EMBEDDING THEOREMS FOR CLASSES OF GBL-ALGEBRAS

P. JIPSEN AND F. MONTAGNA

Abstract. The poset sum construction is used to derive embedding theorems for several classes of generalized basic logic algebras (GBL-algebras). In particular it is shown that every n-potent GBL-algebra is embedded in a poset sum of finite n-potent MV-chains, and every normal GBL-algebra is embedded in a poset sum of totally ordered GMV-algebras. Representable normal GBL-algebras have poset sum embeddings where the poset is a forest. We also give a Conrad-Harvey-Holland-style embedding theorem for commutative GBL-algebras, where the poset summands are the real numbers extended with $-\infty$. Finally, an explicit construction of a generic commutative GBL-algebra is given, and it is shown that every normal GBL-algebra embeds in the conucleus image of a GMV-algebra.

1. Introduction

Generalized BL-algebras (GBL-algebras for short, cf [JT02], [GT05]) are divisible residuated lattices, that is, residuated lattices such that if $x \leq y$, then there exist z, u such that $zy = yu = x$. These algebras constitute a generalization of several important classes of algebras. First of all, GBL-algebras include (zero-free subreducts of) Heyting algebras, which are the algebraic counterpart of intuitionistic logic. Moreover, as the name suggests, GBL-algebras are a generalization of (the zero free subreducts of) BL-algebras, which constitute the variety generated by the commutative and integral residuated lattices with $([0, 1], \max, \min, 0, 1)$ as lattice reduct, and with a monoid operation which is continuous on $[0, 1]$, called *continuous t-norm*, cf [Haj], [Ha98], [CEGT]. BL-algebras have been introduced by Hàjek in [Haj] as a general semantics for fuzzy logics. Indeed BL-algebras include Chang's MV-algebras [CDM], product algebras [Haj] and Gödel algebras (i.e., representable Heyting algebras, cf [Haj]). But GBL-algebras are also a generalization of ℓ-groups, which are structures arising from classical algebra, cf [AF] and [Gla]. Indeed, an ℓ-group is a divisible residuated lattice, with residuals $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Divisibility follows from the observation that for all x, y, if $z = xy^{-1}$ and $u = y^{-1}x$ then $zy = yu = x$. Thus GBL-algebras constitute a bridge between algebraic logic and classical algebra.

In this paper we prove several embedding theorems for classes of GBL-algebras. By *embedding theorem* we mean a theorem stating that every algebra of a given class $\mathcal{C}$ embeds into an algebra in $\mathcal{C}$ having a special form. A typical example is a naive version of Stone's theorem stating that every boolean algebra *embeds* into a powerset boolean algebra. Embedding theorems are weak versions of *representation theorems*. By this terminology we mean theorems stating that every algebra of a given class $\mathcal{C}$ is

2000 *Mathematics Subject Classification.* Primary: 06F05, Secondary: 06F15, 06D35, 03G25.
Key words and phrases. Generalized BL-algebras, residuated lattices, basic logic, generalized MV-algebras, lattice-ordered groups, poset sums.

isomorphic to an algebra in $\mathcal{C}$ having a special form. An example of a representation theorem is the strong version of Stone's theorem, which says that every boolean algebra is isomorphic to the algebra of closed and open sets of a totally disconnected and compact topological space. The list of all important representation theorems in algebraic logic (often expressed in terms of an equivalence of categories) would be too long to be included in this introduction. We will only mention a few of them, which are closely related to GBL-algebras, namely, Mundici's equivalence Γ between MV-algebras and abelian ℓ-groups with strong unit [Mu86], recently extended by Dvurečenskij [Dv03] to the non-commutative case, the ordinal sum representation of totally ordered BL-algebras [AM03], also extended by Dvurečenskij [Dvu] to the non-commutative case, or even the representation of finite GBL-algebras as finite poset sums of finite MV-algebras, proved in [JM]. But in the literature of ℓ-groups we also find embedding theorems, for instance Holland's theorem stating that every ℓ-group embeds into the ℓ-group of automorphisms of a totally ordered set, with composition as group operation and with lattice operations defined pointwise, or even the Conrad-Harvey-Holland embedding of any abelian ℓ-group into the abelian ℓ-group of functions from a root system into the reals, cf [AF], [Gla] (in fact, the embedding is an isomorphism if the ℓ-group is divisible in the sense that for every element x and for every positive integer n there is a y such that $y^n = x$, but it is not an isomorphism in general).

Coming to the content of this paper, our aim is to generalize the ordinal sum decomposition of [AM03] or of [Dvu] to classes of GBL-algebras. To this purpose we will use the poset sum construction introduced in [JM], which is a common generalization of ordinal sums and of direct products. The paper is organized as follows: in Section 3 we give a general sufficient condition for embeddability into a poset sum of a family of GBL-algebras. Then in Section 4 we use this condition in order to prove that every n-potent GBL-algebra embeds into the poset sum of a family of finite n-potent MV-chains. Heyting algebras occur as a particular case, because they are just 1-potent bounded GBL-algebras. In Section 5 we prove that every normal[1] GBL-algebra embeds into a poset sum of totally ordered GMV-algebras, and that every commutative GBL-algebra embeds into the poset sum of totally ordered MV-algebras and totally ordered abelian ℓ-groups. In Section 6 we show that representable normal GBL-algebras correspond to poset sums in which the poset is a forest, and we characterize various classes of GBL-algebras in terms of poset sum embeddability. In Section 7 we combine the previous embedding theorems with Hahn's embedding theorem of totally ordered abelian groups, thus proving that the above mentioned classes of GBL-algebras embed into algebras of functions taking values in $\mathbf{R} \cup \{-\infty\}$, whose structure is induced only by the structure of the reals and by some orderings. Finally, in Section 8 we give an explicit construction of a strongly generic commutative GBL-algebra, that is, of a GBL-algebra which generates the full variety of commutative GBL-algebras as a quasivariety. Of course, the countably generated free algebra would be such an example, but our construction is more concrete and explicit.

2. Basic notions

In this section we review some definitions and some known results about residuated lattices, GBL-algebras, GMV-algebras and ordinal sums.

[1] a residuated lattice is said to be *normal* iff every filter of it is a normal filter

2.1. Residuated lattices.

Definition 2.1. A *residuated lattice* (cf e.g. [BT03], [JT02]) is an algebra of the form $(L, \vee, \wedge, \cdot, \backslash, /, e)$ where $(L, \vee, \wedge)$ is a lattice, $(L, \cdot, e)$ is a monoid and $\backslash$ and $/$ are binary operations that are left and right residuals of $\cdot$, i.e., for all $x, y, z \in L$

$$x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y.$$

In the sequel the symbol $\cdot$ will often be omitted.

Definition 2.2. A residuated lattice is: *commutative* if it satisfies $xy = yx$; *integral* if it satisfies $x \leq e$; *bounded* if it has a minimum m (and hence a maximum m/m) and if the signature has (besides the symbols of operations and constants for residuated lattices) an additional constant symbol interpreted as m; *divisible* iff $x \leq y$ implies $y(y\backslash x) = (x/y)y = x$; *cancellative* if $uxv = uyv$ implies $x = y$; *representable* if it is isomorphic to a subdirect product of totally ordered residuated lattices.

Note that *ℓ-groups* (cf [AF], [Gla]) can be presented as residuated lattices satisfying $x(x\backslash e) = e$. Indeed, given an ℓ-group we obtain a cancellative and divisible residuated lattice letting $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Conversely, from a residuated lattice satisfying $x(x\backslash e) = e$ we obtain an ℓ-group by letting $x^{-1} = x\backslash e = e/x$.

In a commutative residuated lattice the operations $x\backslash y$ and y/x coincide and are denoted by $x \to y$.

2.2. GBL-algebras and GMV-algebras.

Definition 2.3. A *GBL-algebra* (cf [JT02] and GT) is a divisible residuated lattice. A *GMV-algebra* is a GBL-algebra satisfying the equation $y/((x\backslash y) \wedge e)) = ((y/x) \wedge e)\backslash y = x \vee y$. An *MV-algebra* is a commutative, integral and bounded GMV-algebra. A *pseudo BL-algebra* (*psBL-algebra* for short, cf [DGJ02]) is an integral and bounded GBL-algebra satisfying $(x\backslash y) \vee (y\backslash x) = (y/x) \vee (x/y) = e$. A *BL-algebra* (cf [Haj]) is a commutative, integral, bounded and representable GBL-algebra. A *Heyting algebra* is a bounded GBL-algebra satisfying $x \cdot y = x \wedge y$. A *Gödel algebra* is a representable Heyting algebra.

Definition 2.4. The *negative cone* of a residuated lattice $\mathbf{L}$ is the algebra $\mathbf{L}^-$ whose domain is $\{x \in \mathbf{L} : x \leq e\}$, whose lattice operations and whose monoid operation are the restrictions to $\mathbf{L}^-$ of the corresponding operations in $\mathbf{L}$ and whose residuals $\backslash^-$ are and $/^-$ are defined by $x\backslash^- y = (x\backslash y) \wedge e$ and $y/^- x = (y/x) \wedge e$, where $\backslash$ and $/$ denote the residuals of $\mathbf{L}$. Thus in particular in the *negative cone of an ℓ-group* $\mathbf{G}$ the residuals are $x\backslash y = (x^{-1}y) \wedge e$ and $y/x = (yx^{-1}) \wedge e$.

In [BCGJT] it is shown that the class of negative cones of ℓ-groups, the class of cancellative and integral GMV-algebras and the class of cancellative and integral GBL-algebras coincide.

Proposition 2.5. (cf [GT05]). *Any integral GMV-algebra satisfies the equation $x\backslash y \vee y\backslash x = y/x \vee x/y = e$. Thus every integral and bounded GMV-algebra is a psBL-algebra.*

The next proposition shows that any integral GMV-algebra can be represented by means of a negative cone of an ℓ-group and a nucleus. Recall that a *nucleus* on a residuated lattice $\mathbf{R}$ is a unary operator γ satisfying the following conditions:

- If $x \leq y$, then $\gamma(x) \leq \gamma(y)$.
- $x \leq \gamma(x)$.
- $\gamma(\gamma(x)) = \gamma(x)$.
- $\gamma(xy) = \gamma(\gamma(x)\gamma(y))$.

Proposition 2.6. (cf [GT05]).

(a) *If $\mathbf{G}^-$ is the negative cone of an ℓ-group and γ is a nucleus on $\mathbf{G}^-$, then the image $\gamma(\mathbf{G}^-)$ of $\mathbf{G}^-$ under γ is a GMV-algebra with respect to the operations: $x \vee_\gamma y = \gamma(x \vee y)$, $x \wedge_\gamma y = x \wedge y$, $x \cdot_\gamma y = \gamma(x \cdot y)$, $x \backslash_\gamma y = x \backslash y$ and $x /_\gamma y = x/y$. The monoid unit is $\gamma(e)$. Moreover since $\mathbf{G}^-$ is a GMV-algebra, by* [GT05], *Theorem 3.4, we have that $\gamma(e) = e$ and γ preserves finite joins.*

(b) ([GT05], Theorem 3.12). *For every integral GMV-algebra $\mathbf{A}$, there are a negative cone $\mathbf{G}^-$ of an ℓ-group and a nucleus γ on $\mathbf{G}^-$, such that $\mathbf{A} = (\gamma(\mathbf{G}^-), \vee_\gamma, \wedge_\gamma, \cdot_\gamma, \backslash_\gamma, /_\gamma, \gamma(e))$, with $\cdot_\gamma, \vee_\gamma, \wedge_\gamma, /_\gamma, \backslash_\gamma$ defined as in (a). Moreover $\gamma(\mathbf{G}^-)$ is a lattice filter of $\mathbf{G}^-$, that is, it is closed upwards and it is closed under $\wedge$. Finally, by* [GT05], *Theorem 3.11, $\mathbf{G}^-$ is generated by $\gamma(\mathbf{G}^-)$ as a monoid.*

(c) *Every GBL-algebra (hence, every GMV-algebra) is a direct product of an ℓ-group and an integral GBL-algebra (respectively GMV-algebra).*

Proposition 2.6 (c) allows us to concentrate on integral GBL-algebras.

Corollary 2.7. *Any totally ordered GMV-algebra is either an ℓ-group, or a bounded and integral GMV-algebra, or the negative cone of an ℓ-group.*

Proof. By Proposition 2.6 (c), any GMV-algebra $\mathbf{A}$ decomposes as a product of an ℓ-group and an integral GMV-algebra. Thus if $\mathbf{A}$ is totally ordered, it is either an ℓ-group or an integral GMV-algebra. In the latter case, by Proposition 2.6 (b), there are a negative cone $\mathbf{G}^-$ of an ℓ-group $\mathbf{G}$ and a nucleus γ on $\mathbf{G}^-$ such that $\mathbf{A} = \gamma(\mathbf{G}^-)$ and $\mathbf{G}^-$ is generated by $\mathbf{A}$ as a monoid. Moreover, $\gamma(G^-)$ is a lattice filter of $\mathbf{G}^-$.

