AN OVERVIEW OF GENERALIZED BASIC LOGIC ALGEBRAS

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Classical propositional logic is one of the earliest formal systems of logic, with its origins in the work of Boole and De Morgan. The algebraic semantics of this logic is given by Boolean algebras. Both, the logic and the algebraic semantics have been generalized in many directions over the last 150 years. In this talk we primarily take the algebraic point of view, but we will also use the powerful framework of algebraic logic to clarify the close relationship between algebra and logic.

In various applications (such as fuzzy logic) the properties of Boolean conjunction are too stringent, hence a new binary connective $\cdot$, usually called *fusion*, is introduced. In Boolean algebra the relationship between conjunction and implication is given by the *residuation equivalences*

$$x \wedge y \leq z \iff x \leq y \to z \iff y \leq x \to z.$$

These equivalences imply many of the properties of $\wedge$ and $\to$ (such as commutativity of $\wedge$, distributivity of $\wedge$ over $\vee$, left-distributivity of $\to$ over $\vee$ and right-distributivity of $\to$ over $\wedge$) so one wishes to retain some aspects of these equivalences, but with conjunction replaced by fusion. To avoid imposing commutativity on the fusion operation one has to introduce two implications:

$$x \cdot y \leq z \iff x \leq y \to z \iff y \leq x \rightsquigarrow z.$$

Since the residuals $\to$, $\rightsquigarrow$ are generalized division operations, it is convenient to use the following alternative notation:

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

Thus $\backslash$ is another name for $\rightsquigarrow$, and $x \to y = y/x$. One advantage is that this notation allows the formulation of a simple *mirror image principle*: Any statement about residuated structures has an equivalent mirror image obtained by reading terms backwards (i.e. replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$), hence it suffices to state results in only one form.

In the most general setting a *residuated poset* $\langle P, \cdot, \backslash, /, \leq\rangle$ is a partially ordered set $\langle P, \leq\rangle$ with three binary operations that satisfy the above equivalences. In many applications to logic there exists additional structure, such as a constant 1 to denote *true*, and suprema and infima for finite subsets of P. Since fusion is usually assumed to be at least associative, we settle on a common foundation of Dilworth's *residuated lattices* [Di39], i.e., algebras of the form $\langle L, \vee, \wedge, \cdot, 1, \backslash, /\rangle$ such that

- $\langle L, \vee, \wedge\rangle$ is a lattice (i.e. $\vee$, $\wedge$ are commutative, associative and mutually absorbtive),
- $\langle L, \cdot, 1\rangle$ is a monoid (i.e. $\cdot$ is associative, with identity element 1),
- $xy \leq z \iff x \leq z/y$ and $xy \leq z \iff y \leq x\backslash z$ for all $x, y, z \in L$.

Since the first two properties are defined by identities, it is important to note that this can also be done for the third: the equivalences are captured by $x(x\backslash z \wedge y) \leq z$, $y \leq x\backslash(xy \vee z)$ and their mirror images. Note that in the absence of parenthesis, we assume that $\cdot$ is performed first, followed by $\backslash, /$ and finally $\vee, \wedge$. Moreover we use the notation $s \leq t$ as abbreviation for the equivalent identity $s = s \wedge t$. Hence residuated lattices form a *variety* (also called *equational class*), denoted by RL. A general introduction to varieties of universal algebras can be found in [BS81].

Note that in particular we do **not** assume that L is bounded or that 1 is the top element or that $\cdot$ is commutative. These additional assumptions are handled by expanding the language with an additional constant 0, and/or adding further identities.

A residuated lattice with a constant 0 (which can denote any element) is called a *full Lambek algebra* or FL-algebra for short, and the variety of all such algebras is denoted by FL. A good introduction to FL and its associated logic can be found in [On03].

Much of the research on generalizations of BL-algebras can be viewed as investigations of certain subvarieties of FL and RL. We list here some of the important ones, together with their various names.

Some subvarieties of FL.

