SOME QUASIVARIETIES OF COMPLEX ALGEBRAS THAT ARE VARIETIES

PETER JIPSEN

For many classes $\mathcal{K}$ of structures one finds that the **SP** closure of the class $\mathbf{Cm}\mathcal{K}$ of complex algebras of members of $\mathcal{K}$ is also closed under **H**. If $\mathcal{K}$ is closed under ultraproducts, e.g. an elementary class, then $\mathbf{SPCm}\mathcal{K}$ is a quasivariety (since $\mathbf{P_u Cm}\mathcal{K} \subseteq \mathbf{CmP_u}\mathcal{K}$). So the above observation says that the quasivariety generated by $\mathcal{K}$ is a variety. In this note we find some conditions on the class $\mathcal{K}$ for which this is always the case. The main result explains for example why **SPCm**(semilattices) is a variety whereas **SPCm**(semigroups) is just a quasivariety.

Let $(f_i : i < \kappa)$ be a sequence of function symbols, each with some fixed arity. For brevity we often write $f(x_0, \ldots, x_{n-1}) = f(\underline{x})$, where f is assumed to have arity n.

Definition 1. *Let $\mathbf{A} = (A, f_i^{\mathbf{A}} : i < \kappa)$ be a universal algebra. For each $f_i^{\mathbf{A}} : A^n \to A$ define $\hat{f}_i : \mathcal{P}(A)^n \to \mathcal{P}(A)$ by*

$$\hat{f}_i(\underline{X}) = \{f_i(\underline{x}) : x_j \in X_j \text{ for all } j < n\}.$$

The *full complex algebra* $\mathrm{Cm}\mathbf{A}$ is defined as $(\mathcal{P}(A), \cup, \sim, \emptyset, \hat{f}_i : i < \kappa)$. Here $\sim$ denotes unary complementation with respect to the largest set A. Note that $\hat{f}$ should really be denoted $\hat{f}^{\mathrm{Cm}\mathbf{A}}$, but we will usually drop the superscripts on both f and $\hat{f}$. If f is a constant (0-ary function symbol) then the above definition reduces to $\hat{f} = \{f\}$. When confusion is unlikely, we also omit the ˆ on operation symbols and identify singleton sets with their unique element.

Full complex algebras are examples of complete and atomic Boolean algebras with normal operators. A *complex algebra* is any subalgebra of a full complex algebra. When considering them in the abstract setting of BAOs, the Boolean operations will be denoted by $+, \cdot, -, 0, 1$.

Actually the notion of a complex algebra is usually defined more generally for relational structures $\mathbf{U} = (U, R_i : i < \kappa)$. In this case one defines

$$\hat{f}_i(\underline{X}) = \{y \in U : R_i(x_1, \ldots, x_n, y) \text{ where } x_j \in X_j \text{ for } j = 1 \ldots n\}.$$

Let $\mathcal{K}$ be a class of structures, and define

$$\mathbf{Cm}\mathcal{K} = \{\mathrm{Cm}\mathbf{A} : \mathbf{A} \in \mathcal{K}\}.$$

As usual, $\mathbf{V}(\mathcal{C})$ denotes the variety or equational class generated by a class of algebras $\mathcal{C}$. Recall that $\mathbf{V}(\mathcal{C}) = \mathbf{HSP}\mathcal{C}$, where $\mathbf{H}\mathcal{C}$, $\mathbf{S}\mathcal{C}$, $\mathbf{P}\mathcal{C}$ are the class of homomorphic images, all subalgebras and all products of members of $\mathcal{C}$.

Problem 2. *Under what conditions on $\mathcal{K}$ is $\mathbf{VCm}\mathcal{K} = \mathbf{SPCm}\mathcal{K}$?*

Recall that a map $h : \mathbf{W} \to \mathbf{U}$ is a relational structure homomorphism if $R_i^{\mathbf{W}}(x_0, \dots, x_n)$ implies $R_i^{\mathbf{U}}(h(x_0), \dots, h(x_n))$ for all $x_0, \dots, x_n \in W$ and all $i < \kappa$. Such a map h is called a *bounded morphism* if, in addition, it also satisfies the condition: for all $i < \kappa$, all $y_0, \dots, y_{n-1} \in U$ and $x_n \in W$

$R_i^{\mathbf{U}}(y_0, \dots, y_{n-1}, h(x_n))$ implies there exist $x_0, \dots, x_{n-1} \in W$ such that $h(x_j) = y_j$ for $j < n$ and $R_i^{\mathbf{W}}(x_0, \dots, x_n)$.

