COMPLEX ALGEBRAS OF RECTANGULAR BANDS

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Abstract. This note proves that the variety generated by complex algebras of rectangular bands is finitely based. This is in contrast to the variety generated by complex algebras of all semigroups.

Given a semigroup $(S, \cdot)$, the complex algebra of S is $\mathrm{Cm}(S) = (\mathcal{P}(S), \cup, \cap, \sim, \emptyset, S, \cdot)$, where $XY = \{xy : x \in X, y \in Y\}$ ($\sim X = S \backslash X$, and in the abstract setting $\emptyset$ and S are denoted by 0 and 1). For a class $\mathcal{K}$ of semigroups, $\mathrm{Cm}\mathcal{K} = \{\mathrm{Cm}(S) : S \in \mathcal{K}\}$. The variety generated by $\mathrm{Cm}\mathcal{K}$ is denoted by $\mathrm{VCm}\mathcal{K}$.

The variety RB of rectangular bands is the class of semigroups defined by the identity $xyx = x$. In the lattice of semigroup varieties it covers the two varieties Lz and Rz of left-zero and right-zero semigroups that are defined by $xy = x$ and $xy = y$ respectively. Bergman [2] showed that $\mathrm{VCm}\mathsf{Lz}$ and $\mathrm{VCm}\mathsf{Rz}$ are finitely based. An equational basis for $\mathrm{VCm}\mathsf{Lz}$ is given by equations for the Boolean part, together with $x(y \cup z) = xy \cup xz$, $(x \cup y)z = xz \cup yz$, $0x = 0 = x0$, $x(yz) = (xy)z$ (this much defines the variety BSG of Boolean semigroups), and $x1 = x = xx$. Contrary to expectation, the variety generated by complex algebras of all semigroups is a nonfinitely based subvariety of BSG (this follows from a result of Andreka and Nemeti [1] that the composition reducts of representable relation algebras generate a nonfinitely based variety), which motivates finding other finitely based varieties of the form $\mathrm{VCm}\mathcal{K}$.

Lemma 1. *Let B be a rectangular band. For all $X, Y, Z \subseteq B$ we have*
(a) *$X \cap YZ \neq \emptyset$ implies $XY \cap Y \neq \emptyset$ and $ZX \cap Z \neq \emptyset$,*
(b) *$X \cap XY \cap YX \neq \emptyset$ implies $X \cap YY \neq \emptyset$.*

Proof. In a rectangular band, $x = yz$ implies $xy = yzy = y$ and $zx = zyz = z$, hence (a) holds.

Suppose $x, x', x'' \in X$ and $y, y' \in Y$ such that $x = x'y = y'x''$. Then $yx = yx'y = y$ and $xy' = y'x''y' = y'$, so $y'y = xy'yx = x$, and (b) follows. $\square$

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Theorem 2. *An equational basis for* VCmRB *is given by the identities for* BSG *together with the following identities:*

(i) $x \le xx$
(ii) $x1y = xy$
(iii) $1x1 \cap y \le 1(x \cap 1y1)1$
(iv) $1(x \cap yz)1 \le 1(xy \cap y)1 \cap 1(zx \cap z)1$
(v) $1(x \cap xy \cap yx)1 \le 1(x \cap yy)1$

Proof. If is well-known (and easy to check) that the identities $x^2 = x$ and $xzy = xy$ hold for all rectangular bands, hence (i) and (ii) hold in CmRB. The term $t(x) = 1x1$ is a unary discriminator for complex algebras of rectangular bands, which means that $t(0) = 0$ and $t(x) = 1$ if $x \ne 0$. Therefore (iii) holds. Finally, (vi) and (v) are (in)equational translations of Lemma 1(a) and (b).

Now suppose A is a subdirectly irreducible Boolean semigroup that satisfies (i)–(v). Then $t(x) = 1x1$ is a congruence element for each $x \in A$, and from (iii) it follows that $x \cap t(y) = 0$ iff $t(x) \cap y = 0$ (i.e. t is self-conjugated). Hence $t(\sim t(x)) \cap t(x) = 0$, and so the complement of a congruence element is also a congruence element. Since A is directly indecomposable, it follows that A is simple, and hence t is a unary discriminator in A. This property and the positive identities (i)–(v) are preserved by canonical extensions, so t is a unary discriminator in A^σ.

