Logic and Universal Algebra:
some basic connections

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Abstract

These lecture notes explore some of the basic connections between the fields of logic and universal algebra. In Part 1 we discuss some of the most fundamental logical issues that arise in universal algebra. We prove two seminal results by Birkhoff: the completeness theorem for equational logic and the definability theorem identifying varieties with equational classes.

The second part of the notes gives an introduction-by-example to the field of algebraic logic, discussing the algebraization of classical propositional logic in the class of Boolean algebras. We prove Stone's representation theorem and show how it relates to the strong completeness theorem for classical logic. We also relate Beth's definability theorem to the property of the class of Boolean algebras that epis are surjective.

Preface

These pages form the lecture notes for a short course that I gave in December 1999 at LUATCS'99, the First Southern African Summer School on Logic, Universal Algebra and Theoretical Computer Science. In the course I tried to explain some of the most basic connections between the first two of these three fields: Logic and Universal Algebra.

Although these notes are more or less self-contained in the sense that all definitions are provided, I would not immediately recommend them for self study. The first part presupposes some familiarity with the basics of universal algebra, and likewise, the second part might not make much sense if the reader has never been exposed to the basic semantics and axiomatics of classical propositional logic. In both cases however, I assume that the first one or two lectures in the parallel courses on Universal Algebra and Logic, respectively, will provide sufficient background knowledge.

Finally, since these notes might be used again, I appreciate comments, both on the content and on the presentation. I would like to thank Eva Hoogland and Ramon Jansana for such comments on an earlier version.

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I Algebra as Logic

In this first Part of the notes we discuss some of the most fundamental logical issues that arise in universal algebra. Universal algebra can be seen as a special branch of model theory, in which we are dealing with structures having operations only (i.e., no relations), and in which the language we use to talk about these structures uses equations only. Now, given such a semantics for a formal language, natural questions for logicians to ask are:

- which classes of structures can be defined in this language?
- is there a natural formal calculus capturing the notion of semantic consequence that comes with the semantics?

In the case of universal algebra, two outstanding results, both due to G. Birkhoff, provide answers to these questions. First, there is the theorem that a class of algebras is a variety if and only if it is definable by a set of equations; and second, we have a completeness result identifying, for a given set of equations E, the set of equations that can be *derived* from E with those that are *semantic consequences* of E. We will prove both results here.

1 Algebraic Syntax

Definition 1.1 (similarity type) An algebraic *similarity type* or *signature* is a pair $\Sigma = (F, \rho)$ where F is a non-empty set and ρ is a function $F \to \mathbb{N}$. Elements of F are called *function symbols* or *operators*; the function ρ assigns to each operator $f \in F$ a finite *arity* or *rank*, indicating the number of arguments that f can be applied to. Nullary function symbols (i.e., of rank zero) are called *constants*. $\lhd$

Usually, we will be sloppy in our notation and terminology, for instance writing that $f \in \Sigma$ instead of $f \in F$, or using infix notation with an associated bracketing convention.

Definition 1.2 (terms and equations) Given an algebraic similarity type Σ and a set X of elements called *variables*, we define the set $T_\Sigma(X)$ of Σ-*terms over* X inductively as the smallest set T containing all variables in X such that $f(t_1, \ldots, t_n)$ is in T whenever $t_1, \ldots, t_n$ are in T and f is a function symbol of rank n. (In particular, $T_\Sigma(X)$ contains all constants.)

An *equation* is a pair (s, t) of terms, usually written as $s \approx t$.

The set of variables occurring in a term s is denoted as $Var(s)$; $Var(s \approx t)$ is defined similarly. $\lhd$

Definition 1.3 (specification) A specification is a pair (Σ, E) where Σ is a similarity type and E is a set of equations that we will usually call *axioms*. As a slight variation we may also call the set E a *specification* of type Σ. $\lhd$

Example 1.4 The signature of groups would consist of one binary, one unary and one nullary function symbol, say $\circ$ (for product), i (for inverse) and e (for the identity element). As a possible specification for groups one could consider the following three axioms.

$(G1)$ $x \circ (y \circ z) \approx (x \circ y) \circ z$
$(G2)$ $e \circ x \approx x$
$(G3)$ $i(x) \circ x \approx e$

The second example is written in a computer science format.

Example 1.5 `Naturals` is a specification designed to describe the algebra $\mathcal{N} = (\mathbb{N}, 0, S, +, *)$ of the natural numbers with the successor function, addition and multiplication. The notation should be self-explanatory.

```
specification Naturals

  sorts nat

  functions
    0      :                 -> nat
    succ   : nat             -> nat
    add    : nat # nat       -> nat
    mul    : nat # nat       -> nat

  equations
    [N1]   : add(x,0)        = x
    [N2]   : add(x,succ(y))  = succ(add(x,y))
    [N3]   : mul(x,0)        = 0
    [N4]   : mul(x,succ(y))  = add(x,mul(x,y))

end Naturals
```

2 Algebras and Varieties

We assume familiarity with the basic notions of universal algebra, but in order to fix our notation we recall some definitions. In this section we fix a similarity type Σ.

Definition 2.1 (Σ-algebras) Let A be some set, and n a natural number; an *n-ary operation* on A is a function from A^n to A. A *Σ-algebra* is a pair $\mathcal{A} = (A, I)$ where A is a non-empty set called the *carrier* of the algebra, and I is an *interpretation*, i.e., a function assigning an n-ary operation $f_{\mathcal{A}}$ on A to each function symbol f of rank n. Such an algebra will also be denoted as $\mathcal{A} = (A, f_{\mathcal{A}})_{f \in \Sigma}$. $\lhd$

In practice, when no confusion arises we will often abuse notation and omit the subscripts referring to the algebra $\mathcal{A}$, writing $\mathcal{A} = (A, f)_{f \in \Sigma}$.

