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From Semirings to Residuated Kleene Lattices

Abstract. We consider various classes of algebras obtained by expanding idempotent semirings with meet, residuals and Kleene-$*$. An investigation of congruence properties (e-permutability, e-regularity, congruence distributivity) is followed by a section on algebraic Gentzen systems for proving inequalities in idempotent semirings, in residuated lattices, and in (residuated) Kleene lattices (with cut). Finally we define (one-sorted) residuated Kleene lattices with tests to complement two-sorted Kleene algebras with tests.

Keywords: Semirings, Kleene algebras, residuated lattices, Kleene algebras with test, action algebras, congruence properties, Gentzen systems.

1. Introduction

The aim of this paper is to give an overview of some classes of algebras related to residuated lattices. Starting with (bounded) idempotent semirings (also known as join-semilattice ordered monoids) we consider the classes obtained by expanding these algebras with a meet operation, with residuals, and/or with a Kleene-$*$ operation.

After recalling some known results about Kleene-algebras, residuated Kleene-algebras, and residuated Kleene lattices, we present some data about enumerations of finite members in these classes of algebras. Examples are given to show that idempotent semirings, and hence Kleene algebras, are not congruence (e-)permutable, congruence (e-)regular or congruence distributive. However the join-semilattice structure makes them congruence meet-semidistributive, and with some mild additional assumptions one obtains subclasses that are congruence distributive.

We present an algebraic Gentzen system for proving inequalities in idempotent semirings, in residuated lattices, and in (residuated) Kleene lattices. In the latter case the Gentzen system is not known to be cut-free.

In the final section we consider residuated Kleene lattices with tests as a one-sorted alternative to Kleene algebras with tests.

Presented by **Name of Editor**; *Received* December 1, 2002

Studia Logica **68**: 1–13, 2001.

2. Definitions and terminology

An algebra $(A, \vee, 0, \cdot, e)$ is a *semiring with* 0 *and* e (or just *semiring* for short) if $(A, \cdot, e)$ is a monoid, $(A, \vee, 0)$ is a commutative monoid and $x(y \vee z) = xy \vee xz$, $(y \vee z)x = yx \vee zx$, and $x0 = 0 = 0x$. Here we are writing $x \cdot y$ as xy, and consider this operation to have priority over $\vee$. The class of all semirings is denoted by SR. Since it is defined by identities, this class forms a variety.

Semirings are common generalizations of rings (where $(A, \vee, 0)$ is an abelian group) and bounded distributive lattices (where $\cdot$ is commutative, and $x(x \vee y) = x = x \vee xy$).

A semiring is called *idempotent* if $x \vee x = x$. In this case $(A, \vee, 0)$ is a lower-bounded join-semilattice, and as usual one defines a partial order $x \leq y$ by $x \vee y = y$. It follows from the distributivity that $\cdot$ is order preserving. The variety of idempotent semirings is denoted by ISR.

We will consider expanding the members of ISR with one or more of the following:

- A meet operation $\wedge$, i. e. $(A, \wedge)$ is a meet-semilattice, and $x \wedge (x \vee y) = x = x \vee (x \wedge y)$. This defines the variety ML of *multiplicative lattices*, also known as *lattice ordered monoids with* 0.

- Residuals $\backslash$, $/$ of the multiplication, i. e. for all $x, y, z \in A$

  $$xy \leq z \iff y \leq x\backslash z \quad \text{and} \quad xy \leq z \iff x \leq z/y.$$

  This defines the class RISR of *residuated idempotent semirings*, also known as *residuated join-semilattices with* 0.

- Kleene-$*$, a unary operation that satisfies

  $(*0)$ $\quad e \vee x \vee x^*x^* = x^*$

  $(*1)$ $\quad xy \leq y \implies x^*y = y$

  $(*2)$ $\quad yx \leq y \implies yx^* = y$

  This defines the quasivariety KA of Kleene algebras.

Further classes are obtained by considering combinations of the expansions above (see Figure 1 and Table 1).

Kleene algebras and many related classes were studied by Conway [4], Kozen [9] and others, since they provide an algebraic framework for regular languages (sets of strings accepted by automata) and for sequential programs.