We now construct a rectangular band B such that A^σ is embedded in $\mathrm{Cm}(B)$. Let $D = At(A^\sigma)$. For $a, b \in D$, define $a \equiv b$ if $a \le bb$.

Claim (a): $\equiv$ is an equivalence relation. From (i) we get $a \equiv a$. If $a \equiv b$, then $a \cap bb \ne 0$, hence $ab \cap b \ne 0$ and $ba \cap b \ne 0$ by (iv). Since b is an atom, we have $b \cap ba \cap ab \ne 0$, and therefore (v) implies $b \cap aa \ne 0$, whence $b \equiv a$. Finally, if $a \le bb$ and $b \le cc$ then $a \le cccc \le c1c = cc$.

Note that for each atom a, $aa = \bigcup\{b \in D : b \le aa\} = \bigcup a/{\equiv}$.

Claim (b): The set $C = \{aa : a \in D\}$ is a rectangular band. It suffices to check that $aabbaa = a1abba1a = a1a = aa$, since $abba \ne 0$ for all atoms a, b.

From the easy structure theory of rectangular bands we conclude that there exist sets L, R and an isomorphism $\alpha : C \cong (L \times R, \circ)$, where the operation $\circ$ is given by $(p, q) \circ (r, s) = (p, s)$.

Let $B = ((L \times D) \times (R \times D), \circ)$. Since $|(c] \cap D|^2 \le |D \times D|$, we can choose, for each $c \in C$, a map $\beta_c : (c] \cap D \hookrightarrow \mathcal{P}(D \times D)$ such that

(1) $a \ne b$ implies $\beta_c(a) \cap \beta_c(b) = \emptyset$ (all $a, b \in D$),
(2) $\{u : (u, v) \in \beta_c(a)\} = D = \{v : (u, v) \in \beta_c(a)\}$ (all $a \in D$),
(3) $\bigcup\{\beta_c(a) : c \ge a \in D\} = D \times D$.

Define $\gamma : D \to \mathrm{Cm}(B)$ by

$$\gamma(a) = \{((\pi_L\alpha(aa), u), (\pi_R\alpha(aa), v)) : (u, v) \in \beta_{aa}(a)\}$$

and extend it to $\hat{\gamma} : A^\sigma \hookrightarrow \mathrm{Cm}(B)$ by distributivity over $\cup$. Then $\hat{\gamma}$ is a Boolean embedding by (1). It remains to show that $\hat{\gamma}(ab) = \gamma(a) \circ \gamma(b)$.

For $a, b \in D$, $ab \le aabb \le a1b = ab$, so $ab = c \in C$. Now

$$\begin{aligned}
\hat{\gamma}(c) &= \bigcup\{\gamma(d) : c \ge d \in D\} \\
&= \bigcup\{\{((\pi_L\alpha(c), u), (\pi_R\alpha(c), v)) : (u, v) \in \beta_c(d)\} : c \ge d \in D\} \\
&= \{((\pi_L\alpha(c), u), (\pi_R\alpha(c), v)) : (u, v) \in D \times D\}
\end{aligned}$$

where we used the observation $c = dd$ in the second step, and we used property (3) in the third step. On the other hand,

$$\begin{aligned}
\gamma(a) \circ \gamma(b) &= \{((\pi_L\alpha(aa), u), (\pi_R\alpha(aa), v)) : (u, v) \in \beta_{aa}(a)\} \\
&\quad \circ \{((\pi_L\alpha(bb), u), (\pi_R\alpha(bb), v)) : (u, v) \in \beta_{bb}(b)\} \\
&= \{((\pi_L\alpha(aa), u), (\pi_R\alpha(bb), v)) : (u, v) \in D \times D\}
\end{aligned}$$

by property (2). But $\pi_L\alpha(c) = \pi_L\alpha(ab) = \pi_L\alpha(aaab) = \pi_L\alpha(aac) = \pi_L(\alpha(aa) \circ \alpha(c)) = \pi_L\alpha(aa)$, and similarly $\pi_R\alpha(c) = \pi_R\alpha(bb)$. Hence $\hat{\gamma}(ab) = \gamma(a) \circ \gamma(b)$. $\square$

References

[1] H. Andréka and I. Németi, *Axiomatization of identity-free equations valid in relation algebra*, Algebra Universalis, **35** (1996) 256–264.

[2] C. Bergman, *Complex algebras of left-zero semigroups*, private communication.