Definition 2.2 (homomorphisms) Let $\mathcal{A} = (A, f_{\mathcal{A}})_{f\in\Sigma}$ and $\mathcal{B} = (B, f_{\mathcal{B}})_{f\in\Sigma}$ be two algebras of the same similarity type. A map $\alpha : A \to B$ is called a homomorphism if for all $f \in \Sigma$, and all $a_1, \ldots, a_n \in A$ (where n is the rank of f):

$$\alpha(f_{\mathcal{A}}(a_1, \ldots, a_n)) = f_{\mathcal{B}}(\alpha a_1, \ldots, \alpha a_n). \tag{1}$$

As a special case, for constants c we get

$$\alpha(c_{\mathcal{A}}) = c_{\mathcal{B}}.$$

A bijective homomorphism is called an *isomorphism.*

We say that $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$, notation: $\mathcal{A} \twoheadrightarrow \mathcal{B}$, if there is a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Given a class K of algebras, $\mathbb{H}\mathsf{K}$ denotes the class of homomorphic images of algebras in K. We will be sloppy about the distinction between isomorphic algebras. $\lhd$

Definition 2.3 (subalgebra) Let $\mathcal{A}$ be an algebra, and B a subset of the carrier A. If B is closed under every operation $f_{\mathcal{A}}$, then we call the structure $\mathcal{B} = (B, f_{\mathcal{A}} \restriction_B)_{f\in\Sigma}$ a *subalgebra* of $\mathcal{A}$. We say that $\mathcal{C}$ is *embeddable* in $\mathcal{A}$, notation: $\mathcal{C} \rightarrowtail \mathcal{A}$, if $\mathcal{C}$ is isomorphic to a subalgebra of $\mathcal{A}$, or, equivalently, if there is an *embedding* or injective homomorphism from $\mathcal{C}$ to $\mathcal{A}$.
Given a class K of algebras, $\mathbb{S}\mathsf{K}$ denotes the class of isomorphic copies of subalgebras of algebras in K. $\lhd$

Definition 2.4 (product) Let $(\mathcal{A}_j)_{j\in J}$ be a family of algebras. We define the *product* $\prod_{j\in J} \mathcal{A}_j$ of this family as the algebra $\mathcal{A} = (A, f_{\mathcal{A}})_{f\in\Sigma}$ where A is the cartesian product $\prod_{j\in J} A_j$ of the carriers A_j, and the operation $f_{\mathcal{A}}$ is defined coordinatewise, i.e., for elements $a_1, \ldots, a_n \in \prod_{j\in J} A_j$, $f_{\mathcal{A}}(a_1, \ldots, a_n)$ is the element of $\prod_{j\in J} A_j$ given by:

$$f_{\mathcal{A}}(a_1, \ldots, a_n)(j) = f_{\mathcal{A}_j}(a_1(j), \ldots, a_n(j)).$$

Given a class K of algebras, $\mathbb{P}\mathsf{K}$ denotes the class of isomorphic copies of products of algebras in K. $\lhd$

Definition 2.5 (variety) A class of algebras is a *variety* if it is closed under taking subalgebras, homomorphic images and products. Given a class K of algebras, $\mathbb{V}\mathsf{K}$ denotes the variety generated by K, i.e., the smallest variety containing K. $\lhd$

A famous result of Tarski states that $\mathbb{V}\mathsf{K} = \mathbb{H}\mathbb{S}\mathbb{P}\mathsf{K}$.

Definition 2.6 (subdirect products) An algebra $\mathcal{A}$ is a *subdirect product* of a collection of algebras $(\mathcal{B}_j)_{j\in J}$ if $\mathcal{A}$ is a subalgebra of the product $\prod_{j\in J} \mathcal{B}_j$ such that for each i, if we restrict the projection function $\pi_i : \prod_{j\in J} B_j \to B_i$ to A, we obtain a surjection.

An embedding $\alpha : \mathcal{A} \rightarrowtail \prod_{j\in J} \mathcal{B}_j$ is *subdirect* if $\alpha(\mathcal{A})$ is a subdirect product of the $\mathcal{B}_j$. $\lhd$

An algebra $\mathcal{A}$ is isomorphic to a subdirect product of the family $(\mathcal{B}_i)_{i\in J}$ iff there is an injection $\alpha : \mathcal{A} \rightarrowtail \prod_{j\in J} \mathcal{B}_j$ such that for each $i \in J$, the composition map $\pi_i \circ \alpha : \mathcal{A} \to \mathcal{B}_i$ is surjective.

Definition 2.7 (congruence) Let $\mathcal{A}$ be an algebra for the similarity type Σ. An equivalence relation $\sim$ on A (i.e., a reflexive, symmetric and transitive relation) is a *congruence* if it satisfies, for all $f \in \Sigma$

$$\text{if } a_1 \sim b_1 \ \& \ \ldots \ \& \ a_n \sim b_n, \text{ then } f_{\mathcal{A}}(a_1, \ldots, a_n) \sim f_{\mathcal{A}}(b_1, \ldots, b_n).$$

where n is the rank of f. ◁

It follows from the definition of a congruence that the following is a correct definition:

Definition 2.8 (quotient algebra) Let $\mathcal{A}$ be an Σ-algebra, and $\sim$ a congruence on $\mathcal{A}$. The *quotient algebra of $\mathcal{A}$ by $\sim$* is the algebra whose carrier is the set

$$A/{\sim} = \{[a] \mid a \in A\}$$

of equivalence classes of A under $\sim$, and whose operations are defined by

$$f_{\mathcal{A}/\sim}([a_1], \ldots, [a_n]) = [f_{\mathcal{A}}(a_1, \ldots, a_n)].$$

The map $\nu : A \to A/{\sim}$ given by $\nu(a) = [a]$ is called the *natural homomorphism*; it is easy to see that it is indeed a homomorphism. ◁

3 Equational Logic

Equational *logic* arises if we consider ways of *deducing* new equations from old. Mathematicians usually do not care much about the details of such deductions. For instance, in order to prove that the left inverse of a group element x is also a right inverse, they could write something like:

$$\begin{array}{rcl} x \circ i(x) & = & e \circ x \circ i(x) \\ & = & i(i(x)) \circ i(x) \circ x \circ i(x) \\ & = & i(i(x)) \circ e \circ i(x) \\ & = & i(i(x)) \circ i(x) \\ & = & e. \end{array}$$

Logicians and computer scientists are interested in the precise way such deductions are performed. In this section we define and discuss a formal derivation system for *equational logic*.