We claim that $\mathbf{G}^-$ is totally ordered. First note that $\mathbf{G}^-$ is an integral GMV-algebra, therefore by Proposition 2.5 it satisfies $(x\backslash y) \vee (y \backslash x) = e$. Thus in order to prove that $\mathbf{G}^-$ is totally ordered, it suffices to show that e is join irreducible in $\mathbf{G}^-$. Now suppose $x, y \in G^-$ and $x, y < e$. Then by Proposition 2.6 (b) x and y can be written as products of elements of A, say $x = \prod_{i=1}^n x_i$ and $y = \prod_{j=1}^m y_j$, where at least one x_i and one y_j are less than e. Moreover $x \leq x_i$ and $y \leq y_j$, since $\mathbf{G}^-$ is integral, therefore $x \vee y \leq x_i \vee y_j < e$, because $\mathbf{A}$ is totally ordered.

We continue the proof of Corollary 2.7. If $\gamma(\mathbf{G}^-) = \mathbf{G}^-$, then $\mathbf{A} = \mathbf{G}^-$ is the negative cone of an ℓ-group. Otherwise, there is c such that $c \in G^- \setminus \gamma(G^-)$. Since $\mathbf{G}^-$ is totally ordered, and A is upward closed, c is a lower bound of A. Now for all $x \in G^-$, $\gamma(x)$ is the smallest $y \in \gamma(G^-)$ such that $x \leq y$. Thus $\gamma(c)$ is the minimum of $\mathbf{A}$, and $\mathbf{A}$ is a bounded integral GMV-algebra. □

Another connection between GMV-algebras and negative cones of ℓ-groups is the following: let $\mathbf{G}^-$ be the negative cone of an ℓ-group $\mathbf{G}$, let G^- be the domain of $\mathbf{G}^-$ and let $'$ be a bijection between G^- and a set G' disjoint from G^-. Let $\mathbf{GMV}(\mathbf{G}^-)$ denote the following structure:

- The domain of $\mathbf{GMV}(\mathbf{G}^-)$ is $G^- \cup G'$.

- Let $\cdot, \vee, \wedge, \backslash, /$, denote the operations of $\mathbf{G}^-$ and let e denote its neutral element. Then, observing that every element of $G^- \cup G'$ is either in G^- or has the form x' for some (uniquely determined) $x \in G^-$, the operations $\cdot', \vee', \wedge', \backslash', /'$ of $\mathbf{GMV(G^-)}$ are defined as follows, for all $x, y \in G^-$:
  $x \cdot' y = x \cdot y, \quad x' \cdot' y = (y\backslash x)', \quad x \cdot' y' = (y/x)', \quad x' \cdot' y' = e'$;
  $x \vee' y = x \vee y, \quad x \vee' y' = y' \vee' x = x, \quad x' \vee' y' = (x \wedge y)'$;
  $x \wedge' y = x \wedge y, \quad x \wedge' y' = y' \wedge' x = y', \quad x' \wedge' y' = (x \vee y)'$;
  $x\backslash' y = x\backslash y, \quad x\backslash y' = (y \cdot x)', \quad y'\backslash x = e, \quad x'\backslash y' = x/y$;
  $y/' x = y/x, \quad y'/x = (x \cdot y)', \quad x/y' = e, \quad y'/x' = y\backslash x$.
  Finally, e is both the top element and the neutral element of $\mathbf{GMV(G^-)}$ and e' is its bottom element.

Proposition 2.8. (cf [DDT]). *If $\mathbf{G}^-$ is the negative cone of an ℓ-group, then $\mathbf{GMV(G^-)}$ is an integral and bounded GMV-algebra. Moreover $\mathbf{GMV(G^-)}$ is totally ordered iff $\mathbf{G}^-$ is totally ordered. Finally $\mathbf{G}^-$ is both a subalgebra and a normal filter of $\mathbf{GMV(G^-)}$.*

2.3. **Ordinal sums of integral GBL-algebras.** Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two integral GBL-algebras, and assume that e is join irreducible in $\mathbf{H}_1$ and that $\mathbf{H}_1 \cap \mathbf{H}_2 = \{e\}$. Then the *ordinal sum of type (a)*, $\mathbf{H}_1 \oplus \mathbf{H}_2$, of $\mathbf{H}_1$ and $\mathbf{H}_2$ is defined as follows:

The domain of $\mathbf{H}_1 \oplus \mathbf{H}_2$ is $H_1 \cup H_2$.
The operations in $\mathbf{H}_1 \oplus \mathbf{H}_2$ are as follows:

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ x & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

$$x\backslash y = \begin{cases} x\backslash_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ e & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

$$y/x = \begin{cases} y/_i x & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ e & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

$$x \wedge y = \begin{cases} x \wedge_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ x & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \\ y & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \end{cases}$$

$$x \vee y = \begin{cases} x \vee_i y & \text{if} \quad x, y \in H_i \ (i = 1, 2) \\ x & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \\ y & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \end{cases}$$

If e is not join-irreducible in $\mathbf{H}_1$ and $\mathbf{H}_2$ has a minimum m then $\mathbf{H}_1 \oplus \mathbf{H}_2$ is defined as above with the exception of the join. Indeed, if $x \vee_1 y = e$ in $\mathbf{H}_1$, then the least upper bound of x and y in $\mathbf{H}_1 \oplus \mathbf{H}_2$ is the minimum of $\mathbf{H}_2$. Therefore the join operation in $\mathbf{H}_1 \oplus \mathbf{H}_2$ is defined as follows:

$$x \vee y = \begin{cases} x \vee_2 y & \text{if} \quad x, y \in H_2 \\ x \vee_1 y & \text{if} \quad x, y \in H_1 \text{ and } x \vee_1 y < e \\ m & \text{if} \quad x, y \in H_1 \text{ and } x \vee_1 y = e \\ x & \text{if} \quad y \in H_1 \setminus \{e\}, x \in H_2 \\ y & \text{if} \quad x \in H_1 \setminus \{e\}, y \in H_2 \end{cases}$$

In this case we say that the ordinal sum is *of type (b)*. It is readily seen that in all cases $\mathbf{H}_1 \oplus \mathbf{H}_2$ is an integral GBL-algebra if $\mathbf{H}_1$ and $\mathbf{H}_2$ are (verification is left to the reader).

Note that if e is not join irreducible in $\mathbf{H}_1$ and $\mathbf{H}_2$ has no minimum, then the ordinal sum of type (a) of $\mathbf{H}_1$ and $\mathbf{H}_2$ is not a residuated lattice (it is not even a lattice, because the operation $\vee$ defined as in ordinal sums of type (a) is not a join). In this case, an "extended" ordinal sum may be obtained by taking the ordinal sum of $(\mathbf{H}_1 \oplus \mathbf{W}_1) \oplus \mathbf{H}_2$, where $\mathbf{W}_1$ is the MV-algebra with two elements, the ordinal sum $\mathbf{H}_1 \oplus \mathbf{W}_1$ is of type (b) and the ordinal sum $(\mathbf{H}_1 \oplus \mathbf{W}_1) \oplus \mathbf{H}_2$ is of type (a).

A *filter* of a residuated lattice $\mathbf{A}$ is an upward closed subset F of $\mathbf{A}$ which is closed under the monoid operation and the meet operation, and which contains e. A filter F is said to be *normal* if whenever $x \in F$ and $y \in \mathbf{A}$, then $y\backslash(xy) \in F$ and $(yx)/y \in F$. A normal filter F is said to be a *value* if there exists $a \in \mathbf{A}$ such that F is maximal among all normal filters not containing a. Note that values are precisely the completely meet-irreducible elements in the lattice of normal filters. As an easy consequence of [GOR], we have that an integral GBL-algebra is normal iff for all x, y there is a natural number n such that $xy^n \leq yx$ and $y^n x \leq xy$, cf also [JM]. A GBL-algebra is said to be *n-potent* if it satisfies $x^{n+1} = x^n$, where $x^n = x \cdot \ldots \cdot x$ (n times). Note that n-potent GBL-algebras are normal.

In every residuated lattice, the lattice of normal filters is isomorphic to the congruence lattice: to any congruence θ one associates the normal filter $F_\theta = \uparrow \{x : (x, e) \in \theta\}$. Conversely, given a normal filter F, the set θ_F of all pairs (x, y) such that $x\backslash y \in F$ and $y\backslash x \in F$ is a congruence such that the upward closure of the congruence class of e is F. In particular, the variety of residuated lattices is congruence regular at e.

In [JM] the following result is proved.

Proposition 2.9. (i) *Every subdirectly irreducible integral and normal GBL-algebra is the ordinal sum (either of type (a) or of type (b)) of a proper subalgebra of it and of a non-trivial integral subdirectly irreducible GMV-algebra.*

(ii) *Every n-potent GBL-algebra is commutative and integral.*

Ordinal sums of type (a) can be generalized in an obvious way to the case of infinitely many summands. In this case we consider a totally ordered set I of indices, and for all $i \in I$ we consider an integral GBL-algebra $\mathbf{H}_i$ such that for $i \neq j$, $\mathbf{H}_i \cap \mathbf{H}_j = \{e\}$ and for all i, e is join irreducible in $\mathbf{H}_i$. Then the ordinal sum $\bigoplus_{i \in I} \mathbf{H}_i$ is defined as follows:

- The universe of $\bigoplus_{i \in I} \mathbf{H}_i$ is $\bigcup_{i \in I} H_i$, and the monoid operation is defined by
$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } \; x, y \in H_i \; (i \in I) \\ x & \text{if } \; x \in H_i \setminus \{e\}, y \in H_j \text{ with } i < j \\ y & \text{if } \; y \in H_i \setminus \{e\}, x \in H_j \text{ with } i < j \end{cases}$$
- The partial order on $\bigoplus_{i \in I} \mathbf{H}_i$ is the unique partial order $\leq$ such that e is the top element with respect to $\leq$, the partial order $\leq_i$ on $\mathbf{H}_i$ is the restriction of $\leq$ to $\mathbf{H}_i$, and if $i < j$, then every element of $\mathbf{H}_i \setminus \{e\}$ precedes every element of $\mathbf{H}_j$.
- The lattice operations and the residuals are uniquely determined by $\leq$ and by $\cdot$.

The following representation theorem is proved in [AM03].

Proposition 2.10. *Every totally ordered integral and commutative GBL-algebra* $\mathbf{H}$ *can be represented as an ordinal sum* $\bigoplus_{i\in I}\mathbf{H}_i$ *of commutative, integral and totally ordered GMV-algebras. Moreover* $\mathbf{H}$ *is a BL-algebra iff* I *has a minimum* i_0 *and* $\mathbf{H}_{i_0}$ *is bounded.*

Recently Dvurečenskij has shown that Proposition 2.10 extends to the non-commutative case.

Proposition 2.11. (cf [Dvu]). *Every totally ordered integral GBL-algebra* $\mathbf{H}$ *can be represented as an ordinal sum* $\bigoplus_{i\in I}\mathbf{H}_i$ *of an indexed family of integral and totally ordered GMV-algebras. Moreover* $\mathbf{H}$ *is a psBL-algebra iff* I *has a minimum* i_0 *and* $\mathbf{H}_{i_0}$ *is bounded.*

Notation. In the sequel, given a normal filter F of an integral residuated lattice $\mathbf{A}$, $\mathbf{A}/F$ denotes the quotient of $\mathbf{A}$ modulo the congruence θ_F determined by F and for every $a \in A$, a/F denotes the equivalence class of a modulo θ_F. Moreover for all $G \subseteq A$, G/F denotes the set $\{a/F : a \in G\}$. This notation, as well as the use of $\setminus$ to denote set-theoretic difference, conflicts with the notation used for residuals. However, we believe that this should not create confusion, as elements of a residuated lattices are usually denoted by lowercase letters and sets, filters, etc. are usually denoted by capital letters.

3. Poset sums and a general condition for poset sum embeddability

In the sequel, given a poset $\mathbf{P} = (P, \leq)$, its *dual*, denoted by $\mathbf{P}^d$, is defined as the poset $(P, \geq)$. The next definition is taken from [JM].

Definition 3.1. Let $\mathbf{P} = (P, \leq)$ be a poset and let $(\mathbf{A}_p : p \in P)$ be a collection of residuated lattices. Up to isomorphism we can (and we will) assume that all $\mathbf{A}_p$ share the same neutral element e and that all $\mathbf{A}_p$ which are bounded share the same minimum element 0. Suppose that if p is not minimal, then $\mathbf{A}_p$ is integral and if p is not maximal then $\mathbf{A}_p$ is bounded. The *poset sum* $\bigoplus_{p\in\mathbf{P}}\mathbf{A}_p$ is the algebra defined as follows.

- The domain of $\bigoplus_{p\in P}\mathbf{A}_p$ is the set of all maps h on P such that for all $p \in P$,
  (a) $h(p) \in \mathbf{A}_p$ and
  (b) if $h(p) \neq e$, then for all $q > p$, $h(q) = 0$.
- The monoid operation and the lattice operations are defined pointwise.
- The residuals are defined by

$$(h\backslash g)(p) = \begin{cases} h(p)\backslash_p g(p) & \text{if for all } q < p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$

$$(g/h)(p) = \begin{cases} g(p)/_p h(p) & \text{if for all } q < p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$

  where the subscript $_p$ denotes the realization of operations and of order in $\mathbf{A}_p$.

Note that the function on P that is constantly e is always an element of the poset sum. Sometimes it is convenient to consider the *dual* poset sum, that is, the poset sum $\bigoplus_{p\in\mathbf{P}^d}\mathbf{A}_p$ of the same algebras but with respect to the dual poset $\mathbf{P}^d$. Note that in the dual poset sum condition (b) must be replaced by the following condition.

(b') if $h(p) \neq e$, then for all $q < p$, $h(q) = 0$.

