Lecture 1 – Some Time-Frequency Transformations

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Outline

- What is a Time-Frequency distribution/representation?
- The Wigner Distribution
- The Short-Time Fourier Transform
- The Ambiguity Function
- Uncertainty Principles

Frequency representation

- Suppose we are given some function $f \in L^2(\mathbb{R}^d)$.
- The Fourier transform allows us to realize f as a superposition of pure frequencies, viz.

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle x,\gamma \rangle}\, dx, \quad f(x) = \int_{\mathbb{R}^d} \widehat{f}(\gamma)\, e^{2\pi i \langle x,\gamma \rangle}\, d\gamma.$$

- Parseval's formula says this realization preserves energy.

$$\int_{\mathbb{R}^d} |f(x)|^2\, dx = \int_{\mathbb{R}^d} |\widehat{f}(\gamma)|^2\, d\gamma.$$

- The Fourier transform of f gives a frequency *distribution* of f as well as a *frequency representation* of f.

- It is a distribution in the sense that on any set $S \subseteq \mathbb{R}^d$, the quantity
$$\int_S |\widehat{f}(\gamma)|^2 \, d\gamma$$
represents the proportion of the function's energy in that frequency band.
- Also the Fourier Inversion gives a representation of f in terms of its frequency content.
- A mathematically analogous interpretation takes x as the *position variable* and γ as the *momentum variable*.

What is a Time-Frequency distribution/representation?

- In the context of acoustic signals $f(t)$, it has long been recognized that we do not experience the frequency content of a signal through the usual Fourier paradigm.
- Rather than experiencing a signal as either its *time representation*, $f(t)$, or its *frequency representation*, $\widehat{f}(\gamma)$, a signal consists of different frequencies at different times.
- A simple metaphor for the representation of an acoustic signal jointly in terms of time and frequency is in a musical score (D. Gabor, 1949; de Bruijn, 1965).
- The notion of time-frequency distributions have arisen in the signal processing and quantum physics communities.

Fur Elise
Clavierstuck in A Minor - WoO 59
Ludwig Van Beethoven
Poco moto.
pp
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- When a particular frequency (note) is to be played at a particular time, the composer puts a dot at that place on the time (horizontal) and frequency (vertical) axis.

- The musical score can be thought of as
  - the *energy distribution* of the signal f in time and frequency, and
  - as a *representation* of f in time and frequency variables.
- Adding the square of the intensities of each note in a certain time and frequency range gives the energy in the signal in that region of time-frequency space.
- At the same time, the signal can be reproduced from the musical score.

- We seek to generalize the idea of the musical score (L. Cohen, 1966, 1989).
- For a given $f(x)$, we seek $P(x,\gamma)$, a joint distribution giving the intensity of f at time x and frequency γ.
- What properties do we require that P satisfy?

- We require that P satisfy the *marginals*:

$$\int_{-\infty}^{\infty} P(x,\gamma)\, d\gamma = |f(x)|^2 \text{ and } \int_{-\infty}^{\infty} P(x,\gamma)\, dx = |\widehat{f}(\gamma)|^2.$$

- The energy of f in a given time range $[t_0 - \epsilon, t_0 + \epsilon]$ is

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \int_{-\infty}^{\infty} P(x,\gamma)\, d\gamma\, dx = \int_{t_0-\epsilon}^{t_0+\epsilon} |f(x)|^2\, dx.$$

- The energy of f in a given frequency range $[\gamma_0 - \delta, \gamma_0 + \delta]$ is

$$\int_{\gamma_0-\delta}^{\gamma_0+\delta} \int_{-\infty}^{\infty} P(x,\gamma)\, dx\, d\gamma = \int_{\gamma_0-\delta}^{\gamma_0+\delta} |\widehat{f}(\gamma)|^2\, d\gamma.$$

- And obviously,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,\gamma)\, d\gamma\, dx = \|f\|_2^2.$$

- We would further like to say that the energy of f in the time-frequency region

$$[t_0 - \epsilon, t_0 + \epsilon] \times [\gamma_0 - \delta, \gamma_0 + \delta]$$

is

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \int_{\gamma_0-\delta}^{\gamma_0+\delta} P(x,\gamma)\, d\gamma\, dx.$$

- From a quantum physics perspective, we would like $P(x, \gamma)$ to behave as a true probability distribution.
- The expected value of an observable G is given by

$$\langle Gf, f \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, \gamma)\, P(x, \gamma)\, d\gamma\, dx$$

  where the operator G is given as the function g of the position and momentum operators, i.e., $G = g(X, (-1/2\pi i)d/dx)$, $Xf(x) = xf(x)$.
- These considerations led to the invention of the Wigner distribution.

