Algebraic Gentzen systems and ordered structures

Peter Jipsen, Chapman University, California

- Equational logic
- Gentzen systems
- Algebraic Gentzen systems
- Kleene algebras and GBL-algebras
- Finite GBL-algebras are commutative
- Generalized ordinal sum decompositions

- P. JIPSEN and C. TSINAKIS, *A survey of residuated lattices*, in "Ordered Algebraic Structures" (J. Martinez, editor), Kluwer Academic Publishers, Dordrecht, 2002, 19–56.
- P. JIPSEN, *From semirings to residuated Kleene lattices*, Studia Logica, vol. 76(2) (2004), 291–303.
- F. BELARDINELLI, P. JIPSEN and H. ONO, *Algebraic aspects of cut elimination*, Studia Logica, 77(2) (2004), 209–240.
- P. JIPSEN and F. MONTAGNA, *On the structure of generalized BL-algebras*, Algebra Universalis, to appear.

Some other papers with algebraic semantics for Gentzen systems

- J. M. FONT, V. VERDÚ, *Algebraic Logic for Classical Conjunction and Disjunction*, Studia Logica 50 (1991), 391–419.
- J. M. FONT, R. JANSANA, D. PIGOZZI, *Fully adequate Gentzen systems and the deduction theorem*, Rep. Math. Logic No. 35 (2001), 115–165.
- J. M. FONT, R. JANSANA, D. PIGOZZI, *A survey of abstract algebraic logic*, Abstract algebraic logic, Part II (Barcelona, 1997), Studia Logica 74 (2003), no. 1-2, 13–97.
- F. BOU, À. GARCÍA-CERDAÑA, V. VERDÚ *On two fragments with negation and without implication of the logic of residuated lattices*, (2005) preprint.

The main points of this talk:

Proof theory offers useful tools for **ordered algebraic structures**

- **Formulas** are **terms**
- **Sequents** are **atomic formulas**
- **Gentzen rules** are **quasiequations**

Universal algebra provides semantics for many **Gentzen systems**

This helps with using tools from **automated theorem proving**

Let's recall some standard definitions from **first-order logic**.

A signature (or language) L is the disjoint union of a set L^{f} of **function symbols** and L^{r} of **relation symbols**, each with a fixed non-negative arity.

Function symbols of arity 0 are called **constant symbols**.

The set of **L-terms over variables from X** is denoted by $T_L(X)$.

The set $A_L(X)$ of **atomic L-formulas over X** consists of all equations $t_1 = t_2$ and all $R(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in T_L(X)$ and $R \in L^{\mathrm{r}}$ has arity n.

An L-**structure** $\mathbf{A} = (A, (l^{\mathbf{A}})_{l \in L})$ is a set A with a sequence of operations and relations defined on A, where $l^{\mathbf{A}}$ has arity of l.

An **assignment into $\mathbf{A}$** is a function $h : X \to A$.

h extends uniquely to a **homomorphism** $h : T_L(X) \to \mathbf{A}$.

The notion of **satisfaction** is standard:

- $\mathbf{A} \models s = t$ if $h(s) = h(t)$ for all assignments h into $\mathbf{A}$
- $\mathbf{A} \models R(t_1, \ldots, t_n)$ if $(h(t_1), \ldots, h(t_n)) \in R^{\mathbf{A}}$ "

For a class $\mathcal{K}$ of L-structures, $\mathcal{K} \models \phi$ if $\mathbf{A} \models \phi$ for all $\mathbf{A} \in \mathcal{K}$

The **equational theory** of $\mathcal{K}$ is

$\mathrm{Th}_e(\mathcal{K}) = \{\phi \in A_L(X) : \mathcal{K} \models \phi\}$

For $E \subseteq A_L(X)$, let $\mathrm{Mod}(E) = \{\mathbf{A} : \mathbf{A} \models \phi$ for all $\phi \in E\}$

$\mathrm{Th}_e(\mathrm{Mod}(E))$ is the smallest equational theory containing E.

A **substitution** is an assignment into the term algebra $T_L(X)$.

Birkhoff [1935] provided a deductive system for equational theories:

$\mathrm{Th}_e(\mathrm{Mod}(E))$ is the smallest set E' such that $E \subset E'$, and for all $r, s, t, t_1, \ldots, t_n \in T_L(X)$ we have

refl	$(t = t) \in E'$
symm	$(s = t) \in E'$ implies $(t = s) \in E'$
tran	$(r = s), (s = t) \in E'$ implies $(r = t) \in E'$
cong$^{\mathrm{f}}$	$(s_1 = t_1), \ldots, (s_n = t_n) \in E'$ and $f \in L^{\mathrm{f}}$ implies $(f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)) \in E'$
subs	$(s = t) \in E'$ implies $(h(s) = h(t)) \in E'$ for any substitution h
cong$^{\mathrm{r}}$	$(s_1 = t_1), \ldots, (s_n = t_n), R(s_1, \ldots, s_n) \in E'$ and $R \in L^{\mathrm{r}}$ implies $R(t_1, \ldots, t_n) \in E'$

The substitution rule makes equational logic a challenge.

Together with the congruence rule, it justifies rewriting a term by replacing any subterm that matches $h(s)$ by $h(t)$ if $s = t$ is in E.

This makes goal-oriented proof search difficult since terms can grow and shrink arbitrarily many times.

