Lecture 2 – Gabor Analysis

David Walnut
Department of Mathematical Sciences
George Mason University
Fairfax, VA USA

Chapman Lectures, Chapman University, Orange, CA
6-10 November 2017

Outline

- D. Gabor's notion of *information area*
- Discretizing the Short-Time Fourier Transform
- Stability in time-frequency representations
- The Balian-Low Theorem
- Proof of the BLT
- The Amalgam BLT (Heil)
- The Zak transform and proof.

Information Area (Gabor, 1946)

Theorem (Shannon, Whittaker, Kotelnikov, et al.)

Given $f \in L^2(\mathbb{R})$ *such that*

$$\hat{f}(\gamma) = 0 \text{ for } |\gamma| \geq \Omega/2$$

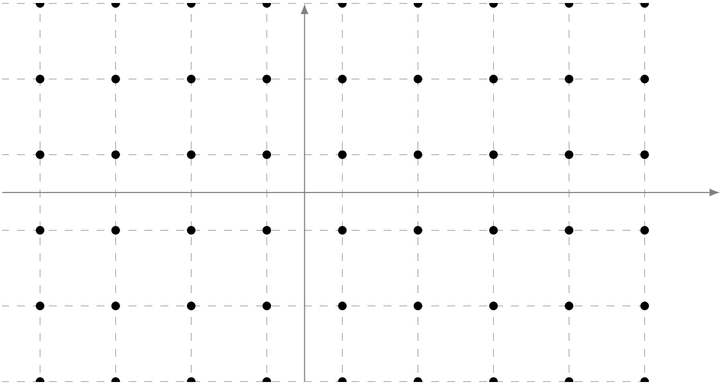
for some $\Omega > 0$*, then*

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{\Omega}\right) \frac{\sin \pi(\Omega t - n)}{\pi(\Omega t - n)}.$$

- A function with bandwidth Ω possesses Ω independent samples per unit time.
- Through a channel with bandwidth Ω, at most Ω independent data values can be transmitted in each unit of time.

Information Area (Gabor, 1946)

- Each independent "packet" of data occupies a region in "time–frequency space" of area 1.



[F]or every type of resonator a characteristic rectangle *of about unit area can be defined in the time/frequency diagram, which corresponds to one "practically" independent reading of the instrument. In order to obtain their number, we must divide up the (time* $\times$ *frequency) area into such rectangles.... The number of these rectangles in any region is the number of independent data which the instrument can obtain from the signal, i.e., proportional to the amount of information. This justifies calling the diagram from now on the "*diagram of information.*"*

– D. Gabor, *Theory of Communication* (1946)

Gabor representation

- We seek a Shannon–type representation of arbitrary functions $f(x)$ in which each of the building blocks occupies a distinct rectangle in time–frequency space of unit area.
- For $\Omega > 0$, let $g(t) = \Omega\, e^{-\pi(t\Omega)^2}, \quad \hat{g}(\omega) = \dfrac{1}{\Omega} e^{-\pi(\omega/\Omega)^2}$.
- Define for $n,\ m \in \mathbb{Z}$,

$$g_{n/\Omega, m\Omega}(t) = g\left(t - \frac{n}{\Omega}\right)\, e^{2\pi i m\Omega t} = T_{\frac{n}{\Omega}} M_{m\Omega} g(t).$$

- Since $g(t)$ *minimizes* the Uncertainty Principle inequality each $g_{n/\Omega, m\Omega}$ occupies a rectangle of unit area in time–frequency space centered at $(n/\Omega, m\Omega)$.
- So we take

$$f(t) \sim \sum_{n,m\in\mathbb{Z}} \langle f, g_{n/\Omega, m\Omega}\rangle\, g_{n/\Omega, m\Omega}(t).$$

- Does the series satisfactorily represent $f(t)$?

What does "satisfactorily" mean?

- **Completeness:** $f(x)$ is uniquely determined by the coefficients $\langle f, g_{n/\Omega, m\Omega} \rangle$.
- **Minimality:** Any representation of $f(x)$ as $f = \sum_{n,m} a_{n,m}\, g_{n/\Omega, m\Omega}$ is unique.
- **Stability:** There exist constants A, $B > 0$ such that for all $f \in L^2(\mathbb{R})$,

$$A \left\| f \right\|^2 \leq \sum_n \left| \langle f, g_{n/\Omega, m\Omega} \rangle \right|^2 \leq B \left\| f \right\|^2 .$$

  This inequality is called the *frame condition*.
- **Time-Frequency Localization:**

$$\left(\int |t\, g(t)|^2 \, dt \right) \times \left(\int |\omega\, \hat{g}(\omega)|^2 \, d\omega \right) < \infty.$$

More on Stability

- If $\{g_{n/\Omega,m\Omega}\}$ is not stable, then either $A = 0$ or $B = \infty$.
- $A = 0$ means there are signals $\mu \in L^2(\mathbb{R})$ such that
  - $\|\mu\|_2 = 1$ and
  - $\sum_{n,m} |\langle \mu, g_{n/\Omega,m\Omega} \rangle|^2 \approx 0$.
- This means that two very different signals f can have almost the same expansion coefficients.
- $B = \infty$ means there are signals $\eta \in H$ such that
  - $\|\eta\| = 1$ and
  - $\sum_{n} |\langle \eta, g_{n/\Omega,m\Omega} \rangle|^2 >>> 1$.
- This means that even if $\|\eta\|_2$ is small, $f + \eta$ can have completely different expansion coefficients than f.
- Therefore, *noise* may be amplified in calculating the expansion coefficients.