Moreover the definition of residuals becomes

$$(h\backslash g)(p) = \begin{cases} h(p)\backslash_p g(p) & \text{if for all } q > p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}$$
$$(g/h)(p) = \begin{cases} g(p)/_p h(p) & \text{if for all } q > p \quad h(q) \leq_p g(q) \\ 0 & \text{otherwise} \end{cases}.$$

In the sequel, we will often omit subscripts when there is no danger of confusion. Note that poset sums generalize both ordinal sums (which occur when $(P, \leq)$ is totally ordered) and direct products (which occur when $\leq$ is just equality on P). In [JM] the following is shown:

Proposition 3.2. (a) *The poset sum of a collection of residuated lattices is a residuated lattice, which is integral (divisible, bounded respectively) when all summands are integral (divisible, bounded respectively).*
(b) *Every finite GBL-algebra can be represented as the poset sum of a finite family of finite MV-chains.*

Our aim is to extend Proposition 3.2 (b) to larger classes of GBL-algebras. As we could not obtain a general representation theorem, we will present some embedding theorems. To begin with, in this section we give a sufficient condition for poset sum embeddability. Recall that by Corollary 2.7, a totally ordered integral GMV-algebra $\mathbf{A}$ is either bounded or the negative cone of an ℓ-group. In the first case we set $\mathbf{A}^* = \mathbf{A}$ and in the second case we set $\mathbf{A}^* = \mathbf{GMV}(\mathbf{A})$. Note that in either case $\mathbf{A}^*$ is a totally ordered and bounded GMV-algebra and that $\mathbf{A}$ is a subalgebra of $\mathbf{A}^*$, cf Proposition 2.8.

Theorem 3.3. *Let $\mathbf{A}$ be an integral GBL-algebra, let Δ be a collection of normal filters of $\mathbf{A}$, let $\preceq$ be a partial order on Δ, and let $\boldsymbol{\Delta} = (\Delta, \preceq)$. Suppose that the following conditions are satisfied.*
(a) *For every $F \in \Delta$, $\mathbf{A}/F$ decomposes as an ordinal sum $\mathbf{B}_F \oplus \mathbf{W}_F$ (of type (a) or (b)), where $\mathbf{B}_F$ is an integral GBL-algebra and $\mathbf{W}_F$ is a totally ordered and integral GMV-algebra.*
(b) *For every $F, G \in \Delta$, if $F \prec G$, then $\{a : a/F \in \mathbf{W}_F\} \subseteq G$.*
(c) *For every $F \in \Delta$ and for every $a \notin F$ there exists $G \in \Delta$ such that $F \preceq G$ and $a/G \in \mathbf{W}_G \setminus \{e\}$.*
(d) *$\bigcap \Delta = \{e\}$.*

Then $\mathbf{A}$ embeds into the (dual) poset sum $\mathbf{A}^{\boldsymbol{\Delta}^d} = \bigoplus_{F \in \boldsymbol{\Delta}^d} \mathbf{W}_F^*$.

Proof. First of all, note that if conditions (a), (b), (c) and (d) hold, then $F \preceq G$ implies $F \subseteq G$. The claim is clear if $F = G$. If $F \prec G$, then by (b), $\{a : a/F \in \mathbf{W}_F\} \subseteq G$. But if $a \in F$, then $a/F = e \in \mathbf{W}_F$. Thus $a \in F$ implies $a \in G$ and the claim follows.

Now for every $a \in \mathbf{A}$, let h_a be the function on Δ defined by

$$h_a(F) = \begin{cases} a/F & \text{if} \quad a/F \in \mathbf{W}_F \\ 0 & \text{otherwise} \end{cases}.$$

We claim that the map $\Phi : a \mapsto h_a$ is an embedding of $\mathbf{A}$ into $\mathbf{A}^{\boldsymbol{\Delta}^d}$. We start from the following observation. For $F \in \Delta$ and for $h, k \in A^{\Delta^d}$, let $h \leq^{\uparrow F} k$ iff $h(G) \leq k(G)$ for all $G \succ F$.

Lemma 3.4. *For all $a, b \in \mathbf{A}$ and for all $F \in \Delta$ we have*

(i) $a/F \in \mathbf{W}_F$ *iff for all* $G \in \Delta$ *with* $F \prec G$, $h_a(G) = e$.

(ii) $h_a \preceq^{\uparrow F} h_b$ *iff* $(a\backslash b)/F \in \mathbf{W}_F$ *(iff* $(b/a)/F \in \mathbf{W}_F$*).*

Proof. (i) If $a/F \in \mathbf{W}_F$, then by (b), $G \succ F$ implies $a \in G$, therefore $h_a(G) = a/G = e$. Conversely, if $a/F \notin \mathbf{W}_F$, then $a \notin F$, and by (c) there exists $G \succeq F$ such that $a/G \in \mathbf{W}_G \setminus \{e\}$. Clearly, $G \neq F$, as $a/F \notin \mathbf{W}_F$ and $a/G \in \mathbf{W}_G$. Thus $G \succ F$ and $h_a(G) = a/G < e$.

(ii) If $(a\backslash b)/F \in \mathbf{W}_F$, then by (b) we have $a\backslash b \in G$ for every $G \succ F$. Thus for all $G \succ F$, $a/G \leq b/G$ and $h_a(G) \leq h_b(G)$. Conversely, suppose that $(a\backslash b)/F \notin \mathbf{W}_F$. Then by the argument used in the proof of (i) we see that there exists $G \succ F$ such that $(a\backslash b)/G \in \mathbf{W}_G \setminus \{e\}$. By the definition of ordinal sum, this is only possible if $b/G \in \mathbf{W}_G \setminus \{e\}$, $a/G \in \mathbf{W}_G$ and $a/G \not\leq b/G$. Hence $h_a(G) \not\leq h_b(G)$. This concludes the proof of Lemma 3.4. □

Continuing with the proof of Theorem 3.3, we verify the following facts.

(1) For $a \in \mathbf{A}$, $\Phi(a) = h_a \in A^{\Delta^d}$. Indeed, for $F \in \Delta$, $h_a(F)$ is either an element of $\mathbf{W}_F$ or 0, therefore $h_a(F) \in \mathbf{W}_F^*$. Moreover if $h_a(F) > 0$, then $a/F \in \mathbf{W}_F$, and by Lemma 3.4 (i) $h_a(G) = e$ for all $G \succ F$. Thus if $h_a(G) < e$, then $h_a(F) = 0$ for all $F \prec G$.

(2) Φ is one-one. Indeed, suppose $a \neq b$. Without loss of generality, we may assume $a\backslash b < e$. Since $\bigcap \Delta = \{e\}$, there exists $G \in \Delta$ such that $a\backslash b \notin G$. Thus by (c) there exists $H \succeq G$ such that $(a\backslash b)/H \in \mathbf{W}_H \setminus \{e\}$, therefore $h_a(H) \not\leq h_b(H)$ and $\Phi(a) \neq \Phi(b)$.

(3) Φ preserves $\vee$, $\wedge$ and $\cdot$. Let us verify first that Φ preserves $\vee$. Let $a, b \in \mathbf{A}$ and let $F \in \Delta$. If $(a \vee b)/F \in \mathbf{W}_F \setminus \{0\}$, then either $a/F \in \mathbf{W}_F \setminus \{0\}$ or $b/F \in \mathbf{W}_F \setminus \{0\}$, and recalling that every element of $\mathbf{W}_F$ is an upper bound of $(\mathbf{A}/F)\backslash \mathbf{W}_F$, we get

$$\Phi(a \vee b)(F) = h_{a \vee b}(F) = (a \vee b)/F = a/F \vee b/F = \Phi(a)(F) \vee \Phi(b)(F).$$

If either $(a \vee b)/F \notin \mathbf{W}_F$ or $(a \vee b)/F = 0$, then $\Phi(a \vee b)(F) = \Phi(a)(F) = \Phi(b)(F) = (\Phi(a) \vee \Phi(b))(F) = 0$.

We verify that Φ preserves $\cdot$. If $a/F, b/F \in \mathbf{W}_F$, then $(a \cdot b)/F \in \mathbf{W}_F$, and $\Phi(a \cdot b)(F) = (a \cdot b)/F = a/F \cdot b/F = (\Phi(a) \cdot \Phi(b))(F)$. Otherwise, if e.g. $a/F \notin \mathbf{W}_F$, then $(a \cdot b)/F \notin \mathbf{W}_F$ and $\Phi(a \cdot b)(F) = \Phi(a)(F) = (\Phi(a) \cdot \Phi(b))(F) = 0$.

The proof for $\wedge$ is similar.

(4) Φ preserves $\backslash$ and $/$. We prove the claim for $\backslash$, the proof for $/$ being quite similar. Suppose first that $a\backslash b \in F$. Then $\Phi(a\backslash b) = e$, $a/F \leq b/F$ and $\Phi(a)(F) \leq \Phi(b)(F)$. Moreover if $F \preceq G$, then by the observation made at the beginning of the proof, $F \subseteq G$, therefore $a\backslash b \in G$ and $a/G \leq b/G$. Thus for all $G \succ F$, $\Phi(a)(G) \leq \Phi(b)(G)$, and by the definition of $\backslash$ in a poset sum, $(\Phi(a)\backslash\Phi(b))(F) = \Phi(a)(F)\backslash\Phi(b)(F) = e$. Next assume $a\backslash b \notin F$ and $(a\backslash b)/F \in \mathbf{W}_F$. Then $\Phi(a\backslash b)(F) = (a\backslash b)/F$. Moreover $(a\backslash b)/F \in \mathbf{W}_F \setminus \{e\}$, therefore by the argument used in the proof of Lemma 3.4, (ii), $a/F, b/F \in \mathbf{W}_F$. Also, by Lemma 3.4, (ii), we have that for all $G \succ F$, $\Phi(a)(G) \leq \Phi(b)(G)$, therefore

$$(\Phi(a)\backslash\Phi(b))(F) = \Phi(a)(F)\backslash\Phi(b)(F) = (a/F)\backslash(b/F) = \Phi(a\backslash b)(F).$$

Finally, if $(a\backslash b)/F \notin \mathbf{W}_F$, then $\Phi(a\backslash b)(F) = 0$. On the other hand, by Lemma 3.4, (ii), there is $G \succ F$ such that $\Phi(a)(G) \not\leq \Phi(b)(G)$, therefore by the definition of $\backslash$ in a (dual) poset sum, $(\Phi(a)\backslash\Phi(b))(F) = 0$. This ends the proof. □

4. A poset sum embedding theorem for n-potent GBL-algebras

In [DL03], Di Nola and Lettieri prove a representation theorem for finite BL-algebras. These algebras can be presented as finite trees whose nodes are labeled by finite MV-algebras. This result is extended to finite GBL-algebras in [JM] (in this case one has to take posets instead of trees). In the current section we partially extend the result to n-potent GBL-algebras. In fact we will prove the following embedding theorem:

Theorem 4.1. *Every n-potent GBL-algebra embeds into the (dual) poset sum of a family of finite and n-potent MV-chains.*

Proof. Let $\Delta(\mathbf{A})$ be the set of all values of $\mathbf{A}$, and let $\boldsymbol{\Delta}(\mathbf{A})$ denote the poset $(\Delta(\mathbf{A}), \subseteq)$, that is, $\Delta(\mathbf{A})$ ordered with respect to set-theoretic inclusion. Then for $F \in \Delta(\mathbf{A})$, $\mathbf{A}/F$ is subdirectly irreducible, because if $b \in \mathbf{A}$ is such that F is maximal among the filters not containing b, then the minimum non-trivial filter of $\mathbf{A}/F$ is the filter generated by b/F. By Proposition 2.9, $\mathbf{A}/F$ decomposes as an ordinal sum $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is a proper subalgebra of $\mathbf{A}/F$ and $\mathbf{W}_F$ is a non-trivial subdirectly irreducible n-potent GMV-algebra. Now by [JM], Lemma 18, $\mathbf{W}_F$, being n-potent and subdirectly irreducible, is (the reduct of) an n-potent MV-chain with $\leq n+1$ elements. In particular, $\mathbf{W}_F$ is bounded and $\mathbf{W}_F^* = \mathbf{W}_F$. Now consider the dual poset sum $\mathbf{A}^{\boldsymbol{\Delta}(\mathbf{A})^d} = \bigoplus_{G \in \boldsymbol{\Delta}(\mathbf{A})^d} \mathbf{W}_F$. Then by Proposition 2.9, $\mathbf{A}^{\boldsymbol{\Delta}(\mathbf{A})^d}$ is a commutative and integral GBL-algebra. Moreover since product is defined pointwise in a poset sum, it is readily seen that $\mathbf{A}^{\boldsymbol{\Delta}(\mathbf{A})^d}$ is n-potent. We claim that $\mathbf{A}$ embeds into $\mathbf{A}^{\boldsymbol{\Delta}(\mathbf{A})^d}$. To this end, it suffices to verify that $\boldsymbol{\Delta}(\mathbf{A})$ and the indexed family $(\mathbf{B}_F, \mathbf{W}_F : F \in \Delta(\mathbf{A}))$ satisfy the assumptions (a), (b), (c) and (d) of Theorem 3.3.

(a) Clear.

(b) For every $F \in \Delta(\mathbf{A})$, $\mathbf{W}_F$ is simple and is a filter of $\mathbf{A}/F$. Hence it is the minimum filter of $\mathbf{A}/F$. Thus if $F \subset G$, then $G/F \supseteq \mathbf{W}_F$, therefore G contains all elements a such that $a/F \in \mathbf{W}_F$.