A substructure $\mathbf{W}$ of a structure $\mathbf{U}$ is an *inner substructure* if the injection map is a bounded morphism (i.e. $R_i^{\mathbf{U}}(x_0 \dots x_n)$ and $x_n \in W$ imply $x_0 \dots x_{n-1} \in W$).

For a class $\mathcal{K}$ of structures, $\mathbb{H}_b\mathcal{K}$ denotes the class of all bounded morphic images and $\mathbb{S}_b\mathcal{K}$ denotes the class of all inner substructures of $\mathcal{K}$. Given a structure $\mathbf{U}$ and a set $X \subseteq U$, we define $\mathrm{Sg}_b^{\mathbf{U}}(X)$ to be the intersection of all inner substructures of $\mathbf{U}$ that contain X, and since inner substructures are closed under intersections, this deserves to be called the *inner substructure generated by* X. For $u \in U$ the *one-generated inner substructure* $\mathrm{Sg}_b^{\mathbf{U}}(\{u\})$ is also denoted by $\mathrm{Sg}_b^{\mathbf{U}}(u)$.

The next result is wellknown, and explains the universal algebraic significance of one-generated inner substructures.

Lemma 3. *Let* $\mathbf{W}$ *be an inner substructure of* $\mathbf{U}$. *Then* $\mathbf{W}$ *is one-generated if and only if* $\mathrm{Cm}\mathbf{W}$ *is subdirectly irreducible.*

Proof. The congruence relations on $\mathrm{Cm}\mathbf{U}$ are in one-to-one correspondence with so-called *congruence ideals*, which are boolean ideals that satisfy an additional condition. In the case of a complex algebra $\mathrm{Cm}\mathbf{U}$, this condition is most easily described in terms of a binary relation ρ defined on U by

$x\rho y$ iff $x = y$ or there exist $\underline{x} \in U^{n+1}$, $i < \kappa$, $j < n$ such that $R_i^{\mathbf{U}}(\underline{x})$, $x = x_j$ and $y = x_n$.

As usual, let $\rho[X] = \{y \in U : x\rho y \text{ for some } x \in X\}$, ρ^{-1} denotes the inverse of ρ, ρ^n the composition of n copies of ρ, and ρ^* the reflexive transitive closure of ρ.

A boolean ideal I of $\mathrm{Cm}\mathbf{U}$ is a *congruence ideal* if for all $X \in I$ we have $\rho[X] \in I$. The congruence ideal generated by an element X in $\mathrm{Cm}\mathbf{U}$ is denoted by $\mathrm{ci}(X)$, and can be computed from below by the formula

$$\mathrm{ci}(X) = \bigcup_{k<\omega} \mathcal{P}(\rho^k[X]).$$

The inner substructure generated by a set $X \subseteq U$ is given by $\mathrm{Sg}_b^{\mathbf{U}}(X) = \rho^{*-1}[X]$. So $\mathbf{W}$ is a one-generated inner substructure of $\mathbf{U}$ iff $W = \rho^{*-1}[\{u\}]$ for some $u \in U$. We now observe that $w \in \rho^{*-1}[\{u\}]$ iff $w\rho^k u$ for some $k < \omega$ iff $\{u\} \in \mathrm{ci}(\{w\})$ iff $\mathrm{ci}(\{u\}) \subseteq \mathrm{ci}(\{w\})$.

This shows that $\mathbf{W}$ is one-generated by $u \in \mathbf{U}$ iff $\mathrm{ci}(\{u\})$ is a minimal nontrivial congruence ideal of $\mathrm{Cm}\mathbf{W}$, which means that $\mathrm{Cm}\mathbf{W}$ is subdirectly irreducible. $\square$

The following result is a dual version of Birkhoff's subdirect embedding theorem.

Lemma 4. *Any structure is a bounded morphic image of the disjoint union of its one-generated inner substructures, i.e.* $\mathbf{U} \in \mathbb{H}_b\mathbb{U}_d\{\mathrm{Sg}_b^{\mathbf{U}}(u) : u \in U\}$.