In order to formulate one of our derivation rules we need the following definition.

Definition 3.1 (simple substitution) Let s and t be terms, an let x be a variable. The term $t[s/x]$, 't with s substituted for x', denotes the term we obtain by replacing every occurrence in t of x by s. Inductively we may define:

$$\begin{array}{rcl} x[s/x] & := & s, \\ y[s/x] & := & y, \\ f(t_1, \ldots, t_n)[s/x] & := & f(t_1[s/x], \ldots, t_n[s/x]). \end{array}$$

◁

Definition 3.2 (equational logic) Let Σ be an algebraic similarity type, and E a Σ-specification. The set of equations that are *derivable* from E is inductively defined by the following schema:

- **(axioms)** The equations in E are derivable from E; they are called *axioms*.
- **(reflexivity)** Every equation $t \approx t$ is derivable from E.
- **(symmetry)** If $t_1 \approx t_2$ is derivable from E, then so is $t_2 \approx t_1$.
- **(transitivity)** If the equations $t_1 \approx t_2$ en $t_2 \approx t_3$ are derivable from E, then so is $t_1 \approx t_3$.
- **(replacement)** Suppose that all equations $t_1 \approx u_1, \ldots, t_n \approx u_n$ are derivable from E, and that f is a function symbol of rank n. Then the equation $f(t_1, \ldots, t_n) \approx f(u_1, \ldots, u_n)$ is derivable from E as well.
- **(substitution)** If $t_1 \approx t_2$ is derivable from E, then so is the equation $t_1[s/x] \approx t_2[s/x]$, for every term s and every variable x occurring in t.

To denote the fact that the equation $t_1 \approx t_2$ is derivable from E, we use the notation: $E \vdash t_1 \approx t_2$.

A *derivation* is a list of equations such that every element is either an axiom or has the form $t \approx t$, or else it can be obtained from earlier elements of the list by applying one of the algebraic derivation rules of (symmetry), (transitivity), (replacement) or (substitution). $\lhd$

Example 3.3 We derive the equation

$$\mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), \mathtt{succ}(0)) = \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(0)))$$

from the specification `Naturals`:

$$\begin{array}{lll}
(1) & \mathtt{add}(\mathtt{x}, \mathtt{succ}(\mathtt{y})) = \mathtt{succ}(\mathtt{add}(\mathtt{x}, \mathtt{y})) & ([\mathtt{N2}]) \\
(2) & \mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), \mathtt{succ}(\mathtt{y})) = \mathtt{succ}(\mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), \mathtt{y})) & (1, \mathrm{subst}) \\
(3) & \mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), \mathtt{succ}(0)) = \mathtt{succ}(\mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), 0)) & (2, \mathrm{subst}) \\
(4) & \mathtt{add}(\mathtt{x}, 0) = \mathtt{x} & ([\mathtt{N1}]) \\
(5) & \mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), 0) = \mathtt{succ}(\mathtt{succ}(0)) & (4, \mathrm{subst}) \\
(6) & \mathtt{succ}(\mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), 0)) = \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(0))) & (5, \mathrm{cong}) \\
(7) & \mathtt{add}(\mathtt{succ}(\mathtt{succ}(0)), \mathtt{succ}(0)) = \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(0))) & (3, 6, \mathrm{trans})
\end{array}$$

This very simple example already makes clear that is advantageous to shorten derivations by introducing *derived rules*.

Definition 3.4 Let Σ be a similarity type, and X a set of variables. A *substitution* is a map $\sigma : X \to T_\Sigma(X)$ mapping variables to terms. Such a substitution can be extended to a map $\hat{\sigma} : T_\Sigma(X) \to T_\Sigma(X)$ by the following inductive definition:

$$\begin{array}{rcl}
\hat{\sigma}(x) & := & \sigma(x) \\
\hat{\sigma}(f(t_1, \ldots, t_n)) & := & f(\hat{\sigma}(t_1, \ldots, \hat{\sigma}(t_n)).
\end{array}$$

We may also use the name 'substitution' for a function mapping terms to terms that satisfies the second of the conditions above; that is, we may confuse σ and $\hat{\sigma}$. $\lhd$

As an example of a derived rule, we mention the following.

Proposition 3.5 ($*$-substitution) *Let E be some specification of type Σ, and let s and t be terms such that $E \vdash s \approx t$. Then for any substitution σ, the equation $\hat{\sigma}(s) \approx \hat{\sigma}(t)$ is derivable from E as well.*

PROOF. By induction on the number of those variables x in $Var(s \approx t)$ for which $\sigma(x) \neq x$. QED

4 Semantics

In this section we make precise how terms and equations can be used to talk about algebras. Obviously, terms refer to elements of algebras; but in order to calculate the meaning of a term, we first need to know what the variables in the term stand for.

