Constructing distributive lattices with a given link lattice.

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Let \( \langle K, \leq \rangle \) be a lattice and suppose that for each \( x \) from \( K \) there is a lattice \( \langle K_x, \leq_x \rangle \) such that

1. if \( y \) covers \( x \) (i.e. \( x \leq y \) and there is no \( z \) such that \( x < z < y \)) in \( K \) then \( K_x \cap K_y \neq \emptyset \);

2. if \( x \leq y \) and \( K_x \cap K_y \neq \emptyset \) then \( K_x \cap K_y \) is a filter of \( K_x \) and an ideal of \( K_y \). Moreover, the orderings \( \leq_x \) and \( \leq_y \) are identical on \( K_x \cap K_y \);

3. \( K_x \cap K_y \subseteq K_{x \wedge y} \cap K_{x \vee y} \), for all \( x, y \) from \( K \).

Herrmann proved [3] that if the above conditions hold then \( \langle \cup_{x \in K} K_x, \leq \rangle \), where \( \leq \) is a transitive closure of the sum of \( \leq_x \) for \( x \in K \), is a lattice, called \( K \)-sum of the family \( \{ K_x \}_{x \in K} \). We shall call the lattice \( K \) the link lattice of the lattice \( \cup_{x \in K} K_x \). Moreover, every modular lattice not containing infinite chains is the \( K \)-sum of its atomistic intervals. In the distributive case it means that a finite distributive lattice \( D \) is the \( K \)-sum of its all maximal Boolean fragments. Thus, if \( D \) has a scarce decomposition [2] then \( D \) is the Wro/nski sum [7] and the Herrmann \( K \)-sum of the same components.

It was also proved by Herrmann [3] that every finite lattice is a link lattice of some finite distributive lattice. However the problem of dimensions of the maximal Boolean fragments of the finite distributive lattice with a given link lattice is still open. We are going to discuss some theorems concerning this problem.

Moreover, we can prove the following theorem:

**Theorem 1** Let \( K \) be a finite lattice. If \( B_0 \oplus B_1 \) is the Wro/nski sum of Boolean lattices \( B_0 \) and \( B_1 \) and there are mappings \( f_0 \) and \( f_1 \) such that \( f_0 \) is
a \textit{\lor}-homomorphism from } K \textit{ into } B_0, f_1 \textit{ is a } \textit{\land}-\textit{homomorphism from } K \textit{ into } B_1 \textit{ and }
\begin{equation}
f_0(0) = 0_0, \quad f_0(1) = 0_1, \quad f_1(0) = 1_0, \quad f_1(1) = 1_1
\end{equation}
then } f_0 \textit{ and } f_1 \textit{ determine some distributive lattice with the link lattice } K. \textit{ Moreover, if the unit of } K \textit{ is the join of atoms of } K \textit{ and the zero of } K \textit{ is the meet of coatoms of } K \textit{ then the above construction determines all finite distributive lattice with the link lattice } K. \textit{

References


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