Definition (Wigner Distribution)

Given $f \in L^2(\mathbb{R}^d)$, the *Wigner distribution* of f, denoted $Wf(x,\gamma)$ is defined by

$$Wf(x,\gamma) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\,\overline{f(x-\frac{t}{2})}\,e^{-2\pi i(\gamma\cdot t)}\,dt.$$

Given f, $g \in L^2(\mathbb{R}^d)$, the *cross-Wigner distribution* is given by

$$W(f,g)(x,\gamma) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\,\overline{g(x-\frac{t}{2})}\,e^{-2\pi i(\gamma\cdot t)}\,dt.$$

- Introduced by E. Wigner in 1932 in the context of quantum mechanical measurements.
- Starting with J. Ville in 1948 it has become popular and useful in signal analysis.

Basic properties of $W(f,g)$

Theorem

The following hold.

- *If* $f,\ g \in L^2(\mathbb{R}^d)$*, then* $W(f,g) \in L^\infty \cap L^2(\mathbb{R}^d)$ *is uniformly continuous on* $\mathbb{R}^{2d}$*, and* Wf *is real-valued.*
- *If* $f_1,\ f_2,\ g_1,\ g_2 \in L^2(\mathbb{R}^d)$*,* Moyal's formula *holds, i.e.*

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\,\overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}.$$

- *If* $f,\ \widehat{f} \in L^1 \cap L^2(\mathbb{R}^d)$ *then* $Wf(x,\omega)$ *satisfies the marginals, i.e.*
  - $\displaystyle\int_{-\infty}^{\infty} Wf(x,\gamma)\, d\omega = |f(x)|^2,$
  - $\displaystyle\int_{-\infty}^{\infty} Wf(x,\gamma)\, dt = |\widehat{f}(\gamma)|^2,$

- If $f(0) \neq 0$ then

$$f(x) = \frac{1}{\overline{f(0)}} \int_{\mathbb{R}^d} Wf(\frac{x}{2}, t)\, e^{2\pi i \langle t, x \rangle}\, dt.$$

- This follows from the fact that the coordinate change

$$(x, y) \mapsto \left(x + \frac{y}{2}, x - \frac{y}{2} \right)$$

is inverted by

$$(x, y) \mapsto \left(\frac{x + y}{2}, x - y \right)$$

and then applying Fourier inversion.

Positivity of *Wf*.

- In order to be a true energy density function, we would like to say that $Wf(x,\gamma) \geq 0$ for all $f \in L^2$ and $(x,\gamma) \in \mathbb{R}^{2d}$. This only holds for functions of a special form, the *generalized gaussians*.
- However, positivity for *Wf* can be achieved by taking averages, i.e., $Wf * \sigma$. This fact can be interpreted as a form of the uncertainty principle.

Theorem

For $a,\ b > 0$ let

$$\sigma_{a,b}(x,\gamma) = e^{-2\pi\left(\frac{x^2}{a}+\frac{\gamma^2}{b}\right)}.$$

Then

- $Wf * \sigma_{a,b} > 0$ *for all* $f \in L^2(\mathbb{R}^d)$ *if and only if* $ab > 1$.
- $Wf * \sigma_{a,b} \geq 0$ *for all* $f \in L^2(\mathbb{R}^d)$ *if and only if* $ab \geq 1$.

- This can be interpreted as saying that the values of $Wf(x,\gamma)$ at a particular point (x,γ) or even in a set $U \subseteq \mathbb{R}^{2d}$ of small area do not have a clear physical meaning.

The Short-Time Fourier Transform

- From the late 1940s, communication engineers took a different (but related) approach to understanding the joint time-frequency content of a signal, the *spectrogram*.