Definition 4.1 (algebraic semantics) Let Σ be an algebraic similarity type, X a set of variables and $\mathcal{A}$ a Σ-algebra. An *assignment* is a function $\mu : X \to A$ associating an element of A with each variable in X. Given such an assignment μ, we can calculate the *meaning* of any term in $T_\Sigma(X)$ inductively as follows:

$$\begin{array}{lcl} \hat{\mu}(x) & = & \mu(x) \\ \hat{\mu}(c) & = & c_{\mathcal{A}} \\ \hat{\mu}(f(t_1,\ldots,t_n)) & = & f_{\mathcal{A}}(\hat{\mu}(t_1),\ldots,\hat{\mu}(t_n)). \end{array}$$

$\lhd$

The latter equality bears an obvious resemblance to the condition (1) defining a homomorphism. In fact, we can turn the meaning function into an actual homomorphism by imposing a Σ-algebraic structure on the set $T_\Sigma(X)$ of terms:

Definition 4.2 (term algebra) Let Σ be an algebraic similarity type, and X a set of variables. The *term algebra of* Σ *over* X is the algebra $\mathcal{T}_\Sigma(X) = (T_\Sigma(X), I)$ where for every function symbol f, $I(f)$ is the operation on $T_\Sigma(X)$ given by

$$I(f)(t_1,\ldots,t_n) = f(t_1,\ldots,t_n).$$

$\lhd$

In other words, the carrier of the term algebra over Σ is the set of Σ-terms over the set of variables X, and the operation $I(f) = f_{\mathcal{T}_\Sigma(X)}$ maps an n-tuple $t_1,\ldots,t_n$ of terms to the term $f(t_1,\ldots,t_n)$. This perspective on Σ-terms as constituting an Σ-*algebra* is extremely useful. For instance, the following observations are immediate.

Proposition 4.3 (substitution as homomorphism) *Any substitution $\sigma : T_\Sigma(X) \to T_\Sigma(X)$ is in fact a homomorphism $\sigma : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(X)$.*

Proposition 4.4 (meaning as homomorphism) *Given any assignment μ, the corresponding meaning function $\hat{\mu}$ is a homomorphism on $\mathcal{T}_\Sigma(X)$.*

As an example of the use of these observations, we now give a nice algebraic formulation of the fact that given an assignment of variables in X, every term in $T_\Sigma(X)$ has a unique meaning. This also offers the opportunity of introducing an important new concept.

Definition 4.5 (universal mapping property) Let K be a class of Σ-algebras, and let $\mathcal{U}$ be an algebra which is generated by a set $Q \subseteq U$. We say that $\mathcal{U}$ has the *universal mapping property* or *UMP* for K over Q if for every algebra $\mathcal{A}$ in K, every map $\alpha : Q \to A$ can be extended to a homomorphism $\beta : \mathcal{U} \to \mathcal{A}$. An alternative formulation is that $\mathcal{U}$ is *freely generated* by Q. $\lhd$

It is easy to see that if $\mathcal{U}$ has the UMP for K over Q, then for every map α from Q to the carrier set of some algebra in K there is a *unique* extension $\hat{\alpha}$ which is a homomorphism. Seen from this perspective, the following proposition states that given an assignment $\mu : X \to A$, every term in X has a *unique* meaning.

Proposition 4.6 (unique meaning) *$\mathcal{T}_\Sigma(X)$ has the UMP over X for the class of all Σ-algebras.*

The following Proposition is needed further on.

Proposition 4.7 *Let K be a class of algebras, and suppose that $\mathcal{U}$ and $\mathcal{U}'$ are two algebras in K that have the universal mapping property for K over the sets Q and Q', respectively. If Q and Q' have the same size, then $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic.*

PROOF. Exercise. QED

Definition 4.8 (Truth and semantic consequence) Let $\mathcal{A}$ be an algebra, $s \approx t$ an equation, and μ an assignment mapping the variables occurring in s and t to elements of A. The equation $s \approx t$ *is true* or *holds in $\mathcal{A}$ under* μ, notation: $\mathcal{A} \models s \approx t[\mu]$, if

$$\hat{\mu}(s) = \hat{\mu}(t).$$

We say that $s \approx t$ *is valid* or *holds* in $\mathcal{A}$, notation: $\mathcal{A} \models s \approx t$, if $\mathcal{A} \models s \approx t[\mu]$ for all assignments μ mapping the variables occurring in s and t to elements of A. A Σ-specification E *holds* in an algebra $\mathcal{A}$, notation: $\mathcal{A} \models E$, if each equation in E holds in $\mathcal{A}$. In such a case we will also say that $\mathcal{A}$ is a *model of E*. A formula or a specification is *valid* in a class K if it is valid in every member of K.

The class of models for E is denoted by $Mod(E)$; classes of this form are called *equationally definable* or simply *equational*. Conversely, the set of equations valid in a class K is called the *equational theory* of K and denoted by $Equ(\mathsf{K})$ (or $Equ_X(\mathsf{K})$ if we want to make the set X of variables explicit).

An equation $s \approx t$ is a *semantic consequence* of a specification E, notation: $E \models s \approx t$, if every model of E is also a model of $s \approx t$. $\lhd$

5 Logical issues

We have set the stage for the two Theorems by Birkhoff that we mentioned in the introduction. The first result states that equational logic is indeed the right formal calculus for the semantic consequence relation.

Theorem 5.1 (Soundness and Completeness) *For any specification E and any equation $s \approx t$*

$$E \vdash s \approx t \text{ iff } E \models s \approx t.$$

The second result identifies varieties and equational classes.

Theorem 5.2 (Equational Definability) *Given a similarity type Σ, a class of Σ-algebras is a variety if and only if it is equationally definable.*

Much of the remainder of the first part of the notes is devoted to proving these two results. Both have an easy side which we will prove in this section, and a difficult direction which requires some more work. We first turn to the soundness direction of Theorem 5.1.

Proposition 5.3 (Soundness) *For any specification E and any equation $s \approx t$, $E \vdash s \approx t$ implies $E \models s \approx t$.*

PROOF. Let $\mathcal{A}$ be a model of E. We have to show that every equation that is derivable from E also holds in A. This is easily done by a straightforward induction on the length of derivations. QED

Soundness can be used for proving that an equation is *not* derivable from a specification.