This is the problem with equational term-rewriting. Depending on E, it can be undecidable, although for many important theories (groups, rings, monoids, loops,...) it provides very efficient normal form algorithms.

An **ordered algebraic structure** is an L-structure $\mathbf{A}$ where

- $\leq$ is a relational symbol in L
- $(\mathbf{A}, \leq)$ is a partially ordered set
- each basic operation on $\mathbf{A}$ is either order-preserving or order-reversing in each argument.

Often $\leq$ is definable by some equation, but we do not assume this.

Following Gorbunov's "Algebraic Theory of Quasivarieties" [1998] a *quasi-identity* is a formula $\alpha_0 \,\&\, \cdots \,\&\, \alpha_{n-1} \implies \alpha_n$ where the α_i are atomic formulas.

Sequent Calculi: $A, B, A_1, \ldots$ denote propositional formulas

Γ, Δ, Σ denote finite (possibly empty) sequences of formulas

A **sequent** is an expression of the form $\Gamma \vdash \Delta$

Informal meaning: a sequent is **valid** if from the assumptions in Γ follows at least one of the formulas in Δ

A **(propositional) Gentzen system** is specified by a set of **axioms** (sequents that are valid by definition) and rules of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad \ldots \qquad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Informal meaning: if all premises $\Gamma_i \vdash \Delta_i$ are valid sequents, then $\Gamma \vdash \Delta$ is also valid.

An **axiom**: $A \vdash A$ (Identity)

Some **structural rules**: $\dfrac{\Gamma, B, A, \Sigma \vdash \Delta}{\Gamma, A, B, \Sigma \vdash \Delta}$ (Exchange $\vdash$)

$\dfrac{\Gamma, \Sigma \vdash \Delta}{\Gamma, A, \Sigma \vdash \Delta}$ (Weakening $\vdash$) $\quad \dfrac{\Gamma \vdash A \quad \Sigma, A, \Delta \vdash B}{\Sigma, \Gamma, \Delta \vdash B}$ (Cut)

Some **logical rules**:

$\dfrac{\Gamma, A, \Sigma \vdash C}{\Gamma, A \wedge B, \Sigma \vdash C}$ ($\wedge \vdash$) $\quad \dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$ ($\vdash \wedge$)

A simple **Gentzen proof**:

$$\dfrac{\dfrac{B \vdash B}{A \wedge B \vdash B}\,(\wedge \vdash) \qquad \dfrac{A \vdash A}{A \wedge B \vdash A}\,(\wedge \vdash)}{A \wedge B \vdash B \wedge A}$$

Arbitrary substitutions on subterms are **not** part of Gentzen proofs.

Some Gentzen systems satisfy the intuitionistic restriction:

Sequences on the right of $\vdash$ have length ≤ 1.

The heavy syntax allows making fine distinctions in rules

but complicates using general purpose theorem provers

At a conference on algebraic logic this begs the question

What is the algebraic version of Gentzen systems?

Use the language of ordered algebraic structures

$\vdash$ becomes $\leq$

formulas become **terms**

$A_1, A_2, \ldots, A_n$ becomes $s_1 \cdot s_2 \cdot \ldots \cdot s_n$ (on left)

$\cdot$ is an associative operation symbol usually called **fusion**

$B_1, B_2, \ldots, B_n$ becomes $t_1 + t_2 + \ldots + t_n$ (on right)

$+$ is an associative operation symbol usually called **fission**

A **sequent** $\Gamma \vdash \Delta$ becomes an **atomic formula** $s \leq t$

A **Gentzen rule** becomes a **quasi-identity**

$$s_1 \leq t_1 \ \& \ldots \& \ s_n \leq t_n \implies s \leq t$$

The **cut rule** is now $\dfrac{r \leq s \qquad s \leq t}{r \leq t}$, i.e. **transitivity** of $\leq$

An **algebraic Gentzen system** G is a set of atomic formulas (axioms) and quasi-identities (rules).

A **proof-tree** of G is a finite rooted tree in which each element is an atomic formula, and if $s_1 \leq t_1, \ldots, s_n \leq t_n$ are all the covers of the elements of $s \leq t$ in the tree, then the quasi-identity

$$s_1 \leq t_1 \;\&\ldots\&\; s_n \leq t_n \implies s \leq t$$

is a substitution instance of a member of G.

In particular, leaves must be instances of axioms

A **Gentzen proof** of $s \leq t$ is a proof-tree with $s \leq t$ as root

If $s \leq t$ is Gentzen provable from G we write $G \vdash_g s \leq t$

E.g. **semilattices** $(A, \wedge)$ are algebras where $\wedge$ is an associative, commutative, idempotent ($x \wedge x = x$) operation

An algebraic Gentzen system G_S for semilattices: $x \leq x$

$$x \leq z \implies x \wedge y \leq z, \qquad y \leq z \implies x \wedge y \leq z$$

$$z \leq x \;\&\; z \leq y \implies z \leq x \wedge y$$

i.e. $x \wedge y$ is a lower bound and exceeds any lower bound of x, y

Easy exercises: $G_S \vdash_g \wedge$ is orderpreserving

$G_S +$ (antisymmetry of $\leq$) $\vdash_g \wedge$ is associative, commutative, idempotent, (transitivity of $\leq$) and $(x \leq y \iff x \wedge y = x)$

In fact all atomic formulas true in semilattices are Gentzen provable

For $G_S^c = G_S +$ transitivity (cut) this is easy to see since then each of the Birkhoff rules for equational logic can be simulated

