Constructing distributive lattices with a given link lattice.

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Let $\langle K, \leq \rangle$ be a lattice and suppose that for each x from K there is a lattice $\langle K_x, \leq_x \rangle$ such that

1. if y covers x (i.e. $x \leq y$ and there is no z such that $x < z < y$) in $\mathcal{K}$ then $K_x \cap K_y \neq \emptyset$;
2. if $x \leq y$ and $K_x \cap K_y \neq \emptyset$ then $K_x \cap K_y$ is a filter of K_x and an ideal of K_y. Moreover, the orderings $\leq_x$ and $\leq_y$ are identical on $K_x \cap K_y$;
3. $K_x \cap K_y \subseteq K_{x \wedge y} \cap K_{x \vee y}$, for all x, y from K.

Herrmann proved [3] that if the above conditions hold then $\langle \bigcup_{x \in K} K_x, \leq \rangle$, where $\leq$ is a transitive closure of the sum of $\leq_x$ for $x \in K$, is a lattice, called K-sum of the family $\{K_x\}_{x \in K}$. We shall call the lattice K the link lattice of the lattice $\bigcup_{x \in K} K_x$. Moreover, every modular lattice not containing infinite chains is the K-sum of its atomistic intervals. In the distributive case it means that a finite distributive lattice $\mathcal{D}$ is the K-sum of its all maximal Boolean fragments. Thus, if $\mathcal{D}$ has a scarce decomposition [2] then $\mathcal{D}$ is the Wro/nski sum [7] and the Herrmann K-sum of the same components.

It was also prooved by Herrmann [3] that every finite lattice is a link lattice of some finite distributive lattice. However the problem of dimensions of the maximal Boolean fragments of the finite distributive lattice with a given link lattice is still open. We are going to discuss some theorems concerning this problem.

Moreover, we can prove the following theorem:

Theorem 1 *Let K be a finite lattice. If $\mathcal{B}_0 \oplus \mathcal{B}_1$ is the Wro/nski sum of Boolean lattices $\mathcal{B}_0$ and $\mathcal{B}_1$ and there are mappings f_0 and f_1 such that f_0 is* *a* $\vee$*-homomorphism from* K *into* B_0*,* f_1 *is a* $\wedge$*-homomorphism from* K *into* B_1 *and*

$$f_0(0) = 0_0,\ f_0(1) = 0_1,\ f_1(0) = 1_0,\ f_1(1) = 1_1$$

then f_0 *and* f_1 *determine some distributive lattice with the link lattice* K*. Moreover, if the unit of* K *is the join of atoms of* K *and the zero of* K *is the meet of coatoms of* K *then the above construction determines all finite distributive lattice with the link lattice* K*.*

References

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