Example 5.4 Consider the specification `Plus` in the similarity type consisting of the function symbols `0`, `succ` and `add`. Its equations are `[N1]` and `[N2]` given in Example 1.5:

```
[N1]  : add(x,0)        = x
[N2]  : add(x,succ(y)) = succ(add(x,y))
```

We claim that from this specification we cannot prove the commutativity of `add`; that is, we will show that

$$\mathtt{Plus} \not\vdash \mathtt{add(x,y)} = \mathtt{add(y,x)}.$$

In order to prove this, consider the algebra $\mathcal{C} = \langle \mathbb{N} \times \mathbb{N}, \mathtt{0}_{\mathcal{C}}, \mathtt{succ}_{\mathcal{C}}, \mathtt{add}_{\mathcal{C}} \rangle$, with

$$\begin{array}{lcl} \mathtt{0}_{\mathcal{C}} & = & \langle 0,0 \rangle \\ \mathtt{succ}_{\mathcal{C}}(\langle h,f \rangle) & = & \langle h, f+1 \rangle \\ \mathtt{add}_{\mathcal{C}}(\langle h_1,f_1 \rangle, \langle h_2,f_2 \rangle) & = & \begin{cases} \langle h_1, f_1+f_2 \rangle & \text{if } h_2 = 0 \\ \langle h_1+h_2, f_2 \rangle & \text{if } h_2 \neq 0 \end{cases} \end{array}$$

This structure is not as complicated as it may look; think of an element $\langle h,f \rangle$ of C as a stack consisting of f *fragile* boxes standing on a stack of h *heavy* boxes. The 'addition'

$\mathtt{add}_{\mathcal{C}}(\langle h_1, f_1\rangle, \langle h_2, f_2\rangle)$ consists of placing the stack $\langle h_2, f_2\rangle$ on top of the stack $\langle h_1, f_1\rangle$. If the upper stack contains heavy boxes (that is, if $h_2 \neq 0$), then all the fragile boxes in the lower stack are destroyed; hence, $h_1 + h_2$ heavy and f_2 fragile boxes remain.

It is obvious that $\mathtt{add}_{\mathcal{C}}$ is not commutative, and it is fairly straightforward to show that the axioms [N1] and [N2] hold in $\mathcal{C}$ (we leave it to the reader to prove this). From this it follows that $\mathtt{add}(\mathtt{x},\mathtt{y}) = \mathtt{add}(\mathtt{y},\mathtt{x})$ is not a *semantic* consequence of `Plus`, so by Soundness we may conclude that it cannot be derived from `Plus` either.

We finish this section with mentioning the easy side of Birkhoff's definability theorem.

Proposition 5.5 *Equational classes are varieties.*

Proof. It is straightforward to prove that equational classes are closed under taking homomorphic images, subalgebras and products. QED

6 Term models

Definition 6.1 (provable equivalence) Given a specification E, define the relation $\equiv_E$ of *provable equivalence* between terms as follows: $s \equiv_E t$ if $E \vdash s \approx t$. $\lhd$

The following is not hard to see: the derivation system of equational logic is tailored towards making $\equiv_E$ a congruence on the term algebra.

Proposition 6.2 *The relation $\equiv_E$ is a congruence on $\mathcal{T}_\Sigma(X)$.*

We leave a proof of this Proposition to the reader, but note in passing that the rule (substitution) is not used in the proof.

Definition 6.3 (term model for E) Let Σ be an algebraic similarity type, X a set of variables, and E a specification. The quotient algebra $\mathcal{T}_\Sigma(X)/{\equiv_E}$ is called the *term model for E*. The set $\{[x] \mid x \in X\}$ is denoted as $\bar{X}$. $\lhd$

Example 6.4 In the specification `Naturals`, the class $[\mathtt{x}]$ contains the terms $\mathtt{add}(\mathtt{x},\mathtt{0})$ and $\mathtt{mul}(\mathtt{x},\mathtt{add}(\mathtt{mul}(\mathtt{add}(\mathtt{y},\mathtt{succ}(\mathtt{z})),\mathtt{0}),\mathtt{succ}(\mathtt{0})))$.

As examples of the interpretations of the operations we mention:

$$
\begin{aligned}
\mathtt{add}_{\mathcal{Q}}([\mathtt{y}],[\mathtt{succ}(\mathtt{succ}(\mathtt{0}))]) &= [\mathtt{add}(\mathtt{y},\mathtt{succ}(\mathtt{succ}(\mathtt{0})))], \\
\mathtt{succ}_{\mathcal{Q}}([\mathtt{add}(\mathtt{mul}(\mathtt{add}(\mathtt{y},\mathtt{succ}(\mathtt{z})),\mathtt{0}),\mathtt{succ}(\mathtt{0}))]) &= [\mathtt{succ})(\mathtt{succ}(\mathtt{0}))]
\end{aligned}
$$

Theorem 6.5 *Let E be a Σ-specification. Then $\mathcal{T}_\Sigma(X)/{\equiv_E}$ has the UMP for the class $Mod(E)$ over $\bar{X}$.*

Proof. First note that $\bar{X}$ indeed *generates* $T_\Sigma(X)/{\equiv_E}$. Let $\mathcal{A}$ be a model of E, and let $\alpha : \bar{X} \to A$ be some map. We have to prove that there is a homomorphism $\beta : \mathcal{T}_\Sigma(X)/{\equiv_E} \to \mathcal{A}$ that agrees with α on $\bar{X}$.