Write $G_S^c \vdash_g s \equiv t$ if $G_S^c \vdash_g s \leq t$ and $G_S^c \vdash_g t \leq s$

Then $\equiv$ is reflexive, symmetric and transitive (uses cut)

(cong$^{\mathrm{f}}$) follows from $\wedge$ being orderpreserving:

$r \equiv s$ implies $r \leq s$ and $s \leq r$, so $r \wedge t \leq s \wedge t$ and $s \wedge t \leq r \wedge t$, hence $r \wedge t \equiv s \wedge t$

Again for 2nd arg: $t \equiv u \implies s \wedge t \equiv s \wedge u$, use transitivity of $\equiv$

(cong$^{\mathrm{r}}$) follows from transitivity of $\leq$:

$r \equiv s$, $t \equiv u$ and $r \leq t$ implies $s \leq r$ and $t \leq u$, so $s \leq u$.

(subs) holds since given a Gentzen proof of $s \leq t$ and substitution h, apply h to all formulas to get a Gentzen proof of $h(s) \leq h(t)$

The same argument works for **any algebraic Gentzen system with cut** if one can prove that all function symbols are orderpreserving or orderreversing in each argument

However even without cut the result holds

Cut-elimination: $\mathrm{Th}_e(\mathrm{Mod}(G_S)) = \mathrm{Th}_e(\text{Semilattices})$

This can be proved by syntactic or algebraic means

If transitivity is used in a proof-tree, rearrange it locally to move the application of this rule closer to a leaf

Or construct an algebra in which $s \leq t$ is true if and only if $s \leq t$ is Gentzen provable without using transitivity

Cut-elimination is useful since the cut rule is problematic in goal-directed proof search (there are infinitely many intermediate "cut-terms" s that could lead to a proof of $r \leq t$)

An algebraic Gentzen system G has the **subterm property** if for any rule in G the variables in the premises are a subset of the variables in the conclusion.

This property ensures that if the conclusion matches an atomic formula, then the premises are uniquely determined

A rule that fails to have the subterm property (e.g. cut) makes proof search infinitely branching nondeterministic

If all the premises of the rules in G are also structurally simpler than the conclusion, proof search is finite

Hence there is a decision procedure for $G \vdash_g s \leq t$

Together with antisymmetry this gives an equational descision procedure for the quasivariety determined by G

Of course the equational theory of semilattices is easily decidable by a simple rewrite system (order all variables, delete duplicates)

But Gentzen system techniques cover many logics **and their algebraic counterparts**.

For **lattices**, add the rules for $\vee$ (dual to $\wedge$) $x \leq x$

$$\frac{x \le z}{x \wedge y \le z} \qquad \frac{y \le z}{x \wedge y \le z} \qquad \frac{z \le x \quad z \le y}{z \le x \wedge y}$$

$$\frac{z \le x}{z \le x \vee y} \qquad \frac{z \le y}{z \le x \vee y} \qquad \frac{x \le z \quad y \le z}{x \vee y \le z}$$

This is the standard definition of lattices in terms of greatest lower bound and least upper bound

Although the first rule is equivalent to $x \wedge y \le x$, it is written as a rule to avoid using the cut rule in proofs

The Gentzen system essentially simulates Whitman's [1941] solution to the word problem for free lattices

Freese, Jezek and Nation [1993] proved that there is **no** confluent terminating rewrite system for lattices

For **distributive lattices** Font and Verdu [1991] give a Gentzen system that has the following algebraic form:

$$x \wedge y \leq y \qquad\qquad x \wedge y = y \wedge x$$

$$\frac{x \leq z}{x \wedge y \leq z} \quad \frac{x \leq y \quad x \wedge y \leq z}{x \leq z} \quad \frac{x \wedge y \leq w \quad x \wedge z \leq w}{x \wedge (y \vee z) \leq w}$$

$$\frac{z \leq x}{z \leq x \vee y} \qquad \frac{z \leq y}{z \leq x \vee y} \qquad \frac{x \leq z \quad y \leq z}{x \vee y \leq z}$$

In the third rule we see $\wedge$ used instead of “,”

They also show that the corresponding distributive logic is **not** protoalgebraic in the sense of Blok-Pigozzi

Font and Verdu (see also Font, Jansana, Pigozzi [2003]) define **matrix models** for Gentzen systems as $(\mathbf{A}, R)$, where $\mathbf{A}$ is an algebra and R is a relation between finite sequences of A and elements of A such that if

$$\frac{\Gamma_1 \vdash \phi_1 \qquad \ldots \qquad \Gamma_n \vdash \phi_n}{\Gamma \vdash \phi}$$

is a Gentzen rule and if h is any assignment into $\mathbf{A}$ with $(h(\Gamma_i), h(\phi_i)) \in R$ then $(h(\Gamma), h(\phi)) \in R$