Briefly, given programs p, q,

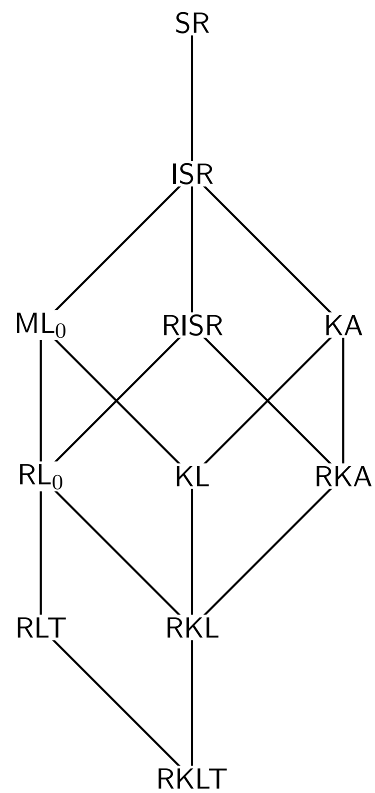


Figure 1. Some expansions of semirings

SR = semirings = rings $(\vee, 0, \cdot, e)$ without additive inverses
ISR = idempotent semirings = SR with $\vee$ idempotent
ML_0 = multiplicative lattices with 0 = ISR with meet $\wedge$
RISR = residuated idempotent semirings = ISR with residuals $\backslash$, $/$
KA = Kleene algebras = ISR with Kleene-$*$
RL_0 = residuated lattices with 0 = ML_0 with residuals
KL = Kleene lattices = KA with meet
RKA = residuated Kleene algebras = KA with residuals
RKL = residuated Kleene lattices = RL_0 with Kleene-$*$
RLT = residuated lattices with tests = RL_0 with $\{x \mid x \leq e\}$ a Bool. alg.
RKLT = residuated Kleene lattices with test = RLT with Kleene-$*$

Table 1.

- pq means running p followed by q,
- $p \vee q$ means running p or q,
- p^* means running p repeatedly 0 or more times.

Residuated Kleene algebras and residuated Kleene lattices have also been called *action algebras* by Pratt [16] and *action lattices* by Kozen [10] respectively, and they are algebraic versions of *action logic*. The standard example of a Kleene algebra is given by the collection of regular sets on an alphabet Σ under the natural set-theoretic operations of union, concatenation and iterated concatenation. This is in fact a residuated Kleene lattice, and moreover is distributive and closed under complementation.

Any join-complete idempotent semiring with completely join-preserving multiplication can be expanded to a residuated Kleene lattice: it suffices to define

$$x \wedge y = \bigvee \{z : z \leq x \text{ and } z \leq y\}$$

$$x \backslash y = \bigvee \{z : xz \leq y\} \qquad x/y = \bigvee \{z : zy \leq x\}$$

$$x^0 = e \qquad x^n = xx^{n-1} \qquad x^* = \bigvee_{n\in\omega} x^n$$

In particular, any finite idempotent semiring expands to a unique finite residuated Kleene lattice.

Note that $(*0)$ says that x^* is reflexive ($e \leq x^*$), transitive ($x^*x^* \leq x^*$) and $x \leq x^*$. Suppose y has the same properties: $e \vee x \vee yy = y$. Then

$$x \leq y \implies xy \leq yy \leq y \overset{(*1)}{\implies} x^*y \leq y \implies x^* \leq x^*e \leq x^*y \leq y.$$

Therefore x^* is the smallest reflexive transitive element above x, i. e. the reflexive transitive closure of x.

Motivation for adding residuals to Kleene algebras

Let Σ^* be the free monoid on a set Σ, and consider the powerset algebra $(\wp(\Sigma^*), \cup, \varnothing, \cdot, \{\lambda\}, ^*)$, where $X \cdot Y = \{xy \mid x \in X, y \in Y\}$ and $X^* = \bigcup_{n\,\in\,\omega} X^n$. Define $A = \{0, e, a, 1\}$ and $h : \wp(\Sigma^*) \to A$ by $h(\varnothing) = 0$, $h(\{\lambda\}) = e$, $h(X) = a$ for any finite set X, and $h(X) = 1$ otherwise. Then A is a homomorphic image of $\wp(\Sigma^*)$, but $(*1)$ fails in A: $aa \leq a$, since $a^*a = 1a = 1 \not\leq a$. The algebra A displays Conway's "leap" since a^* "leaps" to 1 even though a is transitive and reflexive.

This is the standard example that shows KA is not closed under homomorphic images and hence is not a variety. However, h does not preserve residuals: $X/X = \{\lambda\}$ for any finite set X, but $a/a = a$ in A. In fact Pratt showed that expanding KA with residuals eliminates this problem.

THEOREM 2.1. *[16] RKA is a variety defined by the identities for residuated semirings together with $e \vee x \vee x^*x^* = x^*$, $x^* \leq (x \vee y)^*$ and $(y/y)^* \leq y/y$.*

PROOF. $(*1)$ iff $x \leq y/y \implies x^* \leq y/y$, which implies $(y/y)^* \leq y/y$. Conversely, suppose $(y/y)^* \leq y/y$ holds. Then $x \leq (y/y) \implies x^* \leq (y/y)^* \leq y/y$, so $xy \leq y \implies x^*y \leq y$. ■

Even better, with residuals we have that $(*1)$ and $(*2)$ are equivalent. We have already seen that $(*1)$ implies the quasiequation $e \vee x \vee yy \leq y \implies x^* \leq y$, so it suffices to show that this quasiequation implies $(*2)$: $yx \leq y \implies yx^* = y$.

We always have $e \leq y\backslash y$ and $(y\backslash y)(y\backslash y) \leq y\backslash y$, so if $yx \leq y$ then also $x \leq y\backslash y$. Hence by the quasiequation we conclude that $x^* \leq y\backslash y$, i. e. $yx^* \leq y$.

Motivation for adding meet to RKA: Matrix algebras

For a semiring A, consider the set $A^{n\times n}$ of all $n \times n$ matrices. Let $M_n(A) = (A^{n\times n}, \vee, 0_n, \cdot, e_n)$ be the semiring of matrices, where 0_n is the zero matrix, e_n is the identity matrix,

$$[x_{ij}] \vee [y_{ij}] = [x_{ij} \vee y_{ij}] \qquad \text{and} \qquad [x_{ij}] \cdot [y_{ij}] = [\bigvee_{k=1}^{n} x_{ik}y_{kj}]$$

Thus $\vee$ and $\cdot$ are the usual matrix addition and multiplication. Note that if A is idempotent, then so is $M_n(A)$. Furthermore, if A has a Kleene-$*$ defined on it, this induces a Kleene-$*$ on $M_n(A)$:

Let $X = \begin{bmatrix} S & T \\ U & V \end{bmatrix}$ be a block matrix partitioning of $X \in A^{n\times n}$, and let $W = S \vee TV^*U$. It can be shown that the following definition of X^* is independent of the chosen partition.

$$X^* = \begin{bmatrix} W^* & W^*TV^* \\ V^*UW^* & V^* \vee V^*UW^*TV^* \end{bmatrix}$$

This construction has been used to prove several fundamental results about Kleene algebra (e.g. [9]). The following results are from [10].

LEMMA 2.2. *Let A be a residuated idempotent semiring. Then $M_n(A)$ is residuated if and only if A has finite meets.*

In fact, $[x_{ij}]\backslash[y_{ij}] = [\bigwedge_{k=1}^n x_{ki}\backslash y_{kj}]$ and $[x_{ij}]/[y_{ij}] = [\bigwedge_{k=1}^n x_{ik}/y_{jk}]$. On the other hand, if $M_2(A)$ is residuated and $a, b \in A$, then there exist largest elements x, y, z, w such that

$$\begin{bmatrix} x & y \\ x & y \end{bmatrix} = \begin{bmatrix} e & 0 \\ e & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \le \begin{bmatrix} a & a \\ b & b \end{bmatrix}$$

i. e., x is the largest element such that $x \le a$ and $x \le b$, hence $x = a \wedge b$.

THEOREM 2.3. *[10] If A is a residuated (Kleene) lattice then $M_n(A)$ is also a residuated (Kleene) lattice.*

This matrix semiring construction deserves to be studied closely for residuated lattices and RKL.

PROBLEM 2.4. What varieties of residuated lattices are closed under the construction of matrix algebras?

3. Congruence properties of (expansions of) idempotent semirings

Recall that an algebra is *congruence permutable* if $\theta \circ \psi = \psi \circ \theta$ for all congruences θ, ψ. It is *congruence 3-permutable* if $\theta \circ \psi \circ \theta = \psi \circ \theta \circ \psi$ for all congruences θ, ψ. An algebra with a constant e is *congruence e-permutable* if $e(\theta \circ \psi)x$ implies $e(\psi \circ \theta)x$ for all elements x and all congruences θ, ψ. It is *e-regular* if each congruence is determined by its e-congruence class (i. e., the map $\theta \mapsto [e]_\theta$ is injective). The previous two properties are of interest since Gumm and Ursini [6] showed that a variety of algebras is ideal determined (i. e. the e-congruence classes are characterized by being closed with respect to certain “ideal terms”) if and only if each member is congruence e-permutable and congruence e-regular.

If the congruence lattice of an algebra is distributive, modular, or meet-semidistributive ($\theta \wedge \psi = \theta \wedge \phi \implies \theta \wedge (\psi \vee \phi) = \theta \wedge \psi$) then the algebra is said to be congruence distributive, congruence modular, or congruence meet-semidistributive respectively. Finally, each of these congruence properties is said to hold in a class of algebras if it holds for each member of the class.

It is well-known that lattices are congruence distributive, hence all classes with lattice reducts (ML, RL, KL, RKL, RLT, RKLT) are congruence distributive. On the other hand bounded distributive lattices are not congruence

(e-)permutable or congruence e-regular (where e is the top element), and they can be interpreted into the classes ML, KL, KA, ISR and SR by taking $x \wedge y = x \cdot y$ and $x^* = e$. It therefore follows that these classes are not congruence (e-)permutable or congruence e-regular.

The variety RL and its expansions RKL, RLT, RKLT are congruence permutable and e-regular by a result of Blount and Tsinakis [3] (see also [7]).

Freese and Nation [5] showed that semilattices are congruence meet-semidistributive. Since idempotent semirings have semilattice reducts, the same result applies to ISR. Jónsson [8] showed that 3-permutable congruence lattices are modular. For meet-semidistributive lattices, modularity implies distributivity (since the 5-element modular lattice M_3 is not meet-semidistributive and is a sublattice of any nondistributive modular lattice). Hence we have the following result:

THEOREM 3.1. *A congruence 3-permutable class of semilattice expansions is congruence distributive. In particular, any congruence 3-permutable member of* ISR*,* RISR*,* KA*, or* RKA *is congruence distributive.*

We now give two examples of finite (expansions of) idempotent semirings that show the status of the congruence properties in some of the remaining classes of algebras from Table 1. These examples are the smallest possible, and were found by enumerating finite members in these classes and computing their congruence lattices.

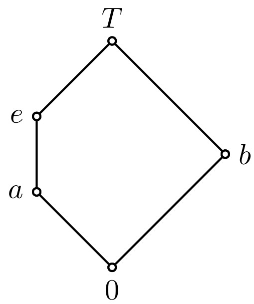
The algebra in Figure 2. is a residuated idempotent semiring with a 4-element congruence lattice that is not a chain. It is easily checked that the two non-comparable congruences θ and ψ (see Figure 2) are not e-permutable. Since ψ has a trivial e-congruence class, it follows also that RISR is not e-regular. The same example can be expanded with a Kleene-* that is definable by a term $x^* = (x \vee e)^n$. Hence RKA is also not congruence (e-)permutable or congruence e-regular.

The algebra in Figure 3. is an idempotent semiring that is not congruence distributive. The congruence lattice is shown to the right of the operation table, with the elements labeled by the nontrivial congruence classes.

The results of this section are summarized in Table 2. It follows that residuated lattices and their expansions are the only classes studied here that are ideal determined in the sense of [6].

PROBLEM 3.2. Decide whether RISR or RKA are congruence distributive.

If we consider varieties that satisfy $x \vee e = e$, then the term $m(x, y, z) = x(x\backslash y) \vee y(y\backslash z) \vee z(z\backslash x)$ is a median term (i. e., satisfies $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$) from which congruence distributivity follows.

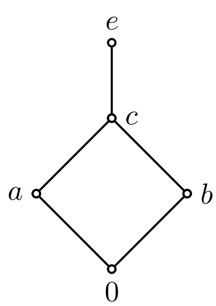


$\cdot$	0	a	e	b	T
0	0	0	0	0	0
a	0	a	a	0	a
e	0	a	e	b	T
b	0	0	b	b	b
T	0	a	T	b	T

$\backslash$	0	a	e	b	T
0	T	T	T	T	T
a	b	T	T	b	T
e	0	a	e	b	T
b	a	a	a	T	T
T	0	a	a	b	T

θ-classes $= \{\{0, b\}, \{a, e, T\}\}$ ψ-classes $= \{\{0, a\}, \{e\}, \{b, T\}\}$

Figure 2. A non-e-permutable residuated idempotent semiring



$\cdot$	0	a	b	c	e
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
e	0	a	b	c	e

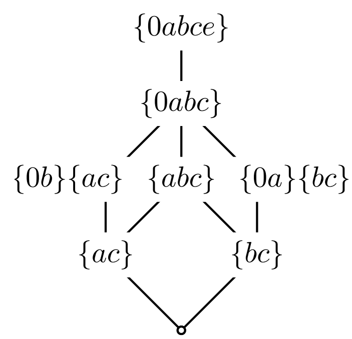


Figure 3. A non-congruence distributive idempotent semiring

	SR	ISR	ML	RISR	KA	RL	KL	RKA	RKL	RLT	RKLT
Cong. permutable	×	×	×	×	×	✓	×	×	✓	✓	✓
Cong. e-permutable	×	×	×	×	×	✓	×	×	✓	✓	✓
Cong. e-regular	×	×	×	×	×	✓	×	×	✓	✓	✓
Cong. distributive	×	×	✓		×	✓	✓		✓	✓	✓

Table 2.

4. A Gentzen system for RKL and some reducts

Gentzen systems are usually defined for logics, and use pairs of sequences of formulas (called sequents) to specify the deduction rules of the logic. Here we take an algebraic approach.

An algebraic Gentzen system is a set G of quasi-inequalities of the form $s_1 \le t_1 \,\&\ldots\&\, s_n \le t_n \implies s_0 \le t_0$, where s_i, t_i are terms. These implications are usually referred to as Gentzen rules and are written in the form $\dfrac{s_1 \le t_1 \ \ldots\ s_n \le t_n}{s_0 \le t_0}$.

For example, a Gentzen system for idempotent semirings is given by the rules below.

$$\frac{}{x \le x} \qquad \frac{}{u0v \le w} \qquad \frac{u \le x \qquad v \le y}{uv \le xy}$$

$$\frac{u \le x}{u \le x \vee y} \quad \frac{u \le y}{u \le x \vee y} \quad \frac{uxv \le w \qquad uyv \le w}{u(x \vee y)v \le w}$$

Rather than using sequences of terms, we are assuming here that $\cdot$ is associative, and we identify xe and ex with x.

For residuals and meet we add the following rules.

$$\frac{uy \le x}{u \le x/y} \qquad \frac{x \le y \qquad uzv \le w}{u(z/y)xv \le w} \qquad \frac{xu \le y}{u \le x\backslash y} \qquad \frac{x \le y \qquad uzv \le w}{ux(y\backslash z)xv \le w}$$

$$\frac{u \le x \qquad u \le y}{u \le x \wedge y} \qquad \frac{uxv \le w}{u(x \wedge y)v \le w} \qquad \frac{uyv \le w}{u(x \wedge y)v \le w}$$

Note that all the rules above are valid quasi-inequalities for residuated lattices.

A *proof-tree for the Gentzen system* G is a finite rooted tree in which each element is an inequality, and if $s_1 \le t_1, \ldots, s_n \le t_n$ are the covers of $s_0 \le t_0$ then the corresponding quasi-inequality is a substitution instance of a member of G. An inequality $s \le t$ is Gentzen provable if there exists a proof-tree with $s \le t$ as the root.

THEOREM 4.1. (Ono and Komori [15]) *An inequality $s \le t$ holds in all residuated lattices if and only if $s \le t$ is Gentzen provable from the rules above.*

Since the premises of each of these rules are determined by the conclusion (i. e., they have the *subterm* property), it is decidable whether an inequality is Gentzen provable.

Corollary 4.2. *The equational theory of residuated lattices is decidable.*

An algebraic proof of this result can be found in [7], and a general approach to algebraic Gentzen systems is presented in [2].

To obtain a Gentzen system for (residuated) Kleene algebras (lattices) we add the rules below.

$$\frac{u \le e}{u \le x^*} \qquad \frac{u \le x}{u \le x^*} \qquad \frac{u \le x^* \quad v \le x^*}{uv \le x^*}$$

$$\frac{u \le y \quad xy \le y}{x^*u \le y} \qquad \frac{u \le y \quad yx \le y}{ux^* \le y} \qquad \frac{x \le u \quad u \le y}{x \le y}$$

The first three rules are equivalent to (∗0), and the next two are equivalent to (∗1) and (∗2). However the last rule is the cut rule which lacks the subterm property, so the decidability of the equational theory of KA, RKA and RKL does not follow.

Problem 4.3. Can the cut rule be eliminated?

Kozen [9] has shown by different methods that the equational theory of KA is decidable (in fact PSPACE complete).

Problem 4.4. Is the equational theory of RKA or RKL decidable?

For RKL, this should be compared to the result that the equational theory of intuitionistic linear logic algebras (ILL = residuated lattices with storage) is undecidable (Lincoln et al [14]).

5. Interpreting Kleene algebras with tests

In order to use Kleene algebras for the analysis of sequential programs, it is useful to have a translation from standard programming constructs to Kleene algebra terms. We briefly recall the basic ideas of relational semantics. Relations on a set of states are used to model the input-output relation of a program. Let S be the set of states that occur during a computation. E.g. a state could be a vector of the current values for the variables that are used in the program. A program p is modeled by a set of pairs of states. The expression $\langle s_1, s_2\rangle \in p$ means that running program p when in state s_1 may produce the state s_2. Programs are allowed to be nondeterministic, so there can be more than one output state for a given input state.

An *atomic program* is a single statement like $a := a + 1$, which corresponds to the relation with pairs of state vectors $\langle s_1, s_2\rangle$, that differ only in the value for the variable a, with this value being one greater in state s_2. The program e is the identity relation on S, and running it has no effect on the state. The program 0 is the empty relation, and it corresponds to aborting the computation.

A *boolean test* is a program b that contains pairs $\langle s, s\rangle$ for every state s in which the test is true. E.g. the test $a > 3$ contains $\langle s, s\rangle$ whenever s is a state in which the variable a is greater than 3.

The *negation* $\neg b$ of a boolean test is the relation $\{\langle s, s\rangle : \langle s, s\rangle \notin b\}$. The standard compound statements of sequential programs are:

- pq which is already a Kleene algebra term
- "if b then p else q" which is translated as $bp \vee (\neg b)q$ and
- "while b do p" which is translated as $(bp)^*(\neg b)$.

Kozen [11] defines *Kleene algebras with tests* to be two-sorted algebras $(K, B, \vee, \cdot, ^*, 0, e, \neg)$ where $B \subseteq K$ and $\neg$ is a unary operation only defined on B such that $(K, \vee, \cdot, ^*, 0, e)$ is a Kleene algebra and $(B, \vee, \cdot, \neg, 0, e)$ is a Boolean algebra (with e as largest element).

These algebras are used in several papers to give equational proofs of correctness of program transformations, compiler optimizations and secure code certification [12][13].

In the remainder of this section we illustrate a one-sorted approach to Kleene algebras with tests. Define $\neg x = ((x \wedge e)\backslash 0) \wedge e$ and consider the identities $\neg\neg x = x \wedge e$ and $(x \wedge e)(y \wedge e) = x \wedge y \wedge e$.

The variety of residuated lattices with 0 that satisfy these identities is referred to as *residuated lattices with tests*. If we also include the *-operation, we obtain the variety RKLT of *residuated Kleene lattices with tests*.

THEOREM 5.1. *Let A be in RKLT, and let $B = \{x \in A : x \leq e\}$. Then $(A, B, \vee, \cdot, ^*, 0, e, \neg)$ is a Kleene algebra with test. Moreover, the standard model for relational semantics is in RKLT.*

While much of the analysis of residuated lattices has focused on the integral case or concerns the negative cones of residuated lattices, members of RKLT are at the opposite end of the spectrum since they have Boolean negative cones.

PROBLEM 5.2. Is the variety of residuated lattices with tests or the variety RKLT decidable?

The standard relational model also satisfies the distributive law since it is a subalgebra of (a reduct of) a relation algebra.

Let $\mathcal{R}$ be the class of all algebras isomorphic to ones whose elements are binary relations and whose operations are union, intersection and composition. Andreka [1] proved that $\mathcal{R}$ is not a variety, but it generates the finitely based variety of distributive lattice-ordered semigroups. Can this result be extended to show that the positive reducts of relation algebras generate the variety of all distributive residuated lattices with tests?

Acknowledgment

I would like to thank Hiroakira Ono and Tomasz Kowalski for organizing the superb AsubL 2002 workshop.

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