Definition

Given $a, b \in \mathbb{R}^d$, we define for $f \in L^2(\mathbb{R}^d)$ the

- *translation operator*, or *time shift operator*, $T_a f(x) = f(x - a)$, and the
- *modulation operator*, or *frequency shift operator*, $M_b f(x) = e^{2\pi i (b \cdot x)} f(x)$.

Given $g \in L^2(\mathbb{R}^d)$, we define the *short-time Fourier transform (STFT)* on $L^2(\mathbb{R}^d)$ by

$$V_g f(x, \gamma) = \int_{\mathbb{R}^d} f(t)\, \overline{g(t - x)}\, e^{-2\pi i (t \cdot \gamma)}\, dt = \langle f, M_\gamma T_x g \rangle.$$

- V_g is also referred to as the *voice transform* or the *windowed Fourier transform* with g being the *window function*.
- $V_g f(x,\gamma) = \widehat{(f)(T_x\bar{g})}(\gamma)$.
- For fixed window g, the function $|V_g f(x,\gamma)|^2$ is called the *spectrogram* of f.

Properties of the STFT

- For $f,\ g \in L^2$, $V_g f(x,\gamma)$ is uniformly continuous on $\mathbb{R}^{2d}$ and bounded with
$$\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2.$$
- For fixed $g \in L^2$, $V_g \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ and moreover
$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}$$
so that if $\|g\| = 1$, V_g is an isometry.
- Moyal's formula is satisfied, i.e., if $f_1,\ f_2,\ g_1,\ g_2 \in L^2(\mathbb{R}^d)$ then
$$\langle V_{g_1}(f_1), V_{g_2}(f_2)\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2\rangle}_{L^2(\mathbb{R}^d)}.$$
- V_g is invertible with
$$f(t) = \frac{1}{\|g\|_2} \iint_{\mathbb{R}^{2d}} V_g f(x,\gamma)\, M_\gamma T_x g(t)\, d\gamma\, dt.$$

The STFT as a TF Distribution

- Many of the fundamental properties of the Wigner distribution follow from those of the STFT by observing that

$$W(f, g^{-})(x, \gamma) = 2^{d}\, e^{4\pi i(x\cdot\gamma)}\, V_{g}f(2x, 2\gamma)$$

where $g^{-}(x) = g(-x)$.

- Note that although

$$Wf(x, \gamma) = 2^{d}\, e^{4\pi i(x\cdot\gamma)}\, V_{f^{-}}f(2x, 2\gamma),$$

the time-frequency distribution $V_{f}f(x, \gamma)$ does not satisfy the marginals.

Radar Ambiguity Function

Definition

The *(radar) ambiguity function* of $f \in L^2(\mathbb{R}^d)$ is given by

$$Af(x,\gamma) = \int_{\mathbb{R}^d} f(t+\frac{x}{2})\,\overline{f(t-\frac{x}{2})}\,e^{-2\pi i(\gamma\cdot t)}\,dt.$$

and given $g \in L^2(\mathbb{R}^d)$, the *cross-ambiguity function* by

$$A(f,g)(x,\gamma) = \int_{\mathbb{R}^d} f(t+\frac{x}{2})\,\overline{g(t-\frac{x}{2})}\,e^{-2\pi i(\gamma\cdot t)}\,dt.$$

- The radar ambiguity function is due to Woodward, 1953.

Why “radar”?

- Suppose that a test signal $f(t)$ is sent from a radar apparatus a distance r from the source and moving at velocity v from the source.
- The echo of f is received with time-shift Δt (proportional to the range) and Doppler shift $\Delta\gamma$ (proportional to the velocity), i.e.
$$e(t) = M_{\Delta\gamma} T_{\Delta t} f.$$
- The echo is compared to arbitrary time-frequency shifts of f, and we compute
$$|\langle e, M_\omega T_x f\rangle| = |Af(x - \Delta t, \omega - \Delta\gamma)|$$
- Maximizing this quantity over all (x, ω) gives estimates for $(\Delta t, \Delta\gamma)$.

Uncertainty Principles

- The *uncertainty principle* can be loosely formulated as follows.

  > *A function and its Fourier transform cannot simultaneously be concentrated on small sets.*

- An alternate but related formulation says the following.