- FL_w-*algebras* [On03]: FL-algebras that satisfy $0 \leq x$ and $x \leq 1$.
- FL_e-*algebras* [On03]: FL-algebras that satisfy $x \cdot y = y \cdot x$ (or equivalently $x\backslash y = y/x$). In this case one usually writes $x \to y$ instead of $x\backslash y$ or y/x.
- DFL = *distributive* FL-*algebras*: FL-algebras that satisfy $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
- RFL = *representable* FL-*algebras*: FL-algebras that are subdirect products of linearly ordered FL-algebras, or equivalently satisfy the identity $1 \leq u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$.
- psMTL = *pseudo monoidal t-norm algebras, or weak-pseudo-BL algebras* [FGI01]: FL_w-algebras that satisfy prelinearity ($x\backslash y \vee y\backslash x = 1$ and $x/y \vee y/x = 1$).
- FL_{ew}-*algebras* [KO01]: algebras that are both FL_e-algebras and FL_w-algebras.
- MTL = *monoidal t-norm algebras* [EG01]: FL_{ew}-algebras that satisfy prelinearity.
- psBL = *pseudo* BL-*algebras* [FGI01], [DGI02]: FL_w-algebras that satisfy divisibility ($x \wedge y = x(x\backslash y) = (y/x)x$).
- BL = *basic logic algebras* [Ha98]: MTL-algebras that satisfy divisibility, or equivalently, commutative prelinear pseudo BL-algebras.
- HA = *Heyting algebras* [BD74]: FL-algebras that satisfy $x \wedge y = xy$, or equivalently FL_w-algebras that are idempotent ($xx = x$).
- psMV = *pseudo* MV-*algebras* [GI01]: pseudo BL-algebras that satisfy $x \vee y = x/(y\backslash x) = (x/y)\backslash x$.
- MV = *multi-valued logic algebras, or Łukasiewicz algebras* [CDM00]: BL-algebras that satisfy $\neg\neg x = x$ or equivalently, commutative pseudo MV-algebras.
- GA = *Gödel logic algebras, or linear Heyting algebras* [Ha98]: BL-algebras that satisfy $x \cdot x = x$ or equivalently, Heyting algebras that satisfy prelinearity.