In the present setting, R is simply the binary relation $\leq$

The original Gentzen systems LJ for **intuitionistic logic** can also be written algebraically for **Heyting algebras**

$$x \leq x \qquad 0 \leq x \qquad x \leq 1 \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z)$$

$$x \wedge y = y \wedge x \qquad x \wedge x = x$$

$$\frac{x \leq y}{x \wedge 1 \leq y} \qquad \frac{x \wedge y \leq z}{y \leq x \rightarrow z} \qquad \frac{u \leq x \quad y \wedge w \leq z}{x \rightarrow y \wedge u \wedge w \leq z}$$

$$\frac{x \wedge w \leq z}{x \wedge y \wedge w \leq z} \qquad \frac{y \wedge w \leq z}{x \wedge y \wedge w \leq z} \qquad \frac{z \leq x \quad z \leq y}{z \leq x \wedge y}$$

$$\frac{z \leq x}{z \leq x \vee y} \qquad \frac{z \leq y}{z \leq x \vee y} \qquad \frac{x \wedge w \leq z \quad y \wedge w \leq z}{(x \vee y) \wedge w \leq z}$$

The equational axioms correspond to the structural rules

They can be used as rewrite rules on subterms

So the Gentzen rules are acting on equivalence classes of terms

The intuitionistic restriction is not so visible, but it is still there:

terms on the right of $\leq$ contain at most one function symbol

A **residuated lattice** $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is a lattice-ordered monoid such that $xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$.

A cut-free Gentzen system is given by Ono and Komori [1985], gives the decidability of the equational theory of residuated lattices and FL-algebras

The **semantic approach to cut elimination** is due to Okada and Terui [1999] using **phase structures**

J. and Tsinakis [2002] give an algebraic Gentzen system for residuated lattices and a purely **algebraic proof of cut-elimination**

Bellardinelli, J. and Ono [2004] define Gentzen structures the same way as matrix models of Font et al and use them to show that cut-elimination follows from a Dedekind-MacNeille style completion

A. Wille [2004] gives an **elegant cut-free algebraic** Gentzen system for **involutive residuated lattices**, short decidability proof

Main point: Proof theory is providing equational decidability results for important algebraic classes

These decision procedures can be expressed in a **purely algebraic language**

But why translate Gentzen systems to universal algebra?

● Algebraic Gentzen systems have **standard algebras** as semantics

— well-known, easy to work with for semantic cut elimination proofs

- Quasiequational logic is a nice fragment of first-order logic
- Easy to prototype Gentzen systems in a standard theorem prover

— no need for tedious programming and optimization

— Gentzen rules **improve performance** of theorem provers compared with standard axiomatizations

— theorem provers are under **active development**, using SAT solvers, BDDs, interactive guidance,...

- Can enter **nonequational queries**, may learn something...

Let's look at two examples: Kleene algebras and generalized BL-algebras

A **Kleene algebra** $(A, \vee, 0, \cdot, 1, ^*)$ is an **idempotent semiring with** 0**,** 1 **and a Kleene** $*$**-operation**. Specifically this means:

$(A, \cdot, 1)$ is a **monoid**,

$(A, \vee, 0)$ is a **join-semilattice with bottom**,

multiplication distributes over all finite joins, i.e. $x0 = 0 = 0x$,

$x(y \vee z) = xy \vee xz, \qquad (y \vee z)x = yx \vee zx, \qquad$ and

* is a **unary operation that satisfies** $\quad 1 \vee x \vee x^*x^* = x^*$

$(*) \quad xy \leq y \implies x^*y = y \qquad yx \leq y \implies yx^* = y$

The **quasivariety** of Kleene algebras is denoted by KA. It is **not** a variety: e.g. there is a 4-element algebra that fails $(*)$ but is a homomorphic image of the Kleene algebra defined on the powerset of a 1-generated free monoid (Conway's leap).

An **algebraic Gentzen system** for Kleene algebras: fusion is associative $\quad x1 = x \quad 1x = x$

$$x \leq x \qquad\qquad x0y \leq z \qquad\qquad \frac{u \leq x \qquad v \leq y}{uv \leq xy}$$

$$\frac{u \leq x}{u \leq x \vee y} \qquad\qquad \frac{u \leq y}{u \leq x \vee y} \qquad\qquad \frac{uxv \leq w \qquad uyv \leq w}{u(x \vee y)v \leq w}$$

$$\frac{u \leq e}{u \leq x^*} \qquad\qquad \frac{u \leq x}{u \leq x^*} \qquad\qquad \frac{u \leq x^* \qquad v \leq x^*}{uv \leq x^*}$$

$$\frac{u \leq y \qquad xy \leq y}{x^*u \leq y} \qquad \frac{u \leq y \quad yx \leq y}{ux^* \leq y} \qquad\qquad \frac{x \leq u \qquad u \leq y}{x \leq y}$$

These rules (excluding cut) have been implemented on a webpage at `www.chapman.edu/~jipsen/kleene`

Problem: is there a cut-free Gentzen system for KA?

KA is known to be decidable, Kozen [1994] (PSPACE complete)

Here is the input for an automated theorem prover (Otter 3.3)

```
op(400, xfy, ;).
op(500, xfy, +).
op(600, xfy, !<). %'not less or equal' symbol
set(auto).    % Pick a strategy automatically
set(output_sequent).
formula_list(usable).
all x (x = x).
all x (-(x !< x)).
```

```
all x (x;e = x).
all x (e;x = x).
all x y z ((x;y);z = x; (y;z)).
all x y z (-(x;f;y !< z)).
all u v x y (u;v !< x;y -> u !< x | v !< y).
all u x y (u !< x+y -> u !< x).
all u x y (u !< x+y -> u !< y).
all u v w x y (u; (x+y);v !< w -> u;x;v !< w | u;y;v
all u x (u !< s(x) -> u !< e).
all u x (u !< s(x) -> u !< x).
all u v x (u;v !< s(x) -> u !< s(x) | v !< s(x)).
all u x y (s(x);u !< y -> u !< y | x;y !< y).
all u x y (u;s(x) !< y -> u !< y | y;x !< y).
end_of_list.
```

