Multicategories in Computer Science

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The concept of a multicategory first appeared in a paper of Lambek in 1979. It was further generalized and studied by Szabo, and Velinov. Lambek was inspired to introduce multicategories by certain analogy between the Gentzen's sequent calculus and Bourbaki's treatment of bilinear maps, but logic is not the only possible suggestive source. In essence any rich enough mathematical area, which involves functions of many variables gives intuitive support for introducing multicategories. Multicategories may appear in several different forms. For example, considered as an algebraic system, a Y−multicategory can be defined as follows:

A $Y-$**multicategory** $\mathcal{Y}$ *is a tuple*

$$\mathcal{Y} = \langle \mathbf{O}, \mathbf{A}, +, \bullet, dom, cod, \gamma, I \rangle$$

such that $Gr(\mathcal{Y}) = \langle \mathbf{O}, \mathbf{A}, +, \bullet, dom, cod \rangle$ *is a mongraph (a graph with additional monoidal operation* $+$ *on the objects and an unit* $\bullet$ *),* $I : \mathbf{O} \to \mathbf{A}$ *is a unary operation selecting the identity arrow associated with each object and* $\gamma : \mathbf{A} \times (\mathbf{O} \times \mathbf{O} \times \mathbf{O}) \times \mathbf{A} \to \mathbf{A}$ *is a partial function (the arrow operation) which, when defined, puts in correspondence an arrow denoted by* $x\lceil A\underline{B}C\rceil y$ *to each tuple* $\langle x, A\underline{B}C, y\rangle$*. The components of a* $Y-$*multicategory fulfil the following axioms.*

For any arrows $x, y, z,$ *and objects* A, B, C, D, E *:*

- $x\lceil A\underline{B}C\rceil y$ *exists iff* $dom(x) = ABC$ *and* $cod(y) = B$*;*
- *if* $x\lceil A\underline{B}C\rceil y$ *exists then* $cod(x\lceil A\underline{B}C\rceil y) = cod(x)$*, and* $dom(x\lceil A\underline{B}C\rceil y) = Adom(y)B$*;*
- $(x\lceil A\underline{B}CDE\rceil y)\lceil Adom(y)C\underline{D}E\rceil z = (x\lceil ABC\underline{D}E\rceil z)\lceil A\underline{B}Cdom(z)E\rceil y$ *provided the described composites exist (commutativity);*
- $(x\lceil A\underline{B}C\rceil y)\lceil AD\underline{cod(z)}EC\rceil z = x\lceil A\underline{B}C\rceil \left(y\lceil D\underline{cod(z)}E\rceil z\right)$ *provided the described composites exist (associativity);*
- $I_{cod(x)}\lceil \underline{cod(x)}\rceil x = x\lceil \underline{dom(x)}\rceil I_{dom(x)} = x.$

Multicategories can be applied in many areas of Computer Science. For example, they can be used to describe and study finite automata, syntax and semantics of formal grammars, derivation systems, term rewriting systems.