Proof. Let $\mathbf{W} = \bigcup\{\mathrm{Sg}_b^{\mathbf{U}}(u) \times \{u\} : u \in U\}$ be a disjoint union of the one-generated inner substructures of $\mathbf{U}$, and define $f : \mathbf{W} \twoheadrightarrow \mathbf{U}$ by $f((x,u)) = x$. Now $R_i^{\mathbf{W}}((x_0,u_0),\dots,(x_n,u_n))$ holds iff $u_j = u_k$ for $j,k \le n$ and $R_i^{\mathrm{Sg}_b^{\mathbf{U}}(u)}(x_0,\dots,x_n)$. The forward implication shows that f is a structure homomorphism. On the other hand, if $x_0,\dots,x_{n-1} \in U$, $(x_n,u) \in V$, and $R_i^{\mathrm{Sg}_b^{\mathbf{U}}(u)}(x_0,\dots,x_n)$, then $R_i^{\mathbf{W}}((x_0,u),\dots,(x_n,u))$ and $f((x_j,u)) = x_j$, hence f is a bounded morphism. $\square$

From the duality between structures with bounded morphisms and complex algebras with homomorphisms, we have the following corollary:

Corollary 5. *For any structure* $\mathbf{U}$, $\mathrm{Cm}\mathbf{U} \in \mathbf{SP}\{\mathrm{Cm}(\mathrm{Sg}_b^{\mathbf{U}}(u)) : u \in U\}$.

A class $\mathcal{K}$ is said to be *closed under one-generated inner substructures* if $\mathrm{Sg}_b^{\mathbf{U}}(u) \in \mathcal{K}$ for all $\mathbf{U} \in \mathcal{K}$ and all $u \in \mathbf{U}$. The next result was originally proved by Yde Venema [Ve] for the class of semilattices.

Theorem 6. *Let $\mathcal{K}$ be a class of structures that is closed under ultraproducts and one-generated inner substructures. Then* $\mathbf{VCm}\mathcal{K} = \mathbf{SPCm}\mathcal{K}$.

Proof. Suppose $\mathcal{K}$ is closed under ultraproducts and one-generated inner substructures, and let $\mathbf{A} \in \mathbf{VCm}\mathcal{K} = \mathbf{HSPCm}\mathcal{K}$. This means there is an algebra $\mathbf{B}$ and structures $\mathbf{U}_i \in \mathcal{K}$ such that $\mathbf{B}$ is a subalgebra of $\prod_{i\in I}\mathrm{Cm}\mathbf{U}_i = \mathrm{Cm}\sum_{i\in I}\mathbf{U}_i$, and $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$. From the duality between BAOs and structures (cf. [Go89]), the canonical structure $\mathrm{Cs}\mathbf{A}$ is an inner substructure of $\mathrm{Cs}\mathbf{B}$, which in turn is a bounded homomorphic image of $\mathrm{CsCm}\sum_{i\in I}\mathbf{U}_i$. By the Fine–van Benthem–Goldblatt Theorem [Go89] 3.6.1, $\mathrm{CsCm}\sum_{i\in I}\mathbf{U}_i$ is a bounded homomorphic image of an ultrapower of $\sum_{i\in I}\mathbf{U}_i$, and by Theorem 2.1.(13) of [Go95], this ultrapower is a bounded morphic image of a disjoint union of ultraproducts of the family $\{\mathbf{U}_i : i \in I\}$. Since $\mathcal{K}$ is closed under ultraproducts, we have shown that $\mathrm{Cs}\mathbf{B}$ is a bounded morphic image of a disjoint union of members of $\mathcal{K}$, say $h : \sum_{j\in J}\mathbf{V}_j \twoheadrightarrow \mathrm{Cs}\mathbf{B}$.

