# A natural interpretation of fuzzy mappings

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We present a new and natural interpretation of fuzzy mappings in a Heyting valued model for intuitionistic set thoery. It comes from the natural interpretasion of fuzzy sets and relations in the same model (Cf. [2][3]).

There has been several different definitions of fuzzy mapping or fuzzy function in the literature, one of which identifies fuzzy mapping with fuzzy relation. Our definition seems to be quite natural and to clear the meaning of the famous Zadeh's Extension Principle, which has been said to be very important in fuzzy theory.

Let $H$ be a complete Heyting algebra and $V^H$ be the $H$-valued model in [1]. The *Heyting value* $||\varphi||$ and the *check set* $\check{x}$ for usual (*crisp* ) set $x(\in V)$ are defined as usual. Basic operations such as *intersection*, *union*, and *complement* of sets, *composition* and *inverse* of relations, etc. are naturally defined in $V^H$.

Every $A \in V^H$ is called an *$H$-fuzzy set*, and every subset of $\check{X}$ in $V^H$ (for crisp $X$) is called an *$H$-fuzzy subset of $X$*. The mapping $\mu_A : X \longrightarrow H;\ x \longmapsto ||\check{x} \in A||$ is called the *membership function of $A$ on $X$*. There is a natural correspondence between $H$-fuzzy subsets of $X$ and mappings from $X$ to $H$, which preserves order and basic set operations.

An $H$-fuzzy subset $R$ of $X \times Y$ is called an *$H$-fuzzy relation from $X$ to $Y$*. Inverses and compositions of $H$-fuzzy relations as well as the defining properties of equivalence relations are showed to exactly correspond to the standard definitions in ordinary fuzzy set theory.

Mappings are considered as special cases of relations as usual. An *$H$-fuzzy mapping $f$ from $X$ to $Y$* is a mapping from $\check{X}$ to $\check{Y}$ in $V^H$, i.e. $||f : \check{X} \longrightarrow \check{Y}|| = \mathbf{1}$. For every $A \in V^H$, the image $f(A)$ is an $H$-fuzzy subset of $Y$, and $\mu_{f(A)}(y) = \bigvee_{x \in X}(\mu_A(x) \wedge \mu_f\langle xy\rangle)$ holds for all $y \in Y$.

For every map $\varphi : X \longrightarrow Y$ between crisp sets, an $H$-fuzzy mapping $\widetilde{\varphi} : \check{X} \longrightarrow \check{Y}$ can be naturally defined. Then for every $A \in V^H$, the image $\widetilde{\varphi}(A)$ is an $H$-fuzzy subset of $Y$, and

$$\mu_{\widetilde{\varphi}(A)}(y) = \bigvee_{x \in X\, \varphi(x)=y} \mu_A(x) = \bigvee_{x \in \varphi^{-1}(y)} \mu_A(x)$$

holds for all $y \in Y$. This equation is identical with the Extension Principle.

## References

[1] M. Shimoda, Categorical aspects of Heyting-valued models for intuitionistic set theory, *Comment. Math. Univ. Sancti Pauli*, 30(1), 17–35, 1981.

[2] M. Shimoda, A natural interpretation of fuzzy sets and fuzzy relations, submitted, 1998.

[3] M. Shimoda, A natural interpretation of fuzzy sets and relations (abstract), *The Bulletin of Symbolic Logic*, 5(1), 132, 1999.