(c) Let $F \in \Delta(\mathbf{A})$ and $a \notin F$. Let G be a filter which is maximal with respect to the properties $G \supseteq F$ and $a \notin G$ (such a filter exists by Zorn's Lemma). Then $G \in \Delta(\mathbf{A})$ and $G \supseteq F$. Moreover since $\mathbf{W}_G$ is the minimum filter of $\mathbf{A}/G$ and a/G belongs to this filter, $a/G \in \mathbf{W}_G \setminus \{e\}$.

(d) By Zorn's Lemma, for every $a < e$, there is a filter F which is maximal with respect to the property that $a \notin F$. Then $F \in \Delta(\mathbf{A})$ and $a \notin F$, therefore $a \notin \bigcap \Delta(\mathbf{A})$, and the claim is proved.

This ends the proof. □

Remark. Theorem 4.1 is an embedding theorem but not a representation theorem, in the sense that not all n-potent GBL-algebras are isomorphic to a poset sum of n-potent MV-algebras. Indeed, any such poset sum has a minimum (the constantly zero function), whereas not all n-potent GBL-algebras are bounded. More generally, any poset sum of bounded residuated lattices is bounded, and this fact imposes a limitation on the class of GBL-algebras which are representable as a poset sum.

Remark. Clearly, 1-potent GBL-algebras are commutative, integral and idempotent residuated lattices, that is, zero-free subreducts of Heyting algebras (note that in 1-potent residuated lattices, product and meet coincide). Thus Theorem 4.1 reduces to an embedding theorem for Heyting algebras and its zero-free subreducts, that is, every Heyting algebra embeds into a poset sum of a family of two-elements Boolean algebras.

5. A poset sum embedding theorem for integral and normal GBL-algebras and for commutative GBL-algebras

The poset sum construction in the previous section does not extend to arbitrary integral and normal GBL-algebras $\mathbf{A}$. Indeed, it is possible that for some $F, G \in \Delta(\mathbf{A})$ with $F \subset G$ and for some $a \in \mathbf{A}$, one has $a/F \in \mathbf{W}_F \setminus \{0, e\}$ and $a/G \in \mathbf{W}_G \setminus \{0, e\}$ (this is the case if $\mathbf{W}_F$ is not simple), therefore $F \subset G$, $h_a(G) < e$ and $h_a(F) > 0$, which is incompatible with the definition of (dual) poset sum.

In order to overcome this problem, we will still consider the set $\Delta(\mathbf{A})$ of values of $\mathbf{A}$, but with a different partial order. More precisely, we set $F \preceq G$ iff either $F = G$ or $G \supseteq \{a : a/F \in \mathbf{W}_F\}$. Clearly $\preceq$ is a partial order. In the sequel $\boldsymbol{\Delta}(\mathbf{A})$ will denote the poset $(\Delta(\mathbf{A}) \preceq)$. (This notation does not conflict with the notation used for n-potent GBL-algebras, because if $\mathbf{A}$ is an n-potent GBL-algebra, then the relations $\preceq$ and $\subseteq$ on $\Delta(\mathbf{A})$ coincide).

Another difference with the n-potent case is that in general if F is a value and we decompose $\mathbf{A}/F$ as an ordinal sum $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, it is possible that $\mathbf{W}_F$ is unbounded and therefore $\mathbf{W}_F^* \neq \mathbf{W}_F$. But with the adjustment to $\boldsymbol{\Delta}(\mathbf{A})$ introduced above, it is still possible to get a poset sum embedding theorem. We begin with the following result.

Lemma 5.1. *Every subdirectly irreducible and normal GMV-algebra* $\mathbf{C}$ *is totally ordered.*

Proof. We can assume without loss of generality that $\mathbf{C}$ is integral, because an ℓ-group is subdirectly irreducible (normal, totally ordered respectively) iff its negative cone is subdirectly irreducible (normal, totally ordered respectively). Thus suppose that $\mathbf{C}$ is integral. We first prove that e is join irreducible in $\mathbf{C}$. Indeed suppose by contradiction that $a, b < e$ and $a \vee b = e$. Let $c < e$ be a generator of the minimum non-trivial filter F of $\mathbf{C}$. Then c belongs both to the filter generated by a and to the filter generated by b (note that such filters are normal, because $\mathbf{C}$ is normal). Then for some n, $a^n \leq c$ and $b^n \leq c$. Now $(a \vee b)^{2n} \leq a^n \vee b^n$, because $\cdot$ distributes over $\vee$, and therefore $(a \vee b)^{2n}$ is a join of products of $2n$ factors of which, for some $i \leq 2n$, i factors are equal to a and $2n - i$ are equal to b. Since either $i \geq n$ or $2n - i \geq n$, we have that each factor is bounded above by $a^n \vee b^n$. Then we deduce $e = (a \vee b)^{2n} \leq a^n \vee b^n \leq c < e$, which is a contradiction. Thus e is join irreducible.

Now by Proposition 2.5, $\mathbf{C}$ satisfies the identity $(a \backslash b) \vee (b \backslash a) = e$. Since e is join irreducible, we conclude that either $a \backslash b = e$ and $a \leq b$ or $b \backslash a = e$ and $b \leq a$. Thus $\mathbf{C}$ is totally ordered and Lemma 5.1 is proved. □

Note that there are totally ordered GMV-algebras that are not normal, see for example Clifford's o-group (cf [Dar], p. 57).

Lemma 5.2. *The class of normal GBL-algebras is closed under quotients, subalgebras and finite products.*

Proof. Let $\mathbf{B}$ be a normal GBL-algebra. Without loss of generality, we may assume that $\mathbf{B}$ is integral (if it is not, then we can safely replace it by its negative cone). Then every quotient of $\mathbf{B}$ is determined by a (necessarily normal) filter F. Now let G be any filter of $\mathbf{B}/F$ and let $G' = \{a \in \mathbf{B} : a/F \in G\}$. It is readily seen that

G' is a filter of $\mathbf{B}$. Thus G' is normal. Next, let $a/F \in G$ and $b/F \in \mathbf{B}/F$. Then $a \in G'$, therefore $b\backslash(ab) \in G'$. It follows that $(b/F)\backslash(a/F \cdot b/F) = (b\backslash(ab))/F \in G$. Similarly we see that $((b/F) \cdot (a/F))/(b/F) \in G$, and G is normal.

The closure under subalgebras and finite products follows from the earlier observation that a GBL-algebra is normal iff for all $x, y \leq e$ there exists n such that $xy^n \leq yx$ and $y^n x \leq xy$. □

An ℓ-group example showing that normality is not preserved under arbitrary products can be found in [Dar] (p. 325; note that the property of normality for ℓ-groups is called *Hamiltonian*).

Theorem 5.3. (i) *Let* $\mathbf{A}$ *be any integral normal GBL-algebra, and let* $\mathbf{\Delta}(\mathbf{A})$ *and* $\mathbf{W}_F$ *(*$F \in \Delta(\mathbf{A})$*) have the usual meaning. Then* $\mathbf{W}_F$ *is totally ordered and* $\mathbf{A}$ *embeds into the dual poset sum* $\mathbf{A}^{\mathbf{\Delta}(\mathbf{A})^d} = \bigoplus_{F \in \mathbf{\Delta}(\mathbf{A})^d} \mathbf{W}_F^*$*. Thus every integral normal GBL-algebra embeds into a (dual) poset sum of totally ordered, integral and bounded GMV-algebras.*

(ii) *Every normal GBL-algebra embeds into a (dual) poset sum of totally ordered integral and bounded GMV-algebras and totally ordered* ℓ*-groups.*

Proof. (i) By Proposition 2.9 (i) for all $F \in \Delta(\mathbf{A})$, $\mathbf{A}/F$ decomposes as $\mathbf{A}/F = \mathbf{B}_F \oplus \mathbf{W}_F$, where $\mathbf{B}_F$ is an integral GBL-algebra and $\mathbf{W}_F$ is a non-trivial subdirectly irreducible integral GMV-algebra. Note that $\mathbf{W}_F$ is normal, because a filter of $\mathbf{W}_F$ is also a filter of $\mathbf{A}/F$, which is normal, since $\mathbf{A}$ is normal and normality is preserved under quotients. Thus by Lemma 5.1, $\mathbf{W}_F$ is totally ordered. By Corollary 2.7, it is either a totally ordered integral and bounded GMV-algebra or the negative cone of a totally ordered ℓ-group. In both cases, $\mathbf{W}_F^*$ is a totally ordered, integral and bounded GMV-algebra. Thus in order to derive the claim it suffices to prove that the poset $\mathbf{\Delta}(\mathbf{A})$ and the indexed family $(\mathbf{B}_F, \mathbf{W}_F : F \in \Delta(\mathbf{A}))$ defined above satisfy conditions (a), (b), (c) and (d) of Theorem 3.3.

(a) Clear.

(b) This follows from the definition of $\preceq$.

(c) Let $F \in \Delta(\mathbf{A})$ and $a \notin F$. If $a/F \in \mathbf{W}_F$, then $a/F \in \mathbf{W}_F \setminus \{e\}$ and we are done. Otherwise, let $G_0 = \{x \in \mathbf{A} : x/F \in \mathbf{W}_F\}$. Then G_0 is a normal filter, $F \subset G_0$ and $a \notin G_0$. By Zorn's Lemma there is a normal filter G which is maximal with respect to the property that that $G_0 \subseteq G$ and $a \notin G$. Then $G \in \Delta(\mathbf{A})$, $a/G \neq e$ and $F \preceq G$. Moreover a/G is in the minimum normal filter of $\mathbf{A}/G$. Since $\mathbf{W}_G$ is a normal filter of $\mathbf{A}/G$, $a/G \in \mathbf{W}_G \setminus \{e\}$.

(d) Clear.

This ends the proof of (i).

(ii) By Proposition 2.6, any GBL-algebra $\mathbf{A}$ decomposes as a product of a commutative and integral GBL-algebra $\mathbf{B}$ and an abelian ℓ-group $\mathbf{G}$. Now by (i), $\mathbf{B}$ embeds into an algebra of the form $\bigoplus_{F \in \mathbf{\Delta}(\mathbf{B})^d} \mathbf{W}_F^*$, where $\mathbf{\Delta}(\mathbf{B})$ is the set of values of $\mathbf{B}$ partially ordered by the relation $\preceq$ defined just before Theorem 5.3 and each $\mathbf{W}_F^*$ is a totally ordered integral and bounded GMV-algebra. Moreover $\mathbf{G}$, being a quotient of $\mathbf{A}$, is normal, as normality is preserved under quotients. By Lemma 5.1, $\mathbf{G}$ has a subdirect embedding into an algebra of the form $\prod_{i \in I} \mathbf{G}_i$, where each $\mathbf{G}_i$ is a totally ordered ℓ-group. Without loss of generality we may assume that $I \cap \Delta(\mathbf{B}) = \emptyset$. Now consider the poset $\mathbf{P} = (\Delta(\mathbf{B}) \cup I, \sqsubseteq)$, where $\sqsubseteq$ is defined by

$x \sqsubseteq y$ iff either $x = y$ or $x, y \in \Delta(\mathbf{B})$ and $x \preceq y$. Thus every element of I is comparable only with itself. Now let for $x \in \Delta(\mathbf{B}) \cup I$, $\mathbf{A}_x = \mathbf{W}_x^*$ if $x \in \Delta(\mathbf{B})$ and $\mathbf{A}_x = \mathbf{G}_x$ if $x \in I$. Then is readily seen that $\bigoplus_{x \in \mathbf{P}^d} \mathbf{A}_x = (\bigoplus_{F \in \mathbf{\Delta(B)}^d} \mathbf{W}_F) \times (\prod_{i \in I} \mathbf{G}_i)$, therefore $\mathbf{A}$ embeds into $\bigoplus_{x \in \mathbf{P}^d} \mathbf{A}_x$. This ends the proof. □

Corollary 5.4. (i) *Every commutative and integral GBL-algebra embeds into a (dual) poset sum of an indexed family of totally ordered MV-algebras.*
(ii) *Every commutative GBL-algebra embeds into a (dual) poset sum of an indexed family of totally ordered MV-algebras and totally ordered abelian ℓ-groups.*

6. Poset sum embedding theorems for classes of GBL-algebras

It is clear that a normal GBL-algebra is integral iff it embeds into a (dual) poset sum of integral totally ordered, normal and bounded GMV-algebras and that a GBL-algebra is commutative iff it embeds into a (dual) poset sum of totally ordered MV-algebras and of totally ordered abelian ℓ-groups. In this section we give similar characterizations for other classes of GBL-algebras. We start from the class of representable GBL-algebras. Our characterization is in terms of poset sum embeddability and not of dual poset sum embeddability. This characterization involves the notion of forest. A *forest* is a poset $(P, \leq)$ such that for all $p \in P$ the set $\{q \in P : q \leq p\}$ is totally ordered. The dual of a forest is a *root system*, that is, a poset $(P, \leq)$ such that for all $p \in P$ the set $\{q \in P : p \leq q\}$ is totally ordered. We prove:

Theorem 6.1. *Let* $\mathbf{A}$ *be a GBL-algebra. The following are equivalent:*
(i) $\mathbf{A}$ *is representable.*
(ii) $\mathbf{A}$ *is embeddable in a poset sum* $\bigoplus_{x \in \mathbf{P}} \mathbf{A}_x$ *such that each* $\mathbf{A}_x$ *is a totally ordered GMV-algebra and the poset* $\mathbf{P}$ *is a forest.*

Proof. By Proposition 2.6 and along the lines of the proof of Theorem 5.3, (ii), we can prove the theorem separately for integral GBL-algebras and for ℓ-groups. Thus suppose first that $\mathbf{A}$ is integral.