Let $W_j = \{w \in V_j : h(w) \in \mathrm{Cs}\mathbf{A}\}$. Then $\sum_{j\in J}\mathbf{W}_j = h^{-1}[\mathrm{Cs}\mathbf{A}]$, and each $\mathbf{W}_j$ is an inner substructure of $\mathbf{V}_j$ since preimages of inner substructures and components of disjoint unions are again inner substructures. By the preceding lemma, for each $j \in J$, $\mathbf{W}_j$ is a bounded morphic image of one-generated inner substructures which, by assumption, are in $\mathcal{K}$ (since they are one-generated inner substructures of $\mathcal{V}_j \in \mathcal{K}$). In summary, we have shown that $\mathrm{Cs}\mathbf{A} \in \mathbb{H}_b\mathbb{U}_d\mathcal{K}$. Applying the duality again, we get $\mathrm{CmCs}\mathbf{A} \in \mathbf{SPCm}\mathcal{K}$. Since $\mathbf{A}$ is a subalgebra of its canonical extension $\mathrm{CmCs}\mathbf{A}$, we finally obtain $\mathbf{A} \in \mathbf{SPCm}\mathcal{K}$. $\square$

If $\mathcal{K}$ is a universal class of algebras, we also prove a converse to the above result.

Theorem 7. *Let $\mathcal{K}$ be a universal class of algebras (i.e. closed under subalgebras and ultraproducts). Then* $\mathbf{VCm}\mathcal{K} = \mathbf{SPCm}\mathcal{K}$ *if and only if $\mathcal{K}$ is closed under one-generated inner substructures.*

Proof. Suppose $\mathcal{K}$ is a class of algebras such that $\mathbf{S}\mathcal{K} = \mathcal{K}$ and $\mathbf{VCm}\mathcal{K} = \mathbf{SPCm}\mathcal{K}$, and assume that $\mathbf{U} \in \mathcal{K}$, but for some $u \in U$, $\mathbf{W} = \mathrm{Sg}_b^{\mathbf{U}}(u) \notin \mathcal{K}$. Since $\mathrm{Cm}\mathbf{U}$ maps homomorphically onto $\mathrm{Cm}\mathbf{W}$, we have $\mathrm{Cm}\mathbf{W} \in \mathbf{HCm}\mathcal{K} \subseteq \mathbf{VCm}\mathcal{K} = \mathbf{SPCm}\mathcal{K}$. By Lemma 3 $\mathrm{Cm}\mathbf{W}$

is subdirectly irreducible, hence $\mathrm{Cm}\mathbf{W} \in \mathbf{SCm}\mathcal{K}$. As $\mathcal{K}$ is a class of algebras, for each $i < \kappa$ the class $\mathbf{SCm}\mathcal{K}$ satisfies the universal formula

$$X_0 \neq \emptyset, \ldots, X_{n-1} \neq \emptyset \Rightarrow \hat{R}_i(X_0, \ldots, X_{n-1}) \neq \emptyset.$$

Now, because $\mathcal{K}$ is assumed to be closed under subalgebras, we have $\mathbf{W} \notin \mathbf{S}\mathcal{K}$. It follows that $\mathbf{W}$ is a substructure but not a subalgebra of $\mathbf{U}$, so there are $x_0, \ldots, x_{n-1} \in W$ and $i < \kappa$ such that $R_i^{\mathbf{W}}(x_0, \ldots, x_{n-1}, x)$ does not hold for any $x \in W$. But this contradicts the fact that $\mathrm{Cm}\mathbf{W} \in \mathbf{SCm}\mathcal{K}$ since if we take $X_j = \{x_j\}$, then the the universal formula above does not hold in $\mathrm{Cm}\mathbf{W}$.

The reverse direction follows immediately from the previous theorem. □

It is easy to check that the varieties of groups, rings, lattices and Boolean algebras are closed under one-generated inner substructures, since their members have no proper inner substructures. Hence the following classes are varieties: $\mathbf{SPCm}$(groups), $\mathbf{SPCm}$(lattices), and $\mathbf{SPCm}$(Boolean algebras). For groups this result is wellknown since it produces the class of group relation algebras. The class of all semilattices is also closed under one-generated inner substructures, since they are the principal filters in a meet semilattice. Therefore $\mathbf{SPCm}$(semilattices) is a variety. However, the class of semigroups is not closed under one-generated inner substructures, as can be seen by examining the constant 2-element semigroup. It follows that $\mathbf{SPCm}$(semigroups) is not closed under $\mathbf{H}$ (but it is a quasivariety).

References

[Go89] R. Goldblatt, *Varieties of Complex Algebras*, Ann. Pure Applied Logic 44 (1989), 173–242.

[Go95] R. Goldblatt, *Elementary Generation and Canonicity for Varieties of Boolean Algebras with Operators*, Algebra Universalis 34 (1995) No. 4, 551–607.

[Ve] Y. Venema, private communication.