A **residuated Kleene algebra** $(A, \vee, 0, \cdot, 1, \backslash, /, ^{*})$ is a Kleene algebra expanded with

residuals $\backslash, /$ of the multiplication, i.e. for all $x, y, z \in A$

(\\) $\quad xy \le z \iff y \le x\backslash z$ and

(/) $\quad xy \le z \iff x \le z/y.$

Although we have **added more** quasiequations to KA, the class RKA of all residuated Kleene algebras is a **variety**:

(\\) is equivalent to $y \le x\backslash(xy \vee z)$ and $x(x\backslash z) \le z$

(/) is equivalent to the mirror images of these, and the implications $(*)$ are equivalent to $x^* \le (x \vee y)^*$ and $(y/y)^* \le y/y$.

Residuated Kleene algebras are also called **action algebras** by Pratt [1990] and Kozen [1994].

Kleene algebras have a long history in Computer Science, with applications in formal foundations of automata theory, regular grammars, semantics of programming languages and other areas.

Elements in a Kleene algebra can be considered as **specifications** or **programs**, with $\cdot$ as sequential composition, $\vee$ as nondeterministic choice, and * as iteration.

Residuals also have a natural interpretation: If we implement an **initial part** p of a specification s, then $px \leq s$ implies $x \leq p\backslash s$, so $p\backslash s$ is the specification for **implementing the remaining part**.

A non-commutative version of a result of Raftery and van Alten [2004] gives another reason for adding residuals:

RKA is **congruence distributive**.

Since residuated lattices are residuated join-semilattices, we can adapt Jipsen and Tsinakis [2002] Thm 6.3 as follows:

Theorem 1. *There are uncountably many minimal nontrivial varieties of residuated join-semilattices and of residuated Kleene algebras.*

It is simple to extend the Gentzen system for Kleene algebras to residuated Kleene algebras, but again it is not known if this system is cut-free.

Problem: Is the equational theory of RKA decidable?

Hajek's **Basic Logic** generalizes multi-valued and fuzzy logic.

Basic Logic algebras (BL-algebras) are the algebraic semantics.

A **generalized BL-algebra** is a residuated lattice that satisfies **divisibility**: $x \le y$ implies $x = uy = yv$ for some u, v.

This condition is equivalent to the implication

$$x \le y \implies x = (x/y)y = y(y\backslash x),$$

which is equivalent to $x \wedge y = ((x \wedge y)/y)y = y(y\backslash(x \wedge y))$.

Hence generalized BL-algebras, or GBL-algebra, form a **variety**.

The subvariety of **integral** GBL-algebras is defined by the simpler equations $x \wedge y = (x/y)y = y(y\backslash x)$.

The variety of **lattice-ordered groups** (or ℓ-groups) is term-equivalent to the subvariety of residuated lattices determined by the equation $x(1/x) = 1$.

A **BL-algebra** is an integral GBL-algebra with a constant 0 denoting the least element, and satisfying **commutativity**: $xy = yx$ and **prelinearity**: $1 \leq (x/y \vee y/x) \wedge (x\backslash y \vee y\backslash x)$.

The name "prelinearity" is justified by the result that subdirectly irreducible commutative prelinear residuated lattices are **linearly ordered**

The variety of GBL-algebras includes the variety of ℓ-groups and the variety of **Brouwerian algebras** (0-free subreducts of Heyting algebras)

Yet GBL-algebras are quite special compared to residuated lattices

E.g., as for ℓ-groups, they have distributive lattice reducts and N. Galatos recently showed that fusion distributes over meet.

Furthermore, Galatos and Tsinakis [2004] prove that any GBL-algebra is a direct product of an ℓ-group and an integral GBL-algebra.

Hence the structure of GBL-algebras can by understood by analysing the structure of ℓ-groups and integral GBL-algebras.

In particular, any finite GBL-algebra is integral.

Problem: Is the equational theory of GBL-algebras decidable? Do they have a cut-free Gentzen system?

For a start: Take the Gentzen system for residuated lattices and add

$$\frac{u \leq x \quad x \leq y \quad y \leq z}{u \leq z(y\backslash x)} \qquad \frac{u \leq x \quad x \leq y \quad y \leq z}{u \leq (x/y)z}$$

$$\frac{x \wedge z \leq w \qquad y \wedge z \leq w}{(x \vee y) \wedge z \leq w} \qquad x \wedge y = y \wedge x$$

With cut this is a Gentzen system for GBL-algebras, but without cut this is unlikely

What types of Gentzen systems admit cut elimination?

K. Terui has some recent algebraic criteria about structural rules that can be added to the Gentzen system of residuated lattices while preserving cut elimination.

Develop general methods to modify Gentzen systems so they become cut-free

For **commutative distributive involutive residuated lattices**, R. Brady [1990] gives a cut-free Gentzen system and proves decidability.

The system uses two levels of syntax, “;” for fusion and “,” for $\wedge$

An algebraic version can use the standard signature of residuated lattices

Hypersequent calculi and display logic are generalizations of Gentzen systems

Do algebraic versions also simplify the syntax and allow implementation in first-order theorem provers?