The Balian–Low Theorem (1981)

Theorem

Given $\Omega > 0$ *and any function* $h(t) \in L^2(\mathbb{R})$*, define*

$$h_{n/\Omega, m\Omega}(t) = h\left(t - \frac{n}{\Omega}\right)\, e^{2\pi i m\Omega t}.$$

If $\{h_{n/\Omega, m\Omega}\}_{n,m\in\mathbb{Z}}$ *satisfies the frame condition, then*

$$\left(\int_{-\infty}^{\infty} |t\, h(t)|^2\, dt\right) \cdot \left(\int_{-\infty}^{\infty} |\omega\, \hat{h}(\omega)|^2\, d\omega\right) = \infty.$$

That is, $h(t)$ *maximizes the Uncertainty Principle.*

Theorem

Given $\Omega, \Omega' > 0$ *and any function* $h(t) \in L^2(\mathbb{R})$*, define*

$$h_{n/\Omega, m\Omega'}(t) = h\left(t - \frac{n}{\Omega}\right)\, e^{2\pi i m\Omega' t}.$$

- *If* $\Omega' > \Omega$ *then* $\{h_{n/\Omega, m\Omega'}\}_{n,m\in\mathbb{Z}}$ *is not complete in* $L^2(\mathbb{R})$ *and hence does not satisfy the frame condition.*
- *If* $\Omega' < \Omega$*, then there exist* $h(t) \in L^2$ *such that*
  - $\left(\int_{-\infty}^{\infty} |t\, h(t)|^2\, dt\right) \cdot \left(\int_{-\infty}^{\infty} |\omega\, \hat{h}(\omega)|^2\, d\omega\right) < \infty$*, and*
  - $\{h_{n/\Omega, m\Omega'}\}_{n,m\in\mathbb{Z}}$ *satisfies the frame condition, but is* overcomplete*.*

Proof of the BLT

- Define the *position*, X, and *momentum*, P, operators as follows
$$Xf(x) = x\, f(x), \quad Pf(x) = \frac{1}{2\pi i}\, f'(x)$$
and note that they are self-adjoint.
- The original formulation of the BLT took a slightly weaker form: *If* $\{g_{n,m}\colon n, m \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *then* $\|Xg\|_2\, \|Pg\|_2 = \infty$.
- This proof (due to G. Battle, 1988) relies on the fact that the operators X and P do not commute, specifically that
$$PX - XP = \frac{1}{2\pi i} I$$
on a dense subset of $L^2(\mathbb{R})$.

- Since $\{g_{n,m} \colon n, m \in \mathbb{Z}\}$ is an orthonormal basis, $\forall f,\ h \in L^2(\mathbb{R})$,

$$\langle f, h\rangle = \sum_{n,m} \langle f, T_n M_m g\rangle \overline{\langle h, T_n M_m g\rangle}.$$

- Consequently, if $g,\ Xg,\ Pg \in L^2(\mathbb{R})$ then

$$\langle Xg, Pg\rangle = \sum_{n,m} \langle Xg, T_n M_m g\rangle \overline{\langle Pg, T_n M_m g\rangle}.$$

- Using orthogonality, we calculate

$$\langle Xg, T_n M_m g\rangle = \langle T_{-n} M_{-m} g, Xg\rangle,$$

$$\langle Pg, T_n M_m g\rangle = \langle T_{-n} M_{-m} g, Pg\rangle.$$

- Substituting we obtain

$$\begin{aligned}\langle Xg, Pg\rangle &= \sum_{n,m} \langle Xg, T_n M_m g\rangle \overline{\langle Pg, T_n M_m g\rangle} \\ &= \sum_{n,m} \langle Pg, T_{-n} M_{-m} g\rangle \overline{\langle Xg, T_{-n} M_{-m} g\rangle} \\ &= \langle Pg, Xg\rangle.\end{aligned}$$

- Finally we arrive at the contradiction

$$0 = \langle Xg, Pg\rangle - \langle Pg, Xg\rangle = \langle (PX - XP)g, g\rangle = \frac{1}{2\pi i}\|g\|_2^2.$$

Amalgam BLT

- A different version of the BLT is due to Heil (1990), and requires the following definition.

