Categorical Equivalence of Varieties and Invariant Relations

K. Denecke and O. Lders

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ABSTRACT

Each variety of algebras of a given type can be considered as a category, where the objects are the algebras in the variety, while the morphisms are the homomorphisms between algebras. In applications the weaker concept of an *equivalence* between categories is used more often than the concept of an isomorphism. For instance, the category $\mathcal{C}$ of finite-dimensional vector spaces over the field K is equivalent with the category of all vector spaces which are dual to the vector spaces from $\mathcal{C}$.

For two varieties V and W to be *categorically equivalent* means precisely the following: there exists a covariant functor F from V to W such that

(i) for all $\underline{A}, \underline{B} \in V$ the functor F defines a bijection between $hom(\underline{A}, \underline{B})$

(ii) for all $\underline{C} \in W$ there is an algebra $\underline{A} \in V$ such that $F(\underline{A})$ is isomorphic to $\underline{C}$.

We may remark that if V and W are categorically equivalent under the categorical equivalence F then $\underline{A}$ is isomorphic to $\underline{B}$ iff $F(\underline{A})$ is isomorphic to $F(\underline{B})$ whenever $\underline{A}, \underline{B} \in V$.

For varieties of R-modules the problem of an algebraic characterization of categorical equivalences was solved a long time ago: The variety of R-modules is categorically equivalent with the variety of S-modules iff the rings R and S are Morita equivalent. In Universal Algebra the best known and most important example of a categorical equivalence is that one between the variety of Boolean algebras (generated by the two-element Boolean algebra) and any variety generated by a single primal algebra, i.e. a finite algebra where arbitrary

operations on the universe are term operations of the algebra ([Hu; 69]). This example shows that from an algebraic point of view categorically equivalent varieties can have quite different properties.

In this paper we will give a different characterization of categorical equivalent varieties using invariant relations.

When the authors started in the late eighties to generalize the categorical equivalence of T. K. Hu between the variety of Boolean algebras and the varieties generated by an arbitrary finite algebra to varieties generated by arbitrary two-element algebras ([Den; 85]) they used duality theory and the "Equivalent Prevarieties Theorem" of B. A. Davey and H. Werner ([Dav-W; 80]). In 1992 R. McKenzie ([McK; 96]) gave a complete algebraic characterization of categorical equivalent varieties using the matrix power construction and idempotent invertible terms. In [Den-L; 95] the authors started to use the Galois-connection between term operations of an algebra and its invariant relations. Our main result is that two varieties which are generated by finite algebras are categorically equivalent iff their algebras of invariant relations are isomorphic. Moreover, we will give several applications.

References

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