All finite GBL-algebras are commmutative

At the conference in Patras, I proved that all representable GBL-algebras are commutative

The results in this section are joint work with Franco Montagna

Lemma 2. *If a is an idempotent in an integral GBL-algebra A, then $ax = a \wedge x$ for all $x \in A$. Hence every idempotent is central, i.e. commutes with every element.*

Proof. Suppose $aa = a$.

Then $ax \leq a \wedge x = a(a\backslash x)$

$= aa(a\backslash x)$

$= a(a \wedge x) \leq ax$. $\square$

In an ℓ-group only the identity is an idempotent, hence idempotents are central in all GBL-algebras.

The next lemma implies that in a finite GBL-algebra the elements above a maximal non-unit idempotent form a chain.

Lemma 3. *Let a be a coatom in an integral GBL-algebra. Then $\{a^k : k = 0, 1, 2, \ldots\}$ is upward closed.*

In a finite residuated lattice, the central idempotent elements form a sublattice that is dually isomorphic to the congruence lattice.

Hence a subdirectly irreducible finite residuated lattice has a unique largest central idempotent $c < 1$.

A **Wajsberg hoop** is an integral commutative residuated lattice that satisfies the identity $x \vee y = (x\backslash y)\backslash y$.

These algebras are term equivalent to 0-free reducts of MV-algebras

It is wellknown that for each positive integer n there is a unique subdirectly irreducible Wajsberg hoop with n elements:

$$1 > a > a^2 > \cdots > a^{n-1} = 0$$

For posets A, B the **ordinal sum** $A \oplus B$ is defined on $A \cup B$ by extending the union of the two partial orders so that all elements of $A \setminus B$ are less than all elements of B.

If A and B are GBL-algebras that are either disjoint or $A \cap B = \{1^A\}$ and 1^A is the least element of B, then the ordinal sum is again a GBL-algebra where for $a \in A$ and $b \in B$ one defines $a \cdot b = b \cdot a = a$.

The next result shows that any finite subdirectly irreducible GBL-algebra decomposes as the ordinal sum of a Wajsberg hoop on top of a smaller GBL-algebra.

Lemma 4. *Let A be a finite subdirectly irreducible GBL-algebra, and let c be its unique largest idempotent below 1. Then A is the ordinal sum of $\downarrow c$ and $\uparrow c$, where c is the identity of $\downarrow c$, and the residuals in the lower component are defined by $x\backslash^{\downarrow} y = x\backslash y \wedge c$ and $x/^{\downarrow} y = x/y \wedge c$. Furthermore $\uparrow c$ is a Wajsberg hoop.*

Since all Wajsberg hoops are commutative, the main result follows by induction on the size of the algebra.

Theorem 5. *Every finite GBL-algebra is commutative.*

More precisely, given a finite subdirectly irreducible GBL-algebra A, we decompose it into the ordinal sum of a smaller GBL-algebra and a Wajsberg hoop.

The smaller GBL-algebra is a subdirect product of subdirectly irreducible homomorphic images, each smaller than A, hence by the inductive hypothesis, they are commutative.

Since ordinal sums preserve commutativity, the result follows.

Note that the theorem also holds if we expand the signature with a constant 0 to the denote the least element of the algebra.

Corollary 6. *The varieties of all GBL-algebras and of all pseudo-BL-algebras do* ***not*** *have the finite model property, i.e. they are not generated by their finite members.*

The subalgebra of idempotent elements

Since $ax = a \wedge x$ for an idempotent a, it is easy to see that the set of idempotents in a GBL-algebra is a sublattice that is closed under multiplication.

We now show that it is also closed under the residuals. It follows from our noncommutative examples below that the set of all central elements in a GBL-algebra is, in general, not a subalgebra.

Theorem 7. *The idempotents in a GBL-algebra form a subalgebra.*

Proof. By the decomposition result of Galatos and Tsinakis [2002], it suffices to prove the result for integral GBL-algebras.

In any residuated lattice $x\backslash(y\backslash z) = (yx)\backslash z$

(since $w \leq x\backslash(y\backslash z) \iff xw \leq y\backslash z$
$\iff yxw \leq z \iff w \leq (yx)\backslash z$)

Let $aa = a$ and $bb = b$ be two idempotents.

$$a\backslash b \leq (a \vee a\backslash b)\backslash(a\backslash b) \leq a\backslash(a\backslash b) = aa\backslash b = a\backslash b, \text{ and}$$

$$a(a\backslash b) = a \wedge b = (a \wedge b)^2 = (a(a\backslash b))^2.$$

By divisibility, we have

$$\begin{aligned} a\backslash b &= (a \vee a\backslash b)((a \vee a\backslash b)\backslash(a\backslash b)) \\ &= (a \vee a\backslash b)(a\backslash b) \\ &= a(a\backslash b) \vee (a\backslash b)^2 \\ &= (a(a\backslash b))^2 \vee (a\backslash b)^2 = (a\backslash b)^2. \end{aligned}$$

The last equality holds in integral algebras because $a(a\backslash b) \leq a\backslash b$.

By symmetry, it follows that $(a/b) = (a/b)^2$, hence the residuals of any two idempotents are again idempotents. $\square$

A **Brouwerian algebra** is a residuated lattice that satisfies

$xy = x \wedge y$.

The previous result shows that any GBL-algebra contains a largest Brouwerian subalgebra, given by the subalgebra of idempotents.

In fact any finite GBL-algebra is completely determined by the poset of join-irreducibles in this subalgebra of idempotents, together with the size of the chain between a join-irreducible and its unique idempotent lower cover.