(i) ⇒ (ii) Let us decompose $\mathbf{A}$ as a subdirect product of totally ordered integral GBL-algebras $(\mathbf{A}_i : i \in I)$. Next let us apply Dvurečenskij's Theorem 2.11, thus getting an ordinal sum decomposition $\mathbf{A}_i = \bigoplus_{j \in J_i} \mathbf{W}_{i,j}$, where each $\mathbf{W}_{i,j}$ is a totally ordered integral GMV-algebra. Thus $\mathbf{W}_{i,j}^*$ is a totally ordered, integral and bounded GMV-algebra. Now let $P = \{(i,j) : i \in I,\ j \in J_i\}$. Define a partial order $\preceq$ on P by $(i,j) \preceq (i',j')$ iff $i = i'$ and $j \leq j'$. Clearly $\mathbf{P} = (P, \preceq)$ is a forest. We associate to each $a \in \mathbf{A}$ the function h_a on P, defined by

$$h_a(i,j) = \begin{cases} e & \text{if } a_i \in W_{i,h} \quad \text{for some } h > j \\ a_i & \text{if } a_i \in W_{i,j} \\ 0 & \text{if } a_i \in W_{i,h} \setminus \{e\} \quad \text{for some } h < j \end{cases}$$

It is readily seen that the map $\Phi : a \mapsto h_a$ is an embedding of $\mathbf{A}$ into $\bigoplus_{(j,i) \in \mathbf{P}} \mathbf{W}_{j,i}^\star$, and this shows (i) ⇒ (ii)

(ii) ⇒ (i) Since representability is preserved under taking subalgebras, it suffices to show that if for all $p \in P$, $\mathbf{A}_p$ is totally ordered and $\mathbf{P} = (P, \leq)$ is a forest, then the algebra $\mathbf{A}^{\mathbf{P}} = \bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ is representable. For $h, k \in \mathbf{A}^{\mathbf{P}}$ and for $p \in P$, define $h \equiv_p k$ iff for all $q \leq p$, $h(q) = k(q)$. Note that in a poset sum, for every operation

$\circ$, $(h \circ k)(p)$ only depends on the restrictions of h and k to the set $\{q \in P : q \leq p\}$. It follows that $\equiv_p$ is a congruence of $\mathbf{A}^{\mathbf{P}}$. Moreover $\bigcap\{\equiv_p : p \in P\}$ is the minimum congruence, because if $h \equiv_p k$ for all $p \in P$, then h and k coincide. Thus $\mathbf{A}^P$ has a subdirect embedding into $\prod_{p \in P}(\mathbf{A}^{\mathbf{P}}/\equiv_p)$, and it suffices to prove that each $\mathbf{A}^{\mathbf{P}}/\equiv_p$ is totally ordered. In other words, it suffices to prove that for every $p \in P$ and for every $h, k \in A^P$, either $h(q) \leq k(q)$ for all $q \leq p$ or $h(q) \geq k(q)$ for all $q \leq p$. Suppose not. Then there are $q, r \leq p$ such that $h(q) < k(q)$ and $k(r) < h(r)$. Since $\mathbf{P}$ is a forest, the set $\{q \in P : q \leq p\}$ is totally ordered, therefore either $q < r$ or $r < q$. Suppose e.g. $q < r$. Then $h(q) < k(q) \leq e$, therefore by the definition of poset sum, $h(s) = 0$ for all $s > q$. In particular, $h(r) = 0 \leq k(r)$, and a contradiction has been reached.

The case where $\mathbf{A}$ is an ℓ-group is easy:

(i) $\Rightarrow$ (ii) Suppose that $\mathbf{A}$ is representable. Consider a subdirect embedding of $\mathbf{A}$ into $\prod_{i \in I} \mathbf{A}_i$, where each $\mathbf{A}_i$ is a totally ordered ℓ-group. Define for $i, j \in I$, $i \leq j$ iff $i = j$. Then $\mathbf{I} = (I, \leq)$ is a forest and $\mathbf{A}$ embeds into $\bigoplus_{i \in \mathbf{I}} \mathbf{A}_i = \prod_{i \in I} \mathbf{A}_i$.

(ii) $\Rightarrow$ (i) If an ℓ-group $\mathbf{A}$ is the poset sum $\bigoplus_{i \in \mathbf{I}} \mathbf{A}_i$ of an indexed family of totally ordered GBL-algebras then it is readily seen that each $\mathbf{A}_i$ must be an ℓ-group. Now a non-trivial ℓ-group is not integral, therefore the definition of poset sum implies that every $i \in I$ must be minimal. Hence for all $i, j \in I$ one has $i \leq j$ iff $i = j$. Thus $\bigoplus_{i \in \mathbf{I}} \mathbf{A}_i = \prod_{i \in I} \mathbf{A}_i$, which is a representable ℓ-group. This ends the proof. $\square$

Several classes of representable GBL-algebras, arising from many-valued logic, have a simple characterization in terms of poset sum embeddability. We collect all of them in the next theorem, whose easy proof is left to the reader.

Theorem 6.2. *A GBL-algebra is*

- *a BL-algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *such that*
  - (a) *each* $\mathbf{A}_p$ *is a totally ordered MV-algebra,*
  - (b) $\mathbf{P} = (P, \leq)$ *is a forest and*
  - (c) *the function on* P *which is constantly equal to* 0 *is in* $\mathbf{A}$*;*
- *an MV-algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *such that conditions* (a) *and* (c) *above hold and*
  - (d) $\mathbf{P} = (P, \leq)$ *is a poset such that* $\leq$ *is the identity on* P*;*
- *an abelian* ℓ*-group iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *such that each* $\mathbf{A}_p$ *is a totally ordered abelian* ℓ*-group and condition* (d) *above holds;*
- n*-potent iff it is embeddable into a poset sum of totally ordered* n*-potent MV-algebras;*
- *a Heyting algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *where condition* (c) *holds and in addition*
  - (e) *every* $\mathbf{A}_p$ *is the two-element MV-algebra* $\mathbf{W}_1$*;*
- *a Gödel algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *where* (b), (c) *and* (e) *hold;*
- *a boolean algebra iff it is isomorphic to a subalgebra* $\mathbf{A}$ *of a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *where* (c), (d) *and* (e) *hold.*

7. Conrad-Harvey-Holland-style embedding theorems for commutative GBL-algebras

A simplified version of the Conrad-Harvey-Holland theorem says that every abelian ℓ-group can be embedded into an ℓ-group of functions from a root system into the set $\mathbf{R}$ of reals, with pointwise sum as group operation. In this section we aim to extend the result to commutative GBL-algebras.

Definition 7.1. Let $\mathbf{\Delta} = (\Delta, \leq)$ be a root system and for every function f from Δ into $\mathbf{R}$, let $Supp(f) = \{\delta \in \Delta : f(\delta) \neq 0\}$. We define a structure $V(\mathbf{\Delta}, \mathbf{R})$ as follows:
(a) The universe of $V(\mathbf{\Delta}, \mathbf{R})$ is the set of all functions f from Δ into $\mathbf{R}$ such that every non-empty subset of $Supp(f)$ has a maximal element.
(b) The group operation is pointwise sum (hence the neutral element is the constantly 0 function $\overline{0}$ and the inverse operation $^{-1}$ is defined, for $f \in V(\mathbf{\Delta}, \mathbf{R})$ and for $\delta \in \Delta$, by $(f^{-1})(\delta) = -f(\delta)$).
(c) The positive cone of $V(\mathbf{\Delta}, \mathbf{R})$ consists of $\overline{0}$ together with all $f \in V(\mathbf{\Delta}, \mathbf{R})$ such that $f(\delta) > 0$ for each maximal element $\delta \in Supp(f)$.

Then we have:

Proposition 7.2. *(Conrad-Harvey-Holland, simplified version, cf* [Gla]*).*
- (a) *The algebra $V(\mathbf{\Delta}, \mathbf{R})$ is an ℓ-group with respect to the operations and to the positive cone introduced in Definition 7.1.*
- (b) *Every abelian ℓ-group $\mathbf{G}$ embeds into an ℓ-group of the form $V(\mathbf{\Delta}, \mathbf{R})$ for a suitable root system $\mathbf{\Delta} = (\Delta, \leq)$.*

Note that lattice operations in $V(\mathbf{\Delta}, \mathbf{R})$ are induced by its positive cone. They may be explicitly defined as follows: let $f, g \in V(\mathbf{\Delta}, \mathbf{R})$ and let $\delta \in \Delta$. If for all $\rho \geq \delta$ we have $f(\rho) = g(\rho)$, then $(f \vee g)(\delta) = (f \wedge g)(\delta) = f(\delta) = g(\delta)$. Otherwise, since $\mathbf{\Delta}$ is a root system and $f, g \in V(\mathbf{\Delta}, \mathbf{R})$, the set $\{\rho \in Supp(g - f) : \delta \leq \rho\}$ has a maximum element, δ_0 say. Then if $g(\delta_0) < f(\delta_0)$ we have $(f \vee g)(\delta) = f(\delta)$ and $(f \wedge g)(\delta) = g(\delta)$. Otherwise we have $(f \vee g)(\delta) = g(\delta)$ and $(f \wedge g)(\delta) = f(\delta)$.

For totally ordered ℓ-groups $\mathbf{G}$, the result was shown first by Hahn:

Proposition 7.3. *(Hahn, simplified version, cf* [Gla]*).*
- (a) *If $\mathbf{\Delta} = (\Delta, \leq)$ is totally ordered, then $V(\mathbf{\Delta}, \mathbf{R})$ is a totally ordered abelian ℓ-group.*
- (b) *Every totally ordered abelian ℓ-group $\mathbf{G}$ embeds into an ℓ-group of the form $V(\mathbf{\Delta}, \mathbf{R})$ for a suitable totally ordered set $\mathbf{\Delta}$.*

Note that the proofs of both Hahn's theorem and of the Conrad-Harvey-Holland theorem provide for an explicit construction of the root system $\mathbf{\Delta}$. More precisely, recall that a *convex subgroup* of an ℓ-group $\mathbf{G}$ is an ℓ-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that for all $h, g \in \mathbf{G}$, if $h \in \mathbf{H}$ and $g \vee g^{-1} \leq h \vee h^{-1}$, then $g \in \mathbf{H}$. We also recall that a *value* of an abelian ℓ-group $\mathbf{G}$ is a convex subgroup $\mathbf{H}$ of $\mathbf{G}$ for which there is $a \in \mathbf{G}$ such that $\mathbf{H}$ is maximal among all convex subgroups not containing a. Then $\mathbf{\Delta}$ may be assumed to be the set $\Delta(\mathbf{G})$ of all values of $\mathbf{G}$, partially ordered by set-theoretic inclusion.

In order to extend the (simplified version of the) Conrad-Harvey-Holland embedding theorem to commutative GBL-algebras, it suffices to extend it to poset sums of totally ordered MV-algebras and totally ordered abelian ℓ-groups. To begin with, we give some embedding theorems for the summands of such poset sums. We already have an embedding theorem for totally ordered abelian ℓ-groups, namely, Hahn's theorem. For totally ordered MV-algebras we will use a variant of Mundici's functor Γ. This functor allows us to represent any MV-algebra as an interval $[e, u]$ of an abelian ℓ group $\mathbf{G}$ such that u is a strong unit of $\mathbf{G}$. However, since integral residuated lattices are regarded as negative cones and not as positive cones, we prefer to represent MV-algebras as intervals of the form $[u^{-1}, e]$ and not of the form $[e, u]$.

We start from an analogue of Hahn's theorem for negative cones.

Definition 7.4. Let $\mathbf{\Delta} = (\Delta, \leq)$ be a totally ordered set. Let $0\!\downarrow$ be the set of all $f \in V(\mathbf{\Delta}, \mathbf{R})$ such that either $f = \overline{0}$ or $f(\max(Supp(f))) < 0$. We define a structure $V^{-}(\mathbf{\Delta}, \mathbf{R})$ as follows:

- The domain of $V^{-}(\mathbf{\Delta}, \mathbf{R})$ is $0\!\downarrow$.
- The monoid operation is pointwise sum and the lattice operations are the restrictions of the lattice operations on $V(\mathbf{\Delta}, \mathbf{R})$.
- The residual $\rightarrow$ is defined as follows: if $g - f \in 0\!\downarrow$ (here $g - f$ denotes the pointwise difference of g and f), then $f \rightarrow g = g - f$. Otherwise, $f \rightarrow g = \overline{0}$.

Hahn's theorem theorem immediately gives the following result.

Proposition 7.5. *(a) If $\mathbf{\Delta}$ is a totally ordered set, then $V^{-}(\mathbf{\Delta}, \mathbf{R})$ is the negative cone of a totally ordered abelian ℓ-group.*
(b) For every negative cone, $\mathbf{G}^{-}$, of a totally ordered abelian ℓ-group $\mathbf{G}$, there is a totally ordered set $\mathbf{\Delta}$ such that $\mathbf{G}^{-}$ embeds into $V^{-}(\mathbf{\Delta}, \mathbf{R})$.