  > *The time-frequency distribution of any function cannot be concentrated on a small set in the time-frequency plane. This is independent of which time-frequency distribution is used.*

Classical Uncertainty Principle

Theorem

Let $f \in L^2(\mathbb{R})$ and let $a, b \in \mathbb{R}$ be given. Then

$$\left(\int_{\mathbb{R}} (x-a)^2\, |f(x)|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}} (\gamma-b)^2\, |\widehat{f}(\gamma)|^2\, d\gamma\right)^{1/2} \geq \frac{1}{4\pi} \|f\|_2^2.$$

Equality holds if and only if $f(x)$ is a time-frequency shift of a gaussian $\varphi(x) = e^{-\pi x^2/c}$.

- If $\|f\|_2 = 1$ then $|f(x)|^2$ and $|\widehat{f}(\gamma)|^2$ can be thought of as probability distributions, and the quantities

$$\min_{a\in\mathbb{R}} \left(\int_{\mathbb{R}} (x-a)^2\, |f(x)|^2\, dx\right)^{1/2}, \ \min_{\gamma\in\mathbb{R}} \left(\int_{\mathbb{R}} (\gamma-b)^2\, |\widehat{f}(\gamma)|^2\, d\gamma\right)^{1/2}$$

as their standard deviations and hence as a measure of concentration of $|f|^2$ and $|\widehat{f}|^2$ about their means.

Proof of the Classical UP

Lemma

Let A and B be self-adjoint operators on a Hilbert space. Then

$$\|(A-a)f\|\|(B-b)f\| \geq \frac{1}{2}|\langle [A,B]f, f\rangle|$$

for all $a,\ b \in \mathbb{R}$, and f in the domain of AB and BA, where $[A,B] = AB - BA$ is the commutator of A and B.

Proof:

$$\begin{aligned} \langle [A,B]f, f\rangle &= \langle [(A-a),(B-b)]f, f\rangle \\ &= \langle (B-b)f, (A-a)f\rangle - \langle (A-a)f, (B-b)f\rangle \\ &= 2i\, Im\langle (B-b)f, (A-a)f\rangle. \end{aligned}$$

Applying Cauchy-Schwarz yields the result.

Apply the above Lemma to the position and momentum operators,

$$Xf(x) = x\, f(x), \text{ and } Pf(x) = \frac{1}{2\pi i} f'(x)$$

which are self-adjoint and whose commutator is

$$[X, P]f = \frac{1}{2\pi i}(xf'(x) - (xf)'(x)) = -\frac{1}{2\pi i} f(x).$$

Finally observe that

$$\|(X - a)f\|_2 = \left(\int_{\mathbb{R}^d} (x - a)^2 |f(x)|^2\, dx\right)^{1/2}$$

and

$$\|(P - b)f\|_2 = \|(\widehat{P - b)f}\|_2 = \left(\int_{\mathbb{R}^d} (\gamma - b)^2 |\widehat{f}(\gamma)|^2\, d\gamma\right)^{1/2}$$

Uncertainty Principle of Donoho and Stark

Theorem (D. Donoho-P. Stark, 1989)

Suppose that for some sets T, $\Omega \subseteq \mathbb{R}^d$, $f \in L^2(\mathbb{R}^d)$ satisfies

$$\left(\int_{T^c} |f(x)|^2\, dx\right)^{1/2} < \epsilon_T \|f\|_2, \text{ and } \left(\int_{\Omega^c} |\widehat{f}(\gamma)|^2\, d\gamma\right)^{1/2} < \epsilon_\Omega \|\widehat{f}\|_2.$$

Then

$$|T||\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2.$$

- We say that under the hypotheses of the theorem f and $\widehat{f}$ are ϵ_T (resp. ϵ_Ω) *concentrated* on T (resp. Ω).
- The theorem can be interpreted as saying that any function must essentially occupy an area of at least one in time-frequency space.

Uncertainty Principle for the STFT

Theorem (Lieb 1990, Gröchenig 2001)

Let $\|f\|_2 = \|g\|_2 = 1$ *and suppose that for some* $\epsilon > 0$ *and* $U \in \mathbb{R}^{2d}$*,*

$$\iint_U |V_g f(x,\gamma)|^2 \, dx \, d\gamma \geq 1 - \epsilon.$$

Then $|U| \geq 2^d (1-\epsilon)^2$*.*