A generalized ordinal sum construction for integral residuated lattices is defined as follows

Let P be a poset, and let A_i ($i \in P$) be a family of integral residuated lattices, each with a least element denoted by 0. The **poset sum** is defined as

$\bigoplus_{i\in P} A_i = \{a \in \prod_{i\in P} : i < j \implies a_i = 1$ or $a_j = 0\}$.

This subset of the product contains the constant function $\underline{1}$, and is closed under $\wedge$, $\vee$ and $\cdot$.

We define an auxillary operations on the poset sum:

$$(a^{\downarrow})_i = \begin{cases} 0 & \text{if } a_j < 1 \text{ for some } j < i \\ a_i & \text{otherwise} \end{cases}$$

Then $\backslash^{\oplus}, /^{\oplus}$ can be defined on the poset sum as follows:

$$a\backslash^{\oplus} b = (a\backslash b)^{\downarrow}$$

$$a/^{\oplus} b = (a/b)^{\downarrow}$$

Theorem 8. *The variety of integral GBL-algebras is closed under poset sums.*

In fact, for a GBL-algebras, this construction describes all the finite members.

Theorem 9. *All finite GBL-algebras are commutative, and can be constructed by poset sums of finite Wajsberg chains.*

Moreover, there is a 1-1 correspondence between finite GBL-algebras and finite posets labelled with natural numbers.

If the poset is a forest, the GBL-algebra is prelinear.

This result is useful for constructing and counting finite GBL-algebras.

A non-commutative sum-indecomposable GBL-algebras

Examples of noncommutative GBL-algebras:

- any noncommutative ℓ-group G
- the negative cone G^- of G
- any "large enough" principal lattice filter $\uparrow u$ of G^-

However all these examples satisfy the identities

$$x \vee y = x/((x \vee y)\backslash x) = (x/(x \vee y))\backslash x$$

that define generalized MV-algebras (GMV-algebras for short; see Jipsen and Tsinakis [2002], and Dvurecenski [2002] for pseudo MV-algebras, i.e. GMV-algebras with 0).

We now describe some examples to show that GBL-algebras in general are **not** ordinal sums of GMV-algebras.

Readers with a background in ℓ-groups may recognize these examples as certain modified intervals in the Scrimger 2-group.

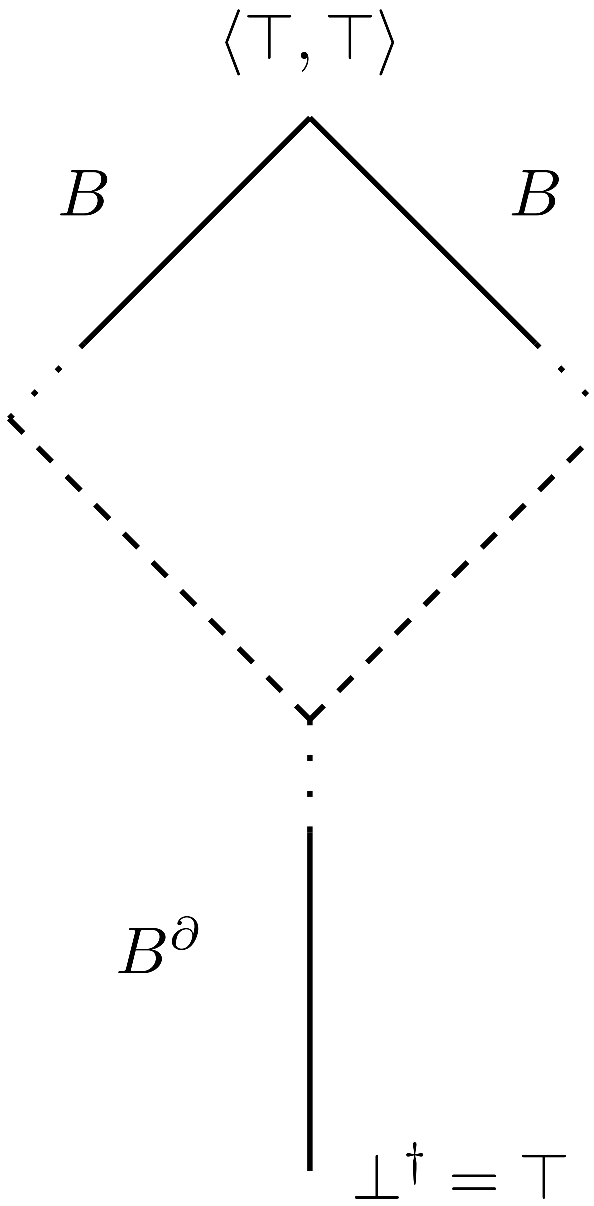
Let B be a residuated lattice with top element $\top$, and denote by B^{∂} the dual poset of the lattice reduct of B.

Let $B^{\dagger}$ be the ordinal sum of B^{∂} and $B \times B$, i.e., every element of B^{∂} is below every element of $B \times B$.

Note that $B^{\dagger}$ is a lattice under this partial order, with bottom element $\top$.

To avoid confusion, we denote this element by $\perp^{\dagger}$.

$B^{\dagger} =$



We define a binary operation $\cdot$ on $B^\dagger$ as follows:

$$\begin{aligned}
\langle a, b\rangle \cdot \langle c, d\rangle &= \langle ac, bd\rangle \\
\langle a, b\rangle \cdot u &= u/a \\
u \cdot \langle a, b\rangle &= b\backslash u \\
u \cdot v &= \top = \bot^\dagger
\end{aligned}$$

To avoid some ambiguity, we use juxtaposition for the monoid operation of B, but will continue to write $\cdot$ for the operation in $B^\dagger$, and $\backslash^\dagger$, $/^\dagger$ for the residuals.

Note that even if B is a commutative residuated lattice, $\cdot$ is in general noncommutative.

The next result shows $\cdot$ is associative and residuated (uses the RL identities $x/(yz) = (x/z)/y$ and $(x\backslash y)/z = x\backslash(y/z)$).

Lemma 10. *For any residuated lattice B with top element, the algebra $B^\dagger$ defined above is a bounded residuated lattice. If B is nontrivial, then $B^\dagger$ is not a GMV-algebra, and if B is subdirectly irreducible, so is $B^\dagger$.*

Proof. Since $\cdot$ is defined pointwise on $B \times B$, it is clearly associative there. The remaining cases (omitting mirror images) are checked as follows:

$$
\begin{aligned}
(\langle a,b\rangle\cdot\langle c,d\rangle)\cdot u &= \langle ac,bd\rangle\cdot u = u/(ac) = (u/c)/a \\
&= \langle a,b\rangle\cdot(u/c) = \langle a,b\rangle\cdot(\langle c,d\rangle\cdot u) \\
(\langle a,b\rangle\cdot u)\cdot\langle c,d\rangle &= (u/a)\cdot\langle c,d\rangle = c\backslash(u/a) = (c\backslash u)/a \\
&= \langle a,b\rangle\cdot(c\backslash u) = \langle a,b\rangle\cdot(u\cdot\langle c,d\rangle) \\
(\langle a,b\rangle\cdot u)\cdot v &= (u/a)\cdot v = \bot^\dagger = \langle a,b\rangle\cdot\bot^\dagger = \langle a,b\rangle\cdot(u\cdot v) \\
(u\cdot\langle a,b\rangle)\cdot v &= (b\backslash u)\cdot v = \bot^\dagger = u\cdot(v/a) = u\cdot(\langle a,b\rangle\cdot v) \\
(u\cdot v)\cdot w &= \bot^\dagger\cdot w = \bot^\dagger = u\cdot\bot^\dagger = u\cdot(v\cdot w)
\end{aligned}
$$

The residuals are defined as follows:

$$\begin{aligned}
\langle a,b\rangle\backslash^\dagger\langle c,d\rangle &= \langle a\backslash c, b\backslash d\rangle & \langle a,b\rangle/^\dagger\langle c,d\rangle &= \langle a/c, b/d\rangle\\
\langle a,b\rangle\backslash^\dagger u &= ua & \langle a,b\rangle/^\dagger u &= \langle\top,\top\rangle\\
u\backslash^\dagger\langle a,b\rangle &= \langle\top,\top\rangle & u/^\dagger\langle a,b\rangle &= bu\\
u\backslash^\dagger v &= \langle\top, u/v\rangle & u/^\dagger v &= \langle u\backslash v,\top\rangle
\end{aligned}$$

The GMV identity $x \vee y = x/((x \vee y)\backslash x)$ fails if we take $x = 1 \in B^\partial$ and $y = \langle\top, b\rangle \in B^2$ for some $b \neq \top$, since $x \vee y = y$ but the right hand side evaluates to $\langle\top,\top\rangle$.

It takes some work to see that the construction preserves subdirect irreducibility. In fact the congruence lattice of $B^\dagger$ is isomorphic to the congruence lattice of B with a new top element added. $\square$

Thus far we have obtained an interesting construction of noncommutative nonlinear subdirectly irreducible residuated lattices

But many such examples (even finite ones) were known.

The strength of this construction comes from the next observation.

Lemma 11. *Let B be a residuated lattice with top element. Then $B^\dagger$ is a GBL-algebra if and only if B is a cancellative GBL-algebra.*

Note that if a residuated lattice has a top element and is either cancellative or a GBL-algebra, then it is in fact integral.

By a result of Bahls, Cole, Galatos, J. and Tsinakis [2003], cancellative integral GBL-algebras are precisely the negative cones of ℓ-groups, so there are many choices for B.

An easy example is obtained if one takes $B = \mathbb{Z}^-$.

Corollary 12. *There exists a GBL-algebra that is noncommutative, subdirectly irreducible, ordinal sum indecomposable, and is not a GMV-algebra.*

This is in contrast with the situation for BL-algebras and basic hoops, where every subdirectly irreducible member is an ordinal sum of MV-algebras or Wajsberg hoops (see Agliano, Ferreirim, Montagna [2003]).

Thus the examples indicate that a structure theorem for GBL-algebras will be **more complicated** than for BL-algebras.

The next (and final) result shows that varieties generated by these examples already occur very low in the lattice of subvarieties

Theorem 13. *If $A = (\mathbb{Z}^-)^\dagger$ then Var(A) is a variety that covers the variety of Boolean algebras.*

Conclusion:

- **Proof theory** has much to offer
- Presenting it **algebraically** may help make it more accessible
- Makes it **easier** to experiment with theorem provers
- **Finite** GBL-algebras and pseudo BL-algebras are **commutative**
- Easy structure theory for finite but not for infinite GBL-algebras

Thank you