Now we treat totally ordered MV-algebras. Recall that a *strong unit* of a lattice ordered abelian group $\mathbf{G}$ with group operation $+$ is an element $u \in \mathbf{G}$ such that for all $g \in \mathbf{G}$ there is a positive integer n such that $g \leq u + \cdots + u$ (n times). Then after reversing the order, Mundici's Γ functor can be rewritten as follows:

Definition 7.6. Let u be a strong unit of an abelian ℓ-group $\mathbf{G}$ with group operation $+$, with neutral element 0 and with inverse operation $-x$. Then $\Gamma(\mathbf{G}, -u)$ denotes the algebra $\mathbf{A} = (A, \odot, \rightarrow, \vee, \wedge, -u, 0)$ where:

- $A = \{x \in \mathbf{G} : -u \leq x \leq 0\}$.
- The lattice operations $\vee$ and $\wedge$ are the restriction of the lattice operations in $\mathbf{G}$.
- For $x, y \in A$, $x \odot y = (x + y) \vee (-u)$ and $x \rightarrow y = (y - x) \wedge 0$.

After reversing the order and restricting our attention to totally ordered MV-algebras, Mundici's equivalence [Mu86] between MV-algebras and lattice ordered abelian groups with a strong unit immediately implies the following result.

Proposition 7.7. *For every totally ordered MV-algebra $\mathbf{A}$ there are a totally ordered abelian ℓ-group $\mathbf{G}$ and a strong unit u of $\mathbf{G}$ such that $\mathbf{A}$ is isomorphic to $\Gamma(\mathbf{G}, -u)$. Hence for every totally ordered MV-algebra $\mathbf{A}$ there are a totally ordered set $\mathbf{\Delta}$ and a strong unit $u \in V(\mathbf{\Delta}, \mathbf{R})$ such that $\mathbf{A}$ embeds into $\Gamma(V(\mathbf{\Delta}, \mathbf{R}), -u)$.*

Note that for every totally ordered abelian ℓ-group $\mathbf{G}$ it is possible to choose a totally ordered set $\mathbf{\Delta}$ such that $\mathbf{G}$ is cofinal in $V(\mathbf{\Delta}, \mathbf{R})$, that is, every element of $V(\mathbf{\Delta}, \mathbf{R})$ has an upper bound in $\mathbf{G}$. This property implies that every strong unit of $\mathbf{G}$ is a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$. Moreover $V(\mathbf{\Delta}, \mathbf{R})$ has a strong unit iff $\mathbf{\Delta}$ has a maximum element. Indeed if u is a strong unit and $\delta = \max(Supp(u))$, then δ must be the maximum of Δ, otherwise if $\sigma > \delta$, then the function f such that $f(\sigma) = 1$ and $f(\rho) = 0$ for $\rho \neq \sigma$ is such that $f \in V(\mathbf{\Delta}, \mathbf{R})$ and for every positive integer n, $f > u + \cdots + u$ (n times). Moreover if $\delta = \max(\Delta)$, then any $u \in V(\mathbf{\Delta}, \mathbf{R})$ such that $u(\delta) > 0$ is a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$.

Remark. For a more general representation theorem of GMV-algebras by means of an algebra of real valued functions, the reader is invited to consult [GRW03].

Since we want that all non-minimal summands in a poset sum share the same minimum, and since 0 is already booked (it is the neutral element of the group of the reals), we will replace $-u$ by $-\infty$ (where we assume that $-\infty \notin \mathbf{R}$ and that $-\infty \notin V(\mathbf{\Delta}, \mathbf{R})$), and we will call the resulting structure $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$. Thus $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is defined as follows:

Definition 7.8. Let $\mathbf{\Delta}$ be a totally ordered set with maximum and let u be a strong unit of $V(\mathbf{\Delta}, \mathbf{R})$. Let $0\downarrow$ be as in Definition 7.4 and let $(-u)\uparrow = \{f \in V(\mathbf{\Delta}, \mathbf{R}) \setminus \{-u\} : \max(Supp(f + u)) > 0\}$. Then $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is defined as follows:

- The domain of $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ is $(0\downarrow \cap (-u)\uparrow) \cup \{-\infty\}$.
- Lattice operations on $0\downarrow \cap (-u)\uparrow$ are the restrictions of lattice operations on $V(\mathbf{\Delta}, \mathbf{R})$, and for all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $f \wedge -\infty = -\infty \wedge f = -\infty$ and $f \vee -\infty = -\infty \vee f = f$.
- For all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $f \cdot -\infty = -\infty \cdot f = -\infty$. Moreover if $f, g \in 0\downarrow \cap (-u)\uparrow$, then
$f \cdot g = \begin{cases} f + g & \text{if} \quad f + g \in (-u)\uparrow \\ -\infty & \text{otherwise} \end{cases}$.
- For all $f \in \Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$, $-\infty \to f = \overline{0}$ and if $f \neq \overline{0}$ and $f \neq -\infty$, then $f \to -\infty = -u - f$. Moreover $\overline{0} \to -\infty = -\infty$, and if $f, g \in 0\downarrow \cap (-u)\uparrow$, then
$f \to g = \begin{cases} g - f & \text{if} \quad g - f \in 0\downarrow \\ \overline{0} & \text{otherwise} \end{cases}$.

Then we have:

Proposition 7.9. *Every totally ordered MV-algebra embeds into an algebra of the form $\Gamma'(V(\mathbf{\Delta}, \mathbf{R}), -u)$ for some totally ordered set $(\Delta, \leq)$ with maximum and for some strong unit u of $V(\mathbf{\Delta}, \mathbf{R})$.*

It follows from Corollary 5.4 and from Propositions 7.3 and 7.9 that every commutative GBL-algebra embeds into a (dual) poset sum $\bigoplus_{p \in \mathbf{P}^d} \mathbf{A}_p$ of algebras $\mathbf{A}_p$ having one of the forms $V(\mathbf{\Delta}_p, \mathbf{R})$ or $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ for some totally ordered set $\mathbf{\Delta}_p$ and for some strong unit u_p of $V(\mathbf{\Delta}_p, \mathbf{R})$. Such poset sums are uniquely determined by the poset $\mathbf{P} = (P, \leq)$, by the totally ordered sets $\mathbf{\Delta}_p$, by the choice, for each p, of one of the forms $V(\mathbf{\Delta}_p, \mathbf{R})$ or $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ and in the last case, by the choice of the strong unit u_p of $V(\mathbf{\Delta}_p, \mathbf{R})$, that is, of a function u_p from Δ_p into $\mathbf{R}$ such that every non-empty subset of $Supp(u_p)$ has a maximum and $u_p(\max(\Delta_p)) > 0$. Thus the only algebraic structure in the definition of such algebras is the group structure of the reals, the rest of the construction essentially depends on order. The algebras of the form shown above will be called *real valued GBL-algebras.*

We want to describe real valued GBL-algebras more closely. First of all, every element F of a real valued GBL-algebra $\bigoplus_{p \in \mathbf{P}^d} \mathbf{A}_p$ is a function which associates to every $p \in P$ either $-\infty$ or a function F_p from Δ_p into $\mathbf{R}$. Up to isomorphism we may safely replace such a function F by the function H from the set $P_\Delta = \{(p,\delta) : p \in P, \delta \in \Delta_p\}$ into $\mathbf{R} \cup \{-\infty\}$ defined by

$$H(p,\delta) = \begin{cases} -\infty & \text{if } \ F(p) = -\infty \\ (F(p))(\delta) & \text{otherwise} \end{cases} .$$

In the sequel, given a function $H(p,\delta)$ on P_Δ such that for all $p \in P$, either for all $\delta \in \Delta_p$, $H(p,\delta) = -\infty$, or for all $\delta \in \Delta_p$, $H(p,\delta) \in \mathbf{R}$, we define H_p as follows: if for all $\delta \in \Delta_p$, $H(p,\delta) = -\infty$, then we set $H_p = -\infty$; otherwise we set H_p to be the function on Δ_p defined by $H_p(\delta) = H(p,\delta)$. Then real valued GBL-algebras can be defined as follows:

Definition 7.10. Let $\mathbf{P} = (P, \leq)$ be a poset and let $\{P_G, P_{MV}\}$ be a partition of P such that every $p \in P_G$ is incomparable with the other elements with respect to $\leq$. Let us label each element of $p \in P_G$ by a totally ordered set $\mathbf{\Delta}_p = (\Delta_p, \leq_p)$ and each $q \in P_{MV}$ by a totally ordered set $\mathbf{\Delta}_q = (\Delta_q, \leq_q)$ with maximum δ_q and by a function $u_q \in V(\mathbf{\Delta}_q, \mathbf{R})$ such that $u(\delta_q) > 0$. Then the *real valued GBL-algebra* associated to the poset $\mathbf{P}$, to the partition $\{P_G, P_{MV}\}$ and to the labeling $\Lambda_G = (\mathbf{\Delta}_p : p \in P_G)$ and $\Lambda_{MV} = (\mathbf{\Delta}_q, u_q : q \in P_{MV})$ is the algebra $\mathbf{A} = \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ defined as follows:

- The domain of $\mathbf{A}$ is the set of all functions H from P_Δ into $\mathbf{R} \cup -\{\infty\}$ such that
  - for all $p \in P_G$, $H(p,\delta) \in \mathbf{R}$ for all $\delta \in \Delta_p$;
  - for all $p \in P_{MV}$ we have that either $H(p,\delta) = -\infty$ for all $\delta \in \Delta_p$ or $H(p,\delta) \in \mathbf{R}$ for all $\delta \in \Delta_p$;
  - if $p \in P_G$ ($p \in P_{MV}$ respectively) then $H_p \in V(\mathbf{\Delta}_p, \mathbf{R})$ ($H_p \in \Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ respectively), and
  - if for some $\delta \in \Delta_p$, $H(p,\delta) \neq 0$, then $H(q,\sigma) = -\infty$ for all $q < p$ and for all $\sigma \in \Delta_q$.
- For every operation $\circ$ of commutative GBL-algebras, let $\circ_p$ denote its realization in $V(\mathbf{\Delta}_p, \mathbf{R})$ if $p \in P_G$ and in $\Gamma'(V(\mathbf{\Delta}_p, \mathbf{R}), -u_p)$ if $p \in P_{MV}$. Then
  - for $\circ \in \{\vee, \wedge, \cdot\}$, for $H, K \in \mathbf{A}$ and for $(p,\delta) \in P_\Delta$, $(H \circ K)(p,\delta) = (H_p \circ_p K_p)(\delta)$;
  - if for all $q > p$ we have that $H_q = -\infty$ implies $K_q = -\infty$ and $H_q, K_q \neq -\infty$ implies that either $H_q = K_q$ or $\max(Supp(H_q - K_q)) < 0$, then $(H \to K)(p,\delta) = (H_p \to_p K_p)(\delta)$; otherwise $(H \to K)(p,\delta) = -\infty$.

The next theorem is a almost a rephrasing of the results of the previous section for commutative GBL-algebras, in terms of embeddability into real-valued GBL-algebras. We use the notation $\mathbf{A} \subseteq \mathbf{B}$ to indicate that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$.

Theorem 7.11. *Every commutative GBL-algebra embeds into a real-valued GBL-algebra of the form* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$*, cf Definition 7.10.*

Moreover, a commutative GBL-algebra is

- *integral iff it embeds into an algebra* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*
  (a) $P_G = \Lambda_G = \emptyset$*;*
- *an ℓ-group iff it embeds into some* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*
  (b) $P_{MV} = \Lambda_{MV} = \emptyset$*;*
- *representable iff it embeds into some* $\mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which*
  (c) $\mathbf{P}$ *is a forest;*

- *a BL-algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (c) *hold and*
  (d) *the constantly* $-\infty$ *function is in* $\mathbf{A}$*;*
- *an MV-algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (d) *hold and*
  (e) *any two distinct elements of* P *are incomparable with respect to* $\leq$*;*
- *a Heyting algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a) *and* (d) *hold and*
  (f) *for all* $H \in \mathbf{A}$ *and for all* $(p, \delta) \in P_\Delta$*,* $H(p, \delta) \in \{-\infty, 0\}$*;*
- *a Gödel algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a), (c), (d) *and* (f) *are satisfied;*
- *a boolean algebra iff it is isomorphic to some* $\mathbf{A} \subseteq \mathbf{GBL}(\mathbf{P}, P_G, P_{MV}, \Lambda_G, \Lambda_{MV})$ *in which* (a), (d), (e) *and* (f) *are satisfied.* □

8. Explicit constructions of generic commutative GBL-algebras

We say that an algebra $\mathbf{A}$ of a variety $\mathcal{V}$ is *generic* for $\mathcal{V}$ if it generates $\mathcal{V}$ as a variety, and *strongly generic* for $\mathcal{V}$ if it generates $\mathcal{V}$ as a quasivariety. In this section we present a commutative and integral countable GBL-algebra which is strongly generic for the variety of commutative and integral GBL-algebras and a commutative GBL-algebra which is strongly generic for the variety of commutative GBL-algebras. Of course, countably generated free algebras do the job, but here we are looking for an explicit and concrete description. We start with commutative and integral GBL-algebras.