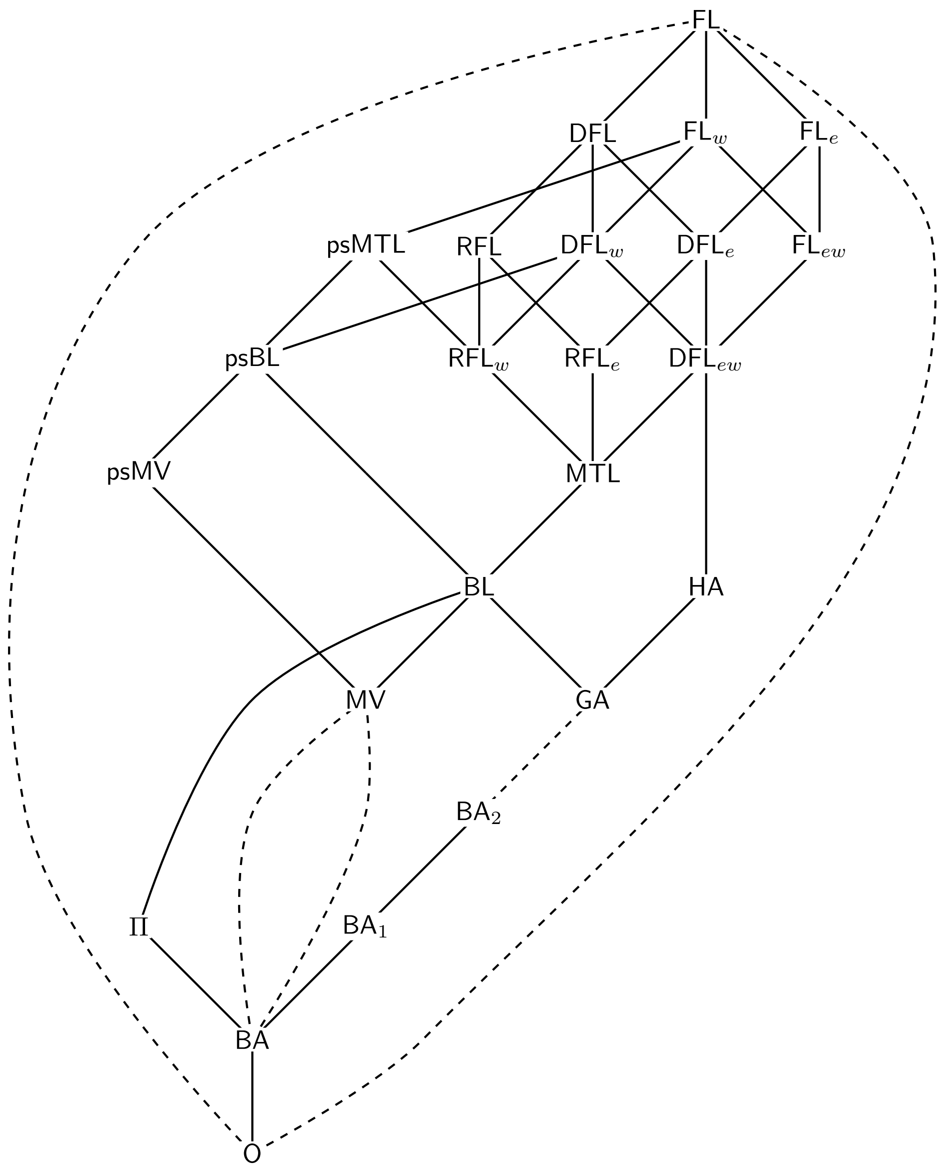

Figure 1. Some subvarieties of FL ordered by inclusion

- Π = *product logic algebras* [Ha98] [Ci01]: BL-algebras that satisfy $\neg\neg x \le (x \to xy) \to y(\neg\neg y)$.
- BA = *Boolean algebras*: Heyting algebras that satisfy $\neg\neg x = x$, or equivalently, MV-algebras that are idempotent ($xx = x$). BA_n = subdirect products of the linearly ordered $n+2$-element Heyting algebra.

Some subvarieties of RL.

- DRL = *distributive residuated lattices*: Residuated lattices that satisfy $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
- IRL = *integral residuated lattices*: Residuated lattices that satisfy $x \le 1$.