Let $\nu : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(X)/{\equiv_E}$ be the natural homomorphism given by $\nu(s) = [s]$. It follows from the universal mapping property of $\mathcal{T}_\Sigma(X)$ over the class of all Σ-algebras that there is some homomorphism $\beta : \mathcal{T}_\Sigma(X) \to \mathcal{A}$ that agrees with $\alpha \circ \nu$ on X. It follows from $\mathcal{A} \models E$ (and soundness!!) that $\beta(s) = \beta(t)$ whenever $E \vdash s \approx t$; thus $s \equiv_E t$ implies $\beta(s) = \beta(t)$. This implies that the following map $\hat{\alpha} : T_\Sigma(X)/{\equiv_E} \to A$ is well-defined:

$$\hat{\alpha}([s]) = \beta(s).$$

We leave it to the reader to proof that $\hat{\alpha}$ is a homomorphism extending α. QED

7 Completeness

This section is devoted to a proof of the completeness direction of Theorem 5.1. The usual method of proving a completeness theorem proceeds by contraposition; one assumes that a certain formula (equation) is *not* provable from a set of formulas (equations) and tries to find a model of the set which is not a model of the formula. It turns out that in the case of equational logic we have a very strong tool in our hands in the form of the term models. We will show that (for big enough X) $\mathcal{T}_\Sigma(X)/{\equiv_E}$ makes *precisely* those equations true that are derivable from E. We first need the following auxiliary result.

Proposition 7.1 *Let E be a Σ-specification, and X and Y sets of variables such that Y is as least as big as X. Then $\mathcal{T}_\Sigma(X)/{\equiv_E}$ can be embedded in $\mathcal{T}_\Sigma(Y)$.*

PROOF. Exercise. QED

Proposition 7.2 *Let E be a Σ-specification. Then any term model for E is a model of E. That is, for any set X we have $\mathcal{T}_\Sigma(X)/{\equiv_E} \models E$.*

PROOF. First consider the case that X contains all variables occurring in any of the equations in E. We have to show that all equations in E hold in $\mathcal{T}_\Sigma(X)/{\equiv_E}$. Hence, if we let μ be an arbitrary assignment on $\mathcal{T}_\Sigma(X)/{\equiv_E}$, and $s \approx t$ an arbitrary equation in E, we have to prove that $\hat{\mu}(s) = \hat{\mu}(t)$.

Since elements of $T_\Sigma(X)/{\equiv_E}$ are equivalence classes of $T_\Sigma(X)$, in each equivalence class $\mu(x)$ we may take a representant, say $\sigma(x)$:

$$\mu(x) = [\sigma(x)].$$

But since σ is a function mapping variables to terms, we may (uniquely) extend it to a *substitution* $\hat{\sigma}$. By a straightforward term induction (or, using the UMP of $\mathcal{T}_\Sigma(X)$) we can prove that for all terms u we have that

$$\hat{\mu}(u) = [\hat{\sigma}(u)]. \tag{2}$$

Now since $s \approx t$ belongs to E, it follows by ($*$-substitution) that $E \vdash \hat{\sigma}(s) \approx \hat{\sigma}(t)$. In other words, we have that $\hat{\sigma}(s) \equiv_E \hat{\sigma}(t)$, whence we obtain that $[\hat{\sigma}(s)] = [\hat{\sigma}(t)]$. Then by (2) it follows that $\hat{\mu}(s) = \hat{\mu}(t)$.

Finally, suppose that X does *not* contain all the variables that occur in the equations in E. Let Y be the extension of X with all these variables; it then follows by the first part of the proof that E holds in $\mathcal{T}_\Sigma(Y)$. But by the previous Proposition $\mathcal{T}_\Sigma(X)/{\equiv_E}$ can be embedded in $\mathcal{T}_\Sigma(Y)$, so by Proposition 5.5 it follows that $\mathcal{T}_\Sigma(X)/{\equiv_E} \models E$. QED

Proposition 7.3 *Let E be a Σ-specification. If X is big enough (say, countable), then $\mathcal{T}_\Sigma(X)/{\equiv_E}$ falsifies every equation that is not derivable from E.*

Proof. Let $s \approx t$ be an equation that is *not* derivable from E. By Theorem 6.5 and our requirement that X is big enough we may assume that s and t are terms in X. In order to show that it does not hold in $\mathcal{T}_\Sigma(X)/{\equiv_E}$, we have to find an assignment μ of variables to elements of $T_\Sigma(X)/{\equiv_E}$ such that $\hat{\mu}(s) \neq \hat{\mu}(t)$.

Consider the map $\mu : T_\Sigma(X) \to T_\Sigma(X)/{\equiv_E}$ given by $\mu(x) = [x]$. It is easy to see that $\hat{\mu}$ is the natural homomorphism; that is, $\hat{\mu}(u) = [u]$ for all terms u. Hence, it follows from $E \not\vdash s \approx t$ by the definition of $\equiv_E$ that $\hat{\mu}(s) \neq \hat{\mu}(t)$ — provided that s and t belong to $T_\Sigma(X)$. QED

Proof of Theorem 5.1. We have proved the soundness direction in Proposition 5.3. The other direction is immediate by the previous two Propositions: if $E \models s \approx t$ then by Proposition 7.2, $\mathcal{T}_\Sigma(X)/{\equiv_E} \models s \approx t$, so by Proposition 7.3, $E \vdash s \approx t$. QED

8 Equational Definability

In this section we will prove Birkhoff's definability theorem identifying varieties with equational classes. In the proof we use a semantic variant of the term model.

Definition 8.1 Let K be a class of algebras and let X be a set of variables. Define the relation $\sim_\mathsf{K}$ on $T_\Sigma(X)$ by

$$s \sim_\mathsf{K} t \ \text{ if } \ \mathsf{K} \models s \approx t.$$

$\lhd$

It is easy to see that $\sim_\mathsf{K}$ is a congruence relation on $\mathcal{T}_\Sigma(X)$.

Definition 8.2 Given a class K of Σ-algebras and a set X of variables, we call the algebra $\mathcal{T}_\Sigma(X)/{\sim_\mathsf{K}}$ the K-*free* algebra over $\bar{X}$. As abbreviated notation for this algebra we use $\mathcal{F}_{\bar{X}}(\mathsf{K})$. $\lhd$

That this 'semantic' perspective is not so very different from the 'syntactic' one of the previous section, follows from the following Proposition.

Proposition 8.3 *For any class K of algebras, the congruences $\sim_\mathsf{K}$ and $\equiv_{Equ(\mathsf{K})}$ coincide. As a corollary, any* K-free *algebra over $\bar{X}$ has the universal mapping property for the class K.*

PROOF. The first part of the Proposition follows from the observation that $\mathsf{K} \models s \approx t$ iff $Equ(\mathsf{K}) \vdash s \approx t$, which we leave for the reader to prove. From the first part it follows that $\mathcal{F}_{\bar{X}}(\mathsf{K})$ has the UMP over $\bar{X}$ for the class of models for $Equ(\mathsf{K})$; from this and the fact that K is contained in the class $Mod(Equ(\mathsf{K}))$ the second part of the Proposition is immediate. QED

This Proposition explains why we call the algebra $\mathcal{F}_{\bar{X}}(\mathsf{K})$ *free*: the fact, that every map from the set $\bar{X}$ to the carrier of an algebra $\mathcal{A}$ in K can be extended to a homomorphism from $\mathcal{F}_{\bar{X}}(\mathsf{K})$ to $\mathcal{A}$, indicates that the elements of $\bar{X}$ are 'free' with respect to K.

Free algebras play a crucial role in our proof of the definability theorem. We will now see what kind of *structural* links there are between a class and its free algebras.

Proposition 8.4 *For every class* K *and every set of variables* X*, we have that*

$$\mathcal{F}_{\bar{X}}(\mathsf{K}) \in \mathbb{SP}\mathsf{K}.$$

In fact, if X *is at least countable,* $\mathcal{F}_{\bar{X}}(\mathsf{K})$ *is a subdirect product of algebras in* $\mathbb{S}\mathsf{K}$.

PROOF. We first prove the second part of the Theorem, and then show how to derive the first part from it.

Let X be a countably infinite set, and consider the set H of equations in X that do not hold in K. Obviously, H is countable as well, and by its definition, for any equation η in H, there is some algebra $\mathcal{A}_\eta$ in K where η fails to hold. We leave it to the reader to check that there is a *countable subalgebra* $\mathcal{B}_\eta$ of $\mathcal{A}_\eta$ on which η fails, and that there thus exists a 'witnessing' *surjective* assignment $\mu : X \to B$ such that

$$\hat{\mu}_\eta(s) \neq \hat{\mu}_\eta(t),$$

where we assume η to be of the form $s \approx t$.

Define $\mathcal{B}$ as the product of the algebras $\mathcal{B}_\eta$; that is, set

$$\mathcal{B} \;=\; \prod_{\eta \in H} \mathcal{B}_\eta,$$

and define the homomorphism $\beta : \mathcal{F}_{\bar{X}}(\mathsf{K}) \to \mathcal{B}$ by

$$\beta([s])(\eta) = \hat{\mu}_\eta(s)$$

(why is this map well-defined?) We claim that β is a subdirect embedding: in order to show this, we have to prove that β is injective, and that each $\pi_\eta \circ \beta$ is surjective, where π_η is the η-th projection map from $\mathcal{B}$ onto $\mathcal{B}_\eta$.

For injectivity of β, suppose that $[s]$ and $[t]$ are distinct elements of $\mathcal{F}_{\bar{X}}(\mathsf{K})$. It follows by definition of $\sim_{\mathsf{K}}$ that $K \not\models s \approx t$, whence the equation $s \approx t$ belongs to H — here we need the fact that both s and t use only variables in X. If we call this equation ζ, it follows by definition of $\mathcal{A}_\zeta$ that $\hat{\mu}_\zeta(s) \neq \hat{\mu}_\zeta(t)$. But by definition of β this implies that $\beta([s]) \neq \beta([t])$. Proving the surjectivity of each homomorphism $\pi_\eta \circ \beta$ is a task we leave to the reader.

But if $\mathcal{F}_{\bar{X}}(\mathsf{K})$ is a subdirect product of algebras in K, it certainly belongs to $\mathbb{SP}K$. The first part of the Proposition then follows from the observation that if $X \subseteq Y$ then $\mathcal{F}_{\bar{X}}(\mathsf{K}) \rightarrowtail \mathcal{F}_{\bar{Y}}(\mathsf{K})$. QED

Proposition 8.5 *For every class* K *and every model* $\mathcal{A}$ *of* $Equ(\mathsf{K})$ *there is a set of variables* X *such that* $\mathcal{A}$ *is a homomorphic image of* $\mathcal{F}_{\bar{X}}(\mathsf{K})$.

Proof. Take a set X of variables that has more elements than A, and let $\mu : \bar{X} \to A$ be a surjective assignment. (Why does such a surjection always exist?) By the freeness of $\mathcal{F}_{\bar{X}}(\mathsf{K})$ for $Mod(Equ(\mathsf{K}))$ over $\bar{X}$, this map can be extended to a homomorphism $\hat{\alpha} : \mathcal{F}_{\bar{X}}(\mathsf{K}) \to \mathcal{A}$. Obviously, $\hat{\alpha}$ is surjective as well. QED

This gives us sufficient material to prove Birkhoff's definability theorem.

Proof of Theorem 5.2. We have shown in Proposition 5.5 that equational classes are varieties.

Now suppose that V is a variety. We claim that $\mathsf{V} = Mod(Equ(\mathsf{V}))$; the inclusion from left to right is immediate, so we concentrate on the other one. Suppose that $\mathcal{A}$ is a model of the equational theory of V. Let X be a set of variables such that $\mathcal{A}$ is a homomorphic image of $\mathcal{F}_{\bar{X}}(\mathsf{V})$ — such a set X exists by Proposition 8.5. But it follows from Proposition 8.4 that $\mathcal{F}_{\bar{X}}(\mathsf{V}) \in \mathbb{SP}\mathsf{V}$. But then we may find $\mathcal{A}$ in $\mathbb{HSP}\mathsf{V} = \mathsf{V}$. This suffices to prove the Theorem. QED

9 Initial models

Computer scientists are particularly interested in one specific term model of a specification, namely the one generated by the *empty* set of variables — of course, this makes only sense if the signature contains constants.

Definition 9.1 Let Σ be an algebraic similarity type containing at least one constant, and E a Σ-specification. The algebra $\mathcal{T}_\Sigma(\varnothing)/{\equiv_E}$ is called the *initial model* for E and denoted as $\mathcal{I}_E$. $\triangleleft$

This terminology is justified by the following observation.

Proposition 9.2 *Let* Σ *be an algebraic similarity containing at least one constant, and* E *a* Σ*-specification. Then for any model* $\mathcal{A}$ *of* E *there is a unique homomorphism from* $\mathcal{I}_E$ *to* $\mathcal{A}$. *Furthermore,* $\mathcal{I}_E$ *is the only model of* E *that has this property.*

Proof. The first part of the Proposition is simply the universal mapping property spelled out for the empty set of variables. The second part of the property follows from Proposition 4.7. QED

Computer scientists use equational logic for, among other things, the specification of abstract data types. Here, usually the starting point is a given Σ-algebra $\mathcal{A}$, and the aim is to find a specification E of type Σ such that $\mathcal{A}$ is (isomorphic to) the initial model for E.

It should be stressed that specifications do not completely axiomatize the equational theory of their initial models. For example, it can be proved that the initial model of the specification `Naturals` is isomorphic to the algebra $\mathcal{N} = (\mathbb{N}, 0, S, +, *)$. But one can show, just like in Example 5.4, that commutativity of `add` cannot be proved in this specification. Nevertheless, for *closed* terms (that is, terms without variables) specifications are complete:

Proposition 9.3 *Let Σ be an algebraic similarity type containing at least one constant, and E a Σ-specification. Then for any pair of closed terms s and t we have*

$$E \vdash s \approx t \textit{ iff } \mathcal{I}_E \models s \approx t.$$

Proof. Exercise. QED

II Logic as Algebra

We now turn the table and approach logic from an algebraic angle; this part of the Notes can be seen as an introduction to Algebraic Logic. One of the main aims of this research field is to gain a better understanding of logics by treating them in universal algebraic terms.

Roughly, given a logic, one tries to find a class of algebras that *algebraizes* the logic in some natural sense. Well-known examples of logics with their associated class of algebras include the following:

classical propositional logic	$\sim$	Boolean algebras
intuitionistic propositional logic	$\sim$	Heyting algebras
propositional modal logic	$\sim$	Boolean algebras with operators
first order logic	$\sim$	cylindric algebras

The idea is then to *translate* properties of the logic into equivalent properties of the associated class of algebras. For instance, the following correspondences have been found:

completeness theorems	$\sim$	representation theorems
interpolation property	$\sim$	amalgamation properties
Beth definability property	$\sim$	surjectiveness of epis

The chief motivation for the algebraization program is that when a logic is algebraizable, the powerful apparatus of universal algebra can be applied to solve logical problems. This method has had a profound influence on the development of logic, particularly in the field of non-classical logic.

Here we will discuss the first and the third example of a correspondence between a logical and an algebraic property. But whereas in the previous part we took a rather general stand, looking at *universal* algebra, here we almost exclusively focus on the algebraization of one specific logic, namely, that of classical propositional logic in the class of Boolean algebras.

10 Classical Propositional Logic

Definition 10.1 (syntax) We fix a countably infinite set V of *(propositional) variables* or *proposition letters*. The *alphabet* of propositional logic consists of these variables, of the propositional constants $\top$ and $\bot$, and of the *connectives* $\neg$ and $\vee$. The set L of propositional *formulas* in this language is defined as the smallest set which (i) includes the set $V \cup \{\top, \bot\}$ of *atomic formulas* and (ii) contains $\neg\varphi$ and $\varphi \vee \psi$ whenever it contains φ and ψ.

As abbreviations we use the connectives $\rightarrow$, $\wedge$ and $\leftrightarrow$, defined as follows: $\varphi \rightarrow \psi$ stands for $\neg\varphi \vee \psi$, $\varphi \wedge \psi$ for $\neg(\neg\varphi \vee \neg\psi)$, and $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We adopt standard bracketing conventions; in particular, $\rightarrow$ and $\leftrightarrow$ bind stronger than $\vee$ and $\wedge$. $\lhd$

Logics can be specified in two ways: semantically or syntactically. Here we will discuss (and algebraize) both approaches, starting with the semantics of propositional logic. Propositional logic is about truth and falsehood; we let $2 = \{0, 1\}$ denote the set of *truth values* 0 (= *false*) and 1 (= *true*).

Definition 10.2 (semantics) A *valuation* is a map assigning truth values to propositional variables; formally, a valuation is a function $v : P \to 2$.

A valuation v can be naturally extended to a meaning function $\hat{v}$ mapping formulas to truth values. Consider the following recursive *truth definition*:

$$\begin{array}{rcl} \hat{v}(\top) & = & 1 \\ \hat{v}(\bot) & = & 0 \\ \hat{v}(\neg\varphi) & = & 1 - \hat{v}(\varphi) \\ \hat{v}(\varphi \vee \psi) & = & \max(\hat{v}(\varphi), \hat{v}(\psi)). \end{array}$$

In the sequel we will often identify v and $\hat{v}$, writing v instead of $\hat{v}$.

We say that a valuation v is a *model* of a formula φ, or that φ is *true* under v, notation: $v \models_C \varphi$, if $v(\varphi) = 1$. If φ is not true under v we say that it is *false* under v. A valuation is a model of a set Φ of formulas if it is a model of each φ in Φ.

Formulas that are true under every valuation are called *tautologies*. Given a set Φ of formulas, we say that φ is a *semantic consequence* of Φ, notation: $\Phi \models_C \varphi$, if every model of Φ is also a model of φ. Tautologies are then the consequences of the empty set; instead of $\varnothing \models_C \varphi$ we write $\models_C \varphi$.

Two formulas φ and ψ are called *(semantically) equivalent* if for all valