 2^{1/2}\varphi(2x-1)) = \mathbf{1}_{[0,1/2]}(x) - \mathbf{1}_{[1/2,1]}(x).$$

Shannon wavelet



$$\hat{\psi}(\gamma) = m_1(\gamma/2)\,\hat{\varphi}(\gamma/2)$$

Meyer wavelet



$$\hat{\psi}(\gamma) = m_1(\gamma/2)\,\hat{\varphi}(\gamma/2)$$

Smooth Compactly supported wavelets

Theorem

Let $\psi(x)$ be such that for some $N \in \mathbf{N}$, both $x^N\psi(x)$ and $\gamma^{N+1}\widehat{\psi}(\gamma)$ are in $L^1(\mathbb{R})$. If $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then $\int_{\mathbb{R}} x^m\,\psi(x)\,dx = 0$ for $0 \le m \le N$.

- Theorem says that smooth wavelets have vanishing moments.
- We also want wavelets with compact support, which means that $m_0(\gamma)$ is a polynomial.

Theorem

Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA with finite scaling filter $h(n)$. Let $\psi(x)$ be the corresponding wavelet. Then for each $N \in \mathbf{N}$,

$$\int_{\mathbb{R}} x^k\, \psi(x)\, dx = 0 \text{ for } 0 \le k \le N-1$$

if and only if $m_0(\gamma)$ can be factored as

$$m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma)$$

for some period 1 trigonometric polynomial $\mathcal{L}(\gamma)$.

Daubechies's Strategy.

- We seek a trig polynomial $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k)\, e^{-2\pi i k\gamma}$ satisfying
$$m_0(\gamma) = \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N \mathcal{L}(\gamma).$$
and the QMF conditions.
- We have
$$|m_0(\gamma)|^2 = \left|\frac{1+e^{-2\pi i\gamma}}{2}\right|^{2N} |\mathcal{L}(\gamma)|^2 = \cos^{2N}(\pi\gamma)\, L(\gamma).$$
- Since $L(\gamma)$ is a real-valued trig polynomial with real coefficients, we arrive at $L(\gamma) = P(\sin^2(\pi\gamma))$ for some polynomial P.

- This polynomial P must satisfy

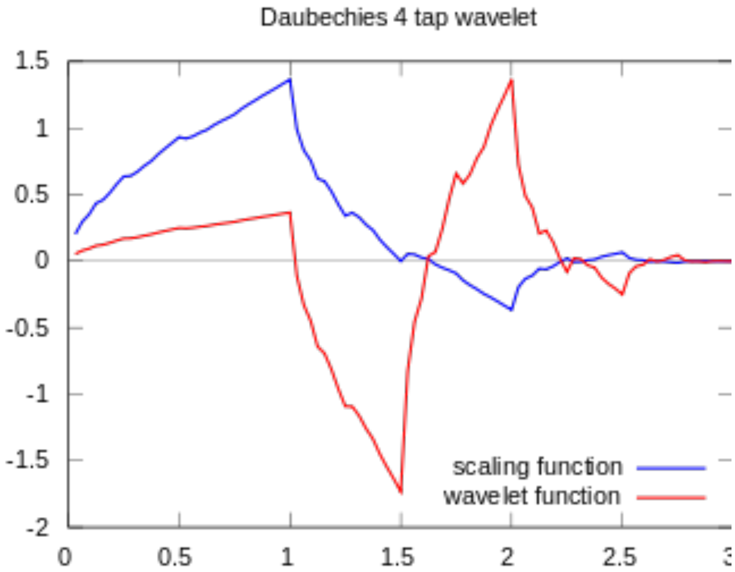
$$1 = (1-y)^N P(y) + y^N P(1-y)$$

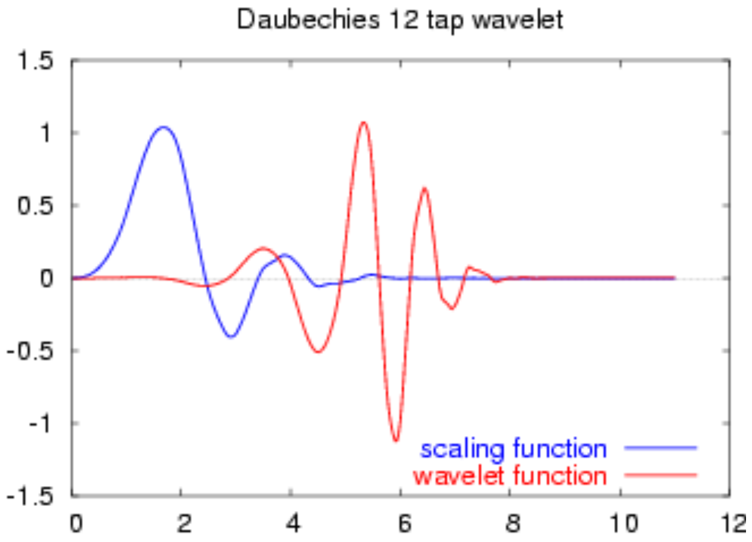
with $P(y) \geq 0$ for all $0 \leq y \leq 1$. and we arrive at

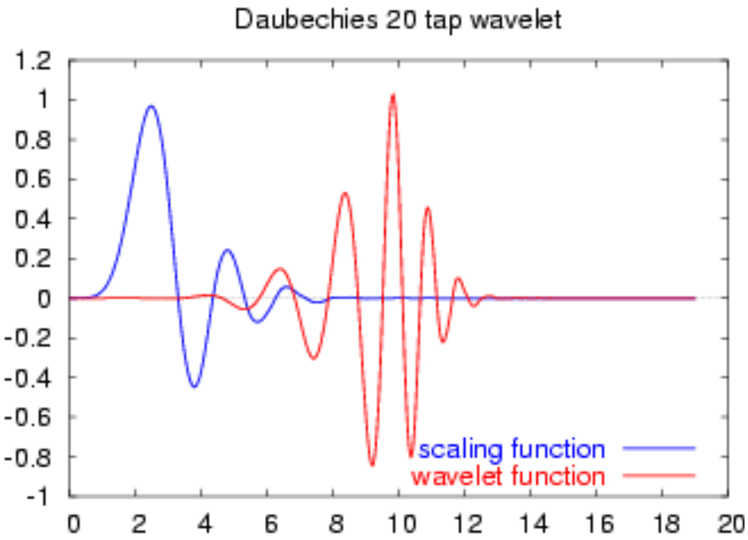
$$P_{N-1}(y) = \sum_{k=0}^{N-1} \binom{2N-1}{k} y^k (1-y)^{N-1-k}.$$

- For example,

$$\begin{aligned} P_0(y) &= 1, \\ P_1(y) &= 1 + 2y, \\ P_2(y) &= 1 + 3y + 6y^2, \\ P_3(y) &= 1 + 4y + 10y^2 + 20y^3. \end{aligned}$$


Daubechies 4 tap wavelet
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
scaling function
wavelet function


Daubechies 12 tap wavelet
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
scaling function
wavelet function


Daubechies 20 tap wavelet
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
2
4
6
8
10
12
14
16
18
20
scaling function
wavelet function