Definition

The *Wiener amalgam space* $W(C_0, \ell^1)$ is defined to be the space of continuous functions on $\mathbb{R}^d$ satisfying

$$\|f\|_{W(C_0,\ell^1)} = \sum_k \|f \cdot \mathbf{1}_{Q+k}\|_\infty < \infty$$

where $Q = [0,1]^d$ denotes the unit square in $\mathbb{R}^d$.

- The Wiener amalgam space combines a local property (continuity) with a global decay property (summability of the local L^∞ norms).

Theorem (Amalgam BLT)

If $\{g_{n,m}\}$ satisfies the frame condition then

$$g \notin W(C_0, \ell^1)$$

and

$$\widehat{g} \notin W(C_0, \ell^1).$$

Zak Transform

Definition

The *Zak transform* of a function $f \in L^2(\mathbb{R})$ is given by

$$Zf(t,\omega) = \sum_{k\in\mathbb{Z}} f(t+k)\, e^{2\pi i k\omega}.$$

- Zf is *quasiperiodic*, i.e.,
  - $Zf(t+1,\omega) = e^{-2\pi i\omega}\, Zf(t,\omega)$, and
  - $Zf(t,\omega+1) = Zf(t,\omega)$.

  Zf is completely determined by its values on $Q = [0,1] \times [0,1]$.
- *Inversion formula:* Given $f \in L^1(\mathbb{R})$,
  - $f(x) = \int_0^1 Zf(x,\omega)\, d\omega$, and
  - $\widehat{f}(\omega) = \int_0^1 Zf(x,\omega)\, e^{-2\pi i x\omega}\, dx$.
- $Z \colon L^2(\mathbb{R}) \to L^2(Q)$ is a unitary operator.

Zak Transform and Gabor systems

Lemma

Given any function $h(t) \in L^2(\mathbb{R})$, *define*

$$h_{n,m}(t) = h(t-n)\ e^{2\pi imt}.$$

Then

$$Z(h_{n,m})(t,\omega) = e^{2\pi imt}\ e^{2\pi in\omega} Zh(t,\omega).$$

- This identity is the main property that makes Zak transform so useful in studying Gabor systems.
- The Zak transform diagonalizes the time-shift, T_n, and frequency-shift, M_m, operators by mapping them to multiplication operators.

Zak Transform and Gabor systems

Theorem

Given $h(t) \in L^2(\mathbb{R})$, the following hold.

- $\{h_{n,m}\}_{n,m\in\mathbb{Z}}$ *is* complete *if and only if* $Zh \neq 0$*, a.e.,*
- $\{h_{n,m}\}_{n,m\in\mathbb{Z}}$ *is* minimal *if and only if* $1/(Zh) \in L^2(Q)$*, and*
- $\{h_{n,m}\}_{n,m\in\mathbb{Z}}$ *satisfies the frame inequality with bounds* $A, B > 0$ *if and only if*

$$A \leq |Zh(x,\omega)|^2 \leq B, \ \text{a.e.}$$

Proof of Amalgam BLT

The proof of this theorem follows easily from the following property of the Zak transform.

Lemma

If Zf is continuous on $\mathbb{R}^2$ then Zf has a zero in Q.

- The proof uses only the quasiperiodicy of Zf.
- If Zf does not vanish then we can write

$$Zf(x,\omega) = |Zf(x,\omega)|\, e^{-2\pi i\varphi(x,\omega)}$$

where φ is *continuous* on $\mathbb{R}^2$.
- Now using quasiperiodicity, we have for all x, $\omega \in \mathbb{R}$,

$$\varphi(1,\omega) = -\omega + k + \varphi(0,\omega), \;\; \varphi(x,1) = n + \varphi(x,0)$$

where the integers k and n do not depend on (x,ω).

- Next we calculate

$$\begin{aligned} 0 &= \varphi(0,0) - \varphi(1,0) + \varphi(1,0) - \varphi(1,1) \\ &\qquad + \varphi(1,1) - \varphi(0,1) + \varphi(0,1) - \varphi(0,0) \\ &= -k - n - 1 + k + n = -1. \end{aligned}$$

Proof of Amalgam BLT

- If $g \in W(C_0, \ell^1)$ then clearly Zg is continuous and hence also has a zero in Q. Hence $\{g_{n,m}\}$ cannot satisfy the frame condition.
- If $\widehat{g} \in W(C_0, \ell^1)$ then since $Z\widehat{g}(x, \omega) = e^{-2\pi i x \omega} Zg(\omega, -x)$, the same argument applies.
- The preceding lemma easily generalizes to $\mathbb{R}^d$, so the Amalgam BLT holds for higher-dimensional Gabor systems as well.
- It is interesting to note that the property $g, \widehat{g} \notin W(C_0, \ell^1)$ neither implies nor is implied by the property that $\|xg(x)\|_2 \|\omega\widehat{g}(\omega)\|_2 = \infty$.