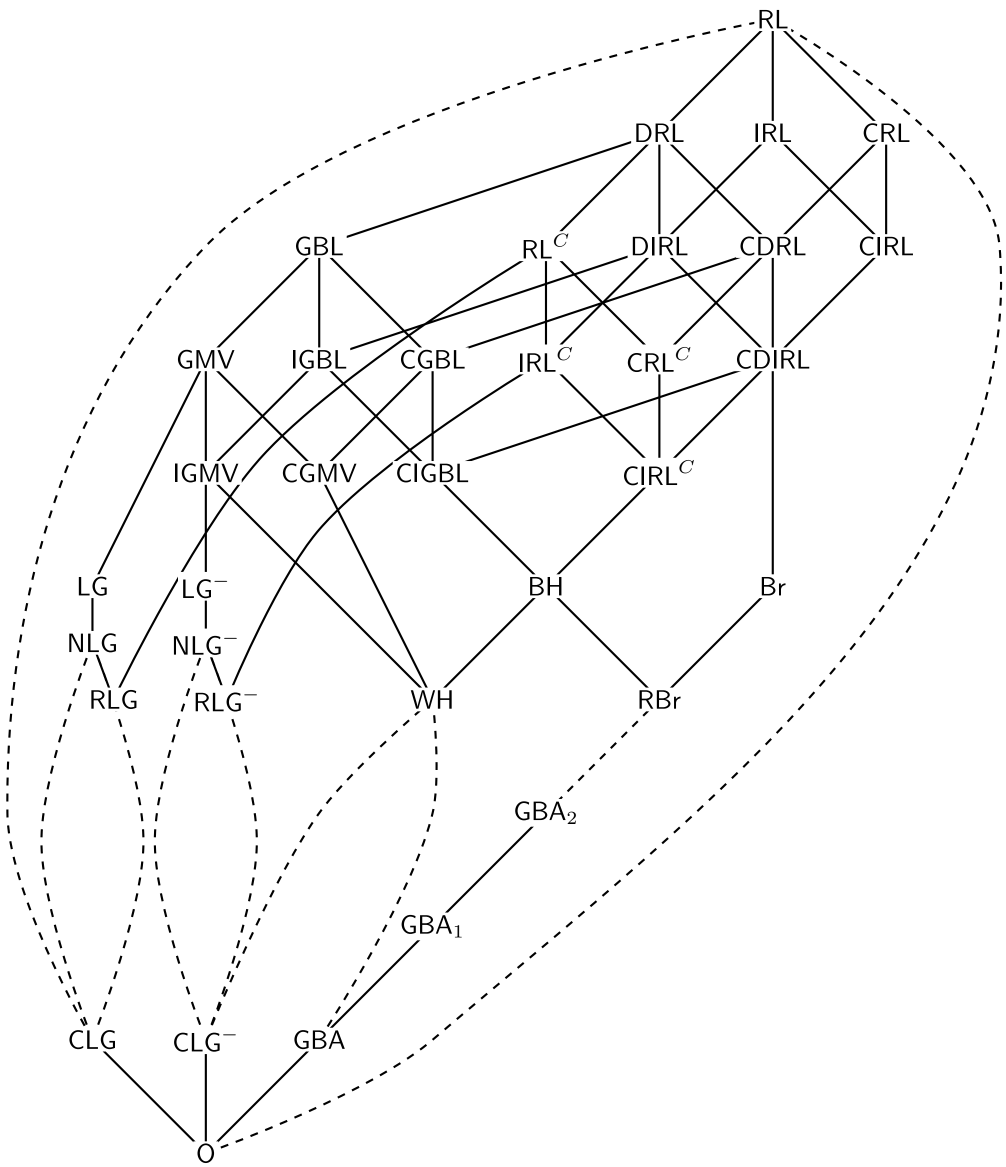

FIGURE 2. Some subvarieties of RL ordered by inclusion

- CRL = *commutative residuated lattices* [HRT02]: Residuated lattices that satisfy $x \cdot y = y \cdot x$ (or equivalently $x\backslash y = y/x$). In this case one usually writes $x \to y$ instead of $x\backslash y$ or y/x.
- RL^C = *representable residuated lattices* [BT]: Residuated lattices that are subdirect products of residuated chains, or equivalently satisfy the identity $1 \le u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$.
- GBL = *generalized* BL-*algebras* [JT02]: Residuated lattices that satisfy $x \wedge y = x(x\backslash(x \wedge y)) = ((x \wedge y)/x)x$.

- GMV = *generalized* MV-*algebras* [JT02] [Ga03]: Residuated lattices that satisfy $x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$.
- Fleas [Ha]: Integral residuated lattices that satisfy prelinearity ($x\backslash y \vee y\backslash x = 1$ and $x/y \vee y/x = 1$).
- GBH = IGBL = *generalized basic hoops, pseudo hoops*: Residuated lattices that satisfy $x \wedge y = x(x\backslash y) = (y/x)x$. It follows that they are integral.
- BH = *basic hoops* [AFM]: Commutative representable generalized basic hoops, i. e., commutative representable residuated lattices that satisfy divisibility ($x \wedge y = x(x\backslash y)$).
- WH = *Wajsberg hoops*: Commutative integral generalized MV-algebras.
- CanRL = *cancellative residuated lattices* [BCGJT]: Residuated lattices that satisfy $x = (x/y)y = y(y\backslash x)$. These identities are equivalent to cancellativity of fusion.
- LG = *lattice-ordered groups or ℓ-groups* [AF88] [GH89]: Residuated lattices that satisfy $1 = x(x\backslash 1)$. NLG = *normal-valued ℓ-groups*, defined by $(x \wedge 1)^2(y \wedge 1)^2 \leq (y \wedge 1)(x \wedge 1)$. RLG = *representable ℓ-groups*, defined by $1 \leq (e\backslash x)yx \vee 1\backslash y$. CLG = *commutative ℓ-groups.*
- LG^- = *negative cones of lattice-ordered groups* [JT02]: Cancellative integral generalized BL-algebras. NLG^- = *negative cones of normal-valued ℓ-groups*, defined by $x^2y^2 \leq yx$ relative to LG^-. RLG^- = *negative cones of representable ℓ-groups*, defined as cancellative integral representable generalized BL-algebras. CLG^- = *negative cones of commutative ℓ-groups*, defined as cancellative basic hoops.
- Br = *Brouwerian algebras*: Residuated lattices that satisfy $x \wedge y = xy$. RBr = *representable Brouwerian algebras*: Brouwerian algebras that satisfy prelinearity, or equivalently, basic hoops that are idempotent ($xx = x$).
- GBA = *generalized Boolean algebras*: Brouwerian algebras that satisfy $x \vee y = (x \to y) \to y$, or equivalently, Wajsberg hoops that are idempotent ($xx = x$). GBA_n = *generalized Boolean algebras of degree n*, defined as subdirect products of the linearly ordered $n+2$-element Brouwerian algebra.

Many further varieties can be obtained from these by combining some of the identities mentioned above. For example the prefixes C, D, I, are used to denote the commutative, distributive and integral identities respectively. The relationships between various subvarieties of RL and FL are shown in Figures 1 and 2. Note that joins in these figures do not in general agree with joins in the lattice of subvarieties.

As indicated in Table 1, there is a close correspondence between certain subvarieties of FL and RL. In logic it is quite usual to have a constant 0 in the language to denote *falsity*. From an algebraic perspective it is in some ways natural to consider the slightly less expressive signature without 0 since, for example, the variety of ℓ-groups is not a subvariety of FL.

The notion of triangular norm has been studied extensively in the theory of probabilistic measures. Since they give rise to generating algebras for several subvarieties, we recall the definition here. A *pseudo-t-norm* is an order-preserving monoid operation on the unit interval $[0, 1]$, with 1 as the identity. A *t-norm* is a commutative pseudo-t-norm. It is said to be continuous if it is a continuous function from $[0, 1]^2$ to $[0, 1]$ in the standard topology of the unit interval. Table 2 summarizes which varieties are generated by algebras based on specific t-norms.