Lemma 8.1. *Every finite GBL-algebra* $\mathbf{A}$ *embeds into a poset sum* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *where each* $\mathbf{A}_p$ *is a finite MV-chain and* $\mathbf{P} = (P, \preceq)$ *is a finite root system.*

Proof. By Proposition 3.2, we know that $\mathbf{A}$ is isomorphic to an algebra of the form $\bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ where $\mathbf{D} = (D, \leq)$ is a finite poset and for all $d \in D$, $\mathbf{B}_d$ is a finite MV-chain. Now let P be the set of all finite non-empty sequences $(d_1, \ldots, d_n)$ of elements of D such that d_1 is a maximal element of D and for $i = 1, \ldots, n-1$, d_i is a cover of d_{i+1}, that is, $d_{i+1} < d_i$ and for all z if $d_{i+1} \leq z \leq d_i$, then either $z = d_i$ or $z = d_{i+1}$. For $p, p' \in P$, define $p \preceq p'$ iff p is an end extension of p', that is, if either $p = p'$ or there is a finite sequence σ of elements of D such that p is the juxtaposition of p' and σ. Clearly, $\mathbf{P} = (P, \preceq)$ is a root system. Now let for $p = (d_1, \ldots, d_n) \in P$, $\mathbf{A}_p = \mathbf{B}_{d_n}$. We define a map Φ from $\bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ into $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ letting for $h \in \bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ and for $p = (d_1, \ldots, d_n) \in P$, $\Phi(h)(p) = h(d_n)$. We claim that Φ is an embedding of $\bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ into $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$. The proof follows from the claims listed below.

Claim (a) If $h \in \bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$, then $\Phi(h) \in \bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$.

Proof of claim (a). For $p = (d_1, \ldots, d_n) \in P$, $\Phi(h)(p) = h(d_n) \in \mathbf{B}_{d_n} = \mathbf{A}_p$. Moreover if $\Phi(h)(p) = h(d_n) < e$ and $p \prec p' = (d_1, \ldots, d_i)$, then $d_i > d_n$, therefore $\Phi(h)(p') = h(d_i) = 0$. This ends the proof of claim (a).

Claim (b) Φ is one-one and preserves $\cdot$, $\vee$ and $\wedge$.

Proof of claim (b). If $h, k \in \bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ and $h \neq k$, then $h(d) \neq k(d)$ for some $d \in D$. Clearly there is $p = (d_1, \ldots, d_n) \in P$ such that $d_n = d$. Therefore $\Phi(h)(p) = h(d) \neq k(d) = \Phi(k)(p)$. Thus Φ is one-one. Moreover for $\circ \in \{\cdot, \vee, \wedge\}$ we have that for

$p = (d_1, \ldots, d_n) \in P$, $\Phi(h \circ k)(p) = (h \circ k)(d_n) = h(d_n) \circ k(d_n) = \Phi(h)(p) \circ \Phi(k)(p)$. This ends the proof of claim (b).

Claim (c). Φ preserves $\to$.

Proof of claim (c). Let $h, k \in \bigoplus_{d \in \mathbf{D}} \mathbf{B}_d$ and let $p = (d_1, \ldots, d_n) \in P$. We first compute $\Phi(h \to k)(p)$. Distinguish two cases:

(c1) If for all $d < d_n$ $h(d) \leq k(d)$, then $\Phi(h \to k)(p) = (h \to k)(d_n) = h(d_n) \to k(d_n)$;

(c2) Otherwise, $\Phi(h \to k)(p) = 0$.

Now we compute $(\Phi(h) \to \Phi(k))(p)$. Again, distinguish two cases:

(c1') If for all $p' \prec p$, $\Phi(h)(p') \leq \Phi(k)(p')$, then $(\Phi(h) \to \Phi(k))(p) = \Phi(h)(p) \to \Phi(k)(p) = h(d_n) \to k(d_n)$.

(c2') Otherwise $(\Phi(h) \to \Phi(k))(p) = 0$.

Thus it suffices to show that (c1) and (c1') are equivalent. Now (c1') reads: for all $p' = (d_1, \ldots, d_n, \ldots, d) \in P$, $h(d) \leq k(d)$, which is clearly equivalent to: for all $d \in D$ with $d < d_n$, $h(d) \leq k(d)$, that is, to (c1). This concludes the proof of Lemma 8.1. □

Definition 8.2. A *final segment* of a poset $\mathbf{P} = (P, \leq)$ is a subset F of P such that if $x \in F$, $y \in P$ and $x \leq y$, then $y \in F$.

Notation. In the sequel we denote by $\mathbf{MV}(\mathbf{Q})$ the MV-algebra with domain $[-1, 0] \cap \mathbf{Q}$ ($\mathbf{Q}$ is the set of rationals), with max and min as lattice operations, with monoid operation $x \cdot y = \max\{x + y, -1\}$ and residual $x \to y = \min\{y - x, 0\}$.

Lemma 8.3. *Let* $\mathbf{A} = \bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ *be a poset sum of integral residuated lattices, and let* F *be a final segment of* $\mathbf{P}$. *Let* $\mathbf{F}$ *be the subposet of* $\mathbf{P}$ *determined by* F, *and let* $\mathbf{B} = \bigoplus_{p \in \mathbf{F}} \mathbf{A}_p$. *Then:*

(a) *The map* Φ *defined, for all* $h \in \mathbf{B}$ *and for all* $p \in P$, *by*

$$\Phi(h)(p) = \begin{cases} h(p) & \text{if } p \in F \\ e & \text{otherwise} \end{cases}$$

is an embedding of $\mathbf{B}$ *into* $\mathbf{A}$.

(b) *Let for* $h \in \mathbf{A}$, $N_h = \{p \in P : h(p) \neq e\}$ *and let* $A_{\text{fin}} = \{h \in \mathbf{A} : N_h \text{ is finite}\}$. *Then* A_{fin} *is the domain of a subalgebra of* $\mathbf{A}$.

(c) *If for* $p \in P$, $\mathbf{B}_p$ *is a subalgebra of* $\mathbf{A}_p$, *then* $\bigoplus_{p \in \mathbf{P}} \mathbf{B}_p$ *is a subalgebra of* $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$.

Proof. (a) First of all, we prove that Φ maps $\mathbf{B}$ into $\mathbf{A}$. Let $h \in \mathbf{A}$ and $p \in P$. Then clearly $\Phi(h)(p) \in \mathbf{A}_p$. Moreover if $p < q$ and $\Phi(h)(p) < e$, then $p \in F$ and $q \in F$, as F is a final segment. Hence $\Phi(h)(q) = h(q) = 0$. It follows that $\Phi(h) \in \mathbf{A}$. That Φ is one-one and that it preserves $\cdot$, $\vee$ and $\wedge$ is clear, as these operations are defined pointwise. Now we prove that Φ preserves $\backslash$. Let $h, k \in \mathbf{A}$ and $p \in P$ be given. If $p \notin F$, then $\Phi(h \backslash k)(p) = (\Phi(h) \backslash \Phi(k))(p) = e$. If $p \in F$ and for all $q \in F$ such that $q < p$ we have $h(q) \leq k(q)$, then for all $q \in P$ with $q < p$ we have $\Phi(h)(q) \leq \Phi(k)(q)$, because if $q \notin F$, then $\Phi(h)(q) = \Phi(k)(q) = e$. Thus in this case, $\Phi(h \backslash k)(p) = (\Phi(h) \backslash \Phi(k))(p) = \Phi(h)(p) \backslash \Phi(k))(p)$. If there is $q \in F$ such that $q < p$ and $h(q) \not\leq k(q)$, then $\Phi(h \backslash k)(p) = (\Phi(h) \backslash \Phi(k))(p) = 0$. This shows compatibility with $\backslash$. The proof that Φ is compatible with / is symmetric, and part (a) is proved.

(b) Just note that $N_{h\cdot k} = N_{h\wedge k} = N_h \cup N_k$, $N_{h\vee k} = N_h \cap N_k$, $N_{h\backslash k} \subseteq N_k$ and $N_{k/h} \subseteq N_k$. Thus if $h, k \in A_{\text{fin}}$, then $h \cdot k$, $h \wedge k$, $h \vee k$, $h\backslash k$ and k/h are in A and $N_{h\cdot k}$, $N_{h\wedge k}$, $N_{h\vee k}$, $N_{h\backslash k}$ and $N_{k/h}$ are all finite. Thus $h \cdot k$, $h \wedge k$, $h \vee k$, $h\backslash k$ and k/h are in A_{fin}. Clearly the function which is constantly equal to e is in A_{fin} and (b) is proved.

(c) Almost trivial. □

Remark. The image of $\mathbf{B}$ under the embedding Φ defined in the proof of Lemma 8.3 (a) is the subalgebra of $\mathbf{A}$ consisting of all $h \in \mathbf{A}$ such that $h(p) = e$ for all $p \notin F$. This subalgebra will be denoted by $\mathbf{A}(F)$ and will be called *the relativization of* $\mathbf{A}$ *to* F.

Notation. In the sequel, given a poset $\mathbf{P}$, $\mathbf{Q}(\mathbf{P})$ will denote the algebra $\bigoplus_{p\in\mathbf{P}} \mathbf{A}_p$ with $\mathbf{A}_p = \mathbf{MV}(\mathbf{Q})$ for every $p \in P$. Moreover, given a poset sum $\mathbf{A} = \bigoplus_{p\in\mathbf{P}} \mathbf{A}_p$, $\mathbf{A}_{\text{fin}}$ will denote the subalgebra of $\mathbf{A}$ whose domain is the set of all $h \in \mathbf{A}$ such that N_h is finite, cf Lemma 8.3, (b).

Theorem 8.4. *Let* $\mathbf{P} = (P, \preceq)$ *be a poset such that every finite root system is isomorphic to a final segment of it. Then every finite GBL-algebra embeds into* $\mathbf{Q}(\mathbf{P})_{\text{fin}}$. *Therefore by Proposition 2.9 (iii),* $\mathbf{Q}(\mathbf{P})_{\text{fin}}$ *generates the variety of commutative and integral GBL-algebras as a quasivariety.*

Proof. Let $\mathbf{A}$ be any finite GBL-algebra. By Proposition 3.2 and by Lemma 8.1, $\mathbf{A}$ embeds into a poset sum $\bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$ such that $\mathbf{D} = (D, \leq)$ is a finite root system, and for $d \in D$, $\mathbf{A}_d$ is a finite MV-chain. Now $\mathbf{D}$ is isomorphic to a final segment of $\mathbf{P}$. Since any finite MV-chain embeds into $\mathbf{MV}(\mathbf{Q})$, by Lemma 8.3 (a) and (c), $\bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$ is a subalgebra of $\mathbf{Q}(\mathbf{P})$. Moreover, after identifying each element $h \in \bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$ with its image under the embedding Φ defined in Lemma 8.3, we have that for $h \in \bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$, $N_h \subseteq D$, therefore N_h is finite and $\bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$ is a subalgebra of $\mathbf{Q}(\mathbf{P})_{\text{fin}}$. Since $\mathbf{A}$ is a subalgebra of $\bigoplus_{d\in\mathbf{D}} \mathbf{A}_d$, the claim is proved. □

By Theorem 8.4, a strongly generic algebra for the variety $\mathcal{CIGBL}$ of commutative an integral GBL-algebras is given by $\mathbf{Q}(\mathbf{P})_{\text{fin}}$, where $\mathbf{P}$ is a poset such that every finite root system embeds in it as a final segment. An example of such a poset is given by the set $\omega^{<\omega}$ of all finite non-empty sequences of natural numbers, partially ordered by the relation $\preceq$ defined by $\sigma \preceq \tau$ iff either $\sigma = \tau$ or σ is an end extension of τ. Let $\mathbf{\Omega} = (\omega^{<\omega}, \preceq)$. Then, recalling that the variety of abelian ℓ-groups is generated as a quasivariety by the ℓ-group $\mathbf{Z}$ of integers, by Theorem 8.4 and by Proposition 2.6, we have:

Theorem 8.5. (a) $\mathbf{Q}(\mathbf{\Omega})_{\text{fin}}$ *is a countable strongly generic algebra for the variety* $\mathcal{CIGBL}$.

(b) $\mathbf{Q}(\mathbf{\Omega})_{\text{fin}} \times \mathbf{Z}$ *is a countable strongly generic algebra for the variety* $\mathcal{CGBL}$ *of commutative GBL-algebras.* □

We now investigate strongly generic models for some notable subvarieties of $\mathcal{CIGBL}$. Strongly generic models for the variety of MV-algebras and for the variety of BL-algebras are easy to obtain: for the variety of MV-algebras, just take $\mathbf{MV}(\mathbf{Q})$, which corresponds to $\mathbf{Q}_{\text{fin}}(\mathbf{P})$ with $\mathbf{P}$ the one-element poset. For the variety of BL-algebras, it follows from [AM03] that a strongly generic model is given by the ordinal sum of ω copies of $\mathbf{MV}(\mathbf{Q})$. This ordinal sum corresponds to the poset sum $\mathbf{Q}(\mathbf{N})_{\text{fin}}$, where $\mathbf{N} = (\omega, \leq)$ is the poset of natural numbers with the usual order.

We now consider the variety of Heyting algebras. This variety is also generated as a quasivariety by their finite members. These are poset sums of copies of the two-element MV-algebra $\mathbf{W}_1$. Now let for every $n > 0$, $\mathbf{W}_n$ denote the MV-chain with $n+1$ elements, and let $\mathbf{W}_n(\mathbf{\Omega}) = \bigoplus_{\sigma\in\mathbf{\Omega}} \mathbf{A}_\sigma$ with $\mathbf{A}_\sigma = \mathbf{W}_n$. Then by Lemma 8.3 we have that every finite Heyting algebra embeds into $\mathbf{W}_1(\mathbf{\Omega})_{\rm fin}$, therefore $\mathbf{W}_1(\mathbf{\Omega})_{\rm fin}$ is strongly generic for the variety of Heyting algebras. By a similar argument we have that the variety $\mathcal{GBL}_2$ of 2-potent GBL-algebras is generated as a quasivariety by $\mathbf{W}_2(\mathbf{\Omega})_{\rm fin}$. This depends on the fact that every 2-potent MV-chain is a subalgebra of $\mathbf{W}_2$. However, it is not true that for every n the algebra $\mathbf{W}_n(\mathbf{\Omega})_{\rm fin}$ is strongly generic for the variety $\mathcal{GBL}_n$ of n-potent GBL-algebras. For instance, let $x' = x \to x^3$ and $2x = (x'\cdot x')'$. Then the identity $x \vee x' = 2(x\vee x')^2$ is not valid in the 3-potent MV-algebra $\mathbf{W}_2$, but is valid in $\mathbf{W}_3(\mathbf{\Omega})_{\rm fin}$.

A countable strongly generic algebra for $\mathcal{GBL}_n$ is obtained as follows: let for every natural number k, $r(k)$ denote the remainder of the division of k by n, and let $w(k) = r(k)+1$. Let for every $\sigma = (k_1,\dots,k_n) \in \omega^{<\omega}$, $\mathbf{A}_\sigma = \mathbf{W}_{w(k_n)}$ and let $\mathbf{W}_{\leq n}(\mathbf{\Omega}) = \bigoplus_{\sigma\in\mathbf{\Omega}} \mathbf{A}_\sigma$. Then we have:

Theorem 8.6. *$\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\rm fin}$ is strongly generic for $\mathcal{GBL}_n$.*

Proof. It suffices to show that any finite n-potent GBL-algebra $\mathbf{A}$ embeds into $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\rm fin}$. By Proposition 3.2 and by Lemma 8.1, we can embed $\mathbf{A}$ into a poset sum $\bigoplus_{p\in\mathbf{P}} \mathbf{A}_p$ where $\mathbf{P}$ is a finite root system and for $p \in P$, $\mathbf{A}_p$ is an MV-chain with cardinality $\leq n+1$. We prove by induction on the cardinality n of P that there is a one-one map Ψ from P into $\omega^{<\omega}$ such that for every $p \in P$ the following conditions hold:

(a) If p is maximal with respect to $\leq$, then $\Psi(p)$ has length 1 (hence it is a maximal element in $\mathbf{\Omega}$).
(b) If p' is a cover of p, then $\Psi(p')$ is a cover of $\Psi(p)$ (thus in particular, $p \leq p'$ iff $\Psi(p) \preceq \Psi(p')$).
(c) Let m be the last element of the sequence $\Psi(p)$. Then $\mathbf{A}_p = \mathbf{W}_{w(m)}$.

For $n = 1$, the claim is easy: let p be the unique element of P, let $h \leq n$ be such that $\mathbf{A}_p = \mathbf{W}_h$, and let $\Psi(p) = (h-1)$ (the sequence whose unique element is $h-1$). Since $w(h-1) = h$, (a), (b) and (c) are satisfied.

Now suppose that the claim is true for every root system of cardinality less than n (with $n > 1$) and consider a root system $\mathbf{P}$ of cardinality n. Let p be a minimal element of P, and consider the subposet $(P', \leq)$ with domain $P' = P\setminus\{p\}$. By the induction hypothesis there is a map Ψ' on $(P', \leq)$ satisfying (a), (b) and (c). We distinguish two cases:

(i) If p is also maximal (thus p is incomparable with the remaining elements), then let h such that $\mathbf{A}_p = \mathbf{W}_h$, let k be big enough such that the one-element sequence $(kn+h-1)$ is not in the range of Ψ' and extend Ψ' to a function Ψ on P letting

$$\Psi(x) = \begin{cases} \Psi'(x) & \text{if} \quad x \neq p \\ (kn+h-1) & \text{if} \quad x = p \end{cases}.$$

It is readily seen that Ψ meets our requirements.

(ii) If p is not maximal, then since $\mathbf{P}$ is a finite root system, there is a unique cover p' of p. Let $\Psi(p') = (k_1,\dots,k_r)$ and let k be big enough so that the sequence

$(k_1, \ldots, k_r, kn+h-1)$ is not in the range of Ψ'. Now extend Ψ' to a function Ψ on P letting

$$\Psi(x) = \begin{cases} \Psi'(x) & \text{if} \quad x \neq p \\ (k_1, \ldots, k_r, kn+h-1) & \text{if} \quad x = p \end{cases}.$$

It is readily seen that Ψ meets our requirements.

Now by (a) and (b) the image $\Psi[P]$ of P under Ψ is a final segment of $\mathbf{\Omega}$ which is isomorphic to $\mathbf{P}$. Moreover by (c) we have $\mathbf{A}_p = \mathbf{A}_{\Psi(p)}$, therefore the relativization $\mathbf{W}_{\leq n}(\mathbf{\Omega})(\Psi[P])$ of $\mathbf{W}_{\leq n}(\mathbf{\Omega})$ to $\Psi([P])$ (cf Lemma 8.3) is a subalgebra of $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\text{fin}}$ which is isomorphic to $\bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$. Therefore $\mathbf{A}$ embeds into $\mathbf{W}_{\leq n}(\mathbf{\Omega})_{\text{fin}}$. This ends the proof. □

9. Normal GBL-algebras and GMV-algebras with a conucleus.

It is well known that every Heyting algebra can be represented as the algebra of open elements of a boolean algebra with an interior operator. In this section we partially extend this result to normal GBL-algebras. More precisely, we show that every normal GBL-algebra embeds into the image of a GMV-algebra under a conucleus.

Definition 9.1. A *conucleus* on a residuated lattice $\mathbf{A}$ is a unary operation σ on $\mathbf{A}$ such that for all $x, y \in A$ the following conditions hold:
$x \leq y$ implies $\sigma(x) \leq \sigma(y)$
$\sigma(x) \leq x$
$\sigma(x) = \sigma(\sigma(x))$
$\sigma(\sigma(x) \cdot \sigma(y)) = \sigma(x) \cdot \sigma(y)$
$\sigma(e) = e$.

Definition 9.2. Let $\mathbf{A}$ be a residuated lattice and σ be a conucleus on $\mathbf{A}$. Then $\sigma(\mathbf{A})$ denotes the structure $(\sigma(A), \cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma, /_\sigma, e)$, where $\sigma(A)$ is the image of A under σ, and for all $x, y \in \sigma(A)$, the operations $\cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma$ and $/_\sigma$ are defined as follows:
$(x \cdot_\sigma y) = x \cdot y$, $x \vee_\sigma y = x \vee y$, $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x /_\sigma y = \sigma(x/y)$.

The next lemma is proved in [MT].

Lemma 9.3. (cf [MT]). *If $\mathbf{A}$ is a residuated lattice and σ is a conucleus on $\mathbf{A}$, then $\sigma(\mathbf{A})$ is a residuated lattice (in particular, $\sigma(A)$ is closed under $\cdot_\sigma, \vee_\sigma, \wedge_\sigma, \backslash_\sigma$ and $/_\sigma$).*

Lemma 9.4. *Let $\mathbf{A} = \bigoplus_{p \in \mathbf{P}} \mathbf{A}_p$ be a poset sum of a family of integral and bounded residuated lattices with common top element e and with common bottom element 0, and let $\mathbf{B} = \prod_{p \in P} \mathbf{A}_p$. Define for all $f \in B$ and for all $p \in P$*

$$\sigma(f)(p) = \begin{cases} f(p) & \text{if } f(q) = e \text{ for all } q < p \\ 0 & \text{otherwise.} \end{cases}$$

Then σ is a conucleus and $\mathbf{A} = \sigma(\mathbf{B})$.

Proof. Clearly, properties (1), (2), (3) and (5) of conuclei are satisfied by σ. We verify property (4), that is, we prove that for all $f, g \in B$ and for all $p \in P$ we have $\sigma(\sigma(f) \cdot \sigma(g))(p) = (\sigma(f) \cdot \sigma(g))(p)$. The claim is clear if either $\sigma(f)(p) = 0$ or $\sigma(g)(p) = 0$. If $\sigma(f)(p) \neq 0$ and $\sigma(g)(p) \neq 0$, then for all $q < p$ we have $(\sigma(f) \cdot \sigma(g))(q) = e$, therefore by the definition of σ it follows that $\sigma(\sigma(f) \cdot \sigma(g))(p) =$

$(\sigma(f) \cdot \sigma(g))(p)$. Thus σ is a conucleus. Now note that for all $f \in B$ we have that $f \in A$ iff $f = \sigma(f)$. It follows that $\bigoplus_{p \in \mathbf{P}} A_p = \sigma(B)$ and for all $f \in B$, $\sigma(f)$ is the greatest element g of A such that $g \leq f$. Thus since the order on $\mathbf{A}$ is the restriction to A of the order on $\mathbf{B}$, $\sigma(\mathbf{B})$ and $\mathbf{A}$ have the same order, and therefore they have the same lattice operations. Moreover, the monoid operation is defined pointwise in both $\sigma(\mathbf{B})$ and $\mathbf{A}$. Hence $\sigma(\mathbf{B})$ and $\mathbf{A}$ coincide as lattice ordered monoids. It follows that residuals in $\sigma(\mathbf{B})$ and $\mathbf{A}$ also coincide, and the claim is proved. □

Theorem 9.5. *Every normal GBL-algebra* $\mathbf{A}$ *embeds into a GBL-algebra of the form* $\sigma(\mathbf{B})$ *for some GMV-algebra* $\mathbf{B}$ *and for some conucleus* σ *on* $\mathbf{B}$.

Proof. By Proposition [GT05], $\mathbf{A}$ can be represented as $\mathbf{A} = \mathbf{C} \times \mathbf{G}$ for some integral and normal GBL-algebra $\mathbf{A}$ and for some ℓ-group $\mathbf{G}$. Moreover $\mathbf{C}$ embeds into a poset sum of the form $\mathbf{D} = \bigoplus_{p \in \mathbf{P}} \mathbf{D}_p$ where for every $p \in P$, $\mathbf{D}_p$ is an integral GMV-algebra. Now by Lemma 9.4, there is an integral GMV-algebra $\mathbf{H}$ and a conucleus τ on $\mathbf{H}$ such that $\mathbf{D} = \tau(\mathbf{H})$. Clearly $\mathbf{A}$ embeds into $\mathbf{D} \times \mathbf{G}$. Now let $\mathbf{F} = \mathbf{H} \times \mathbf{G}$ and let for $(x, y) \in H \times G$, $\sigma(x, y) = (\tau(x), y)$. Clearly $\mathbf{F}$ is a GMV-algebra, σ is a conucleus on $\mathbf{F}$, $\mathbf{D} \times \mathbf{G} = \sigma(\mathbf{F})$ and $\mathbf{A}$ embeds into $\sigma(\mathbf{F})$, as desired. □

Note that the converse of Theorem 9.5 does not hold, that is, the image $\sigma(\mathbf{B})$ of a GMV-algebra $\mathbf{B}$ under a conucleus σ need not be a GBL-algebra. For instance, let $\mathbf{B}$ be the algebra $\mathbf{MV}(\mathbf{Q})$ defined in Section 8. Define a map σ on $\mathbf{B}$ as follows:

$$\sigma(x) = \begin{cases} 1 & \text{if} \quad x = 1 \\ x \wedge \frac{1}{2} & \quad\text{otherwise} \end{cases} .$$

It is readily seen that σ is a conucleus on $\mathbf{B}$. However $\sigma(\mathbf{B})$ is not a GBL-algebra, because $\frac{1}{4} = \frac{1}{2} \wedge_\sigma \frac{1}{4}$, but $\frac{1}{2} \cdot_\sigma (\frac{1}{2} \to_\sigma \frac{1}{4}) = \frac{1}{2} \cdot (\frac{3}{4} \wedge \frac{1}{2}) = \frac{1}{2} \cdot \frac{1}{2} = 0$. Of course for every GMV-algebra $\mathbf{B}$ and for every conucleus σ on $\mathbf{B}$, we have that $\sigma(\mathbf{B})$ is a GBL-algebra iff it satisfies the translation of the divisibility condition, namely the equation

$$(\text{div}_\sigma) \qquad \sigma(x) \cdot_\sigma (\sigma(x) \backslash_\sigma \sigma(y)) \wedge_\sigma e) = (\sigma(y) /_\sigma \sigma(x)) \wedge_\sigma e) \cdot_\sigma \sigma(x) = \sigma(x) \wedge_\sigma \sigma(y).$$

This remark and Theorem 9.5 can be summarized as follows:

Theorem 9.6. (1) *Let* $\mathbf{A}$ *be a normal residuated lattice. Then the following are equivalent:*
(a) $\mathbf{A}$ *is a GBL-algebra.*
(b) $\mathbf{A}$ *embeds into an algebra of the form* $\sigma(\mathbf{B})$ *where* $\mathbf{B}$ *is a GMV-algebra and* σ *is a conucleus on* $\mathbf{B}$ *such that* (div$_\sigma$) *holds.*

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Chapman University, Department of Mathematics and Computer Science, Orange, CA 92866, USA

E-mail address: jipsen@chapman.edu

University of Siena, Department of Mathematics and Computer Science, Pian dei Mantellini 44, 53100 Siena, Italy

E-mail address: montagna@unisi.it