FL	RL	Defining identities
FL_e	CRL	$xy = yx$
DFL	DRL	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
RFL	RL^C	$1 \le u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$
		Below add $0 \le x$ for subvarieties of FL
FL_w	IRL	$x \le 1$
FL_{ew}	CIRL	$xy = yx$, $x \le 1$
psMTL	Fleas	$x\backslash y \vee y\backslash x = 1 = x/y \vee y/x$
MTL	CIRL^C	$xy = yx$, $(x \to y) \vee (y \to x) = 1$
	GBL	$x \wedge y = x(x\backslash(x \wedge y)) = ((x \wedge y)/x)x$
psBL	IGBL	$x \wedge y = x(x\backslash y) = (y/x)x$
BL	BH	$xy = yx$, $x \wedge y = x(x \to y)$, $(x \to y) \vee (y \to x) = 1$
	GMV	$x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$
psMV	IGMV	$x \vee y = x/(y\backslash x) = (x/y)\backslash x$
MV	WH	$xy = yx$, $x \vee y = (x \to y) \to y$
HA	Br	$x \wedge y = xy$
GA	RBr	$x \wedge y = xy$, $(x \to y) \vee (y \to x) = 1$
Π		BL and $\neg\neg x \le (x \to xy) \to y(\neg\neg y)$
BA	GBA	$x \wedge y = xy$, $x \vee y = (x \to y) \to y$

TABLE 1. Definition and correspondence of subvarieties of FL and RL

FL	RL	Generated by
MTL	CIRL^C	all residuated t-norms [JM02]
BL	BH	all continuous t-norms [Ha98] [CEGT00]
MV	WH	Łukasiewicz $xy = \max\{0, x + y - 1\}$ [Ch59]
GA	RBr	Gödel $xy = \min\{x, y\}$ [Ha98]
Π		Product $xy =$ multiplication on $[0, 1]$ [Ha98]

TABLE 2. Some varieties generated by t-norms

The subvarieties of RL are obtained if we do not specify 0 as a constant of the algebra.

The relationship between subvarieties of FL and RL is illuminated by the following reduct and expansion functors: $R_0 : \mathsf{FL} \to \mathsf{RL}$ the forgetful functor that removes 0 from the signature, and $E_0 : \mathsf{RL} \to \mathsf{FL}$ given by $E_0(A) = 0 \oplus A$, the ordinal sum of $\{0\}$ with A on top, and $0x = x0 = 0$. Given a subvariety $\mathcal{V}$ of FL, we obtain a subvariety of RL by defining $R_0\mathcal{V}$ to be the variety generated by $\{R_0 A \mid A \in \mathcal{V}\}$. Similarly, a subvariety $\mathcal{W}$ of RL is mapped to a subvariety of FL by defining $E_0\mathcal{W}$ to be the variety generated by $\{E_0 B \mid B \in \mathcal{W}\}$. Both maps preserve inclusions of varieties.

Any subvariety of RL or FL has a corresponding logic that is sound and complete with respect to the algebraic semantics of the subvariety. Thus many logical questions have algebraic counterparts and vice versa.

In the algebraic setting there are some semantic methods that can shed light on the logical side. For example one can study the subdirectly irreducible members of the variety, or ask about the finite embedding property [BvA02] [BvA], equationally definable principal congruences (EDPC) [BP82], and the existence of a discriminator term [Jo95].

We will see how the congruences in residuated lattices are ideal determined, correspond to convex normal subalgebras, and how this characterization is used to give equational bases for some subvarieties [GU84] [JT02] [vA02]. EDPC is related to the deduction theorem in logic, and discriminator varieties are equivalent to expansions with Baaz's projection Δ [Ba96].

In the logical setting, there are syntactic methods such as cut-free Gentzen systems that have interesting consequences on the algebraic side, for example decidability [OK85] [OT99] and joint embedding properties.

Finally, we will consider the categorical equivalence between lattice ordered groups and cancellative integral generalized BL-algebras [JT02] [BCGJT] [vA], as a special case of the very general Morita equivalence in universal algebra [McK96]. Further we will mention a recent generalization of Mundici [Mu86] and Dvurečenskij's [Dv02] categorical equivalences regarding (pseudo) MV-algebras to an equivalence between lattice ordered groups expanded with a nucleus and generalized MV-algebras. The latter result is due to N. Galatos [Ga03] where he used it to prove the decidability of the equational theory of generalized MV-algebras. A recent implementation, using the decision procedure for lattice ordered groups, due to Holland and McCleary [HM79], can be found at [www.chapman.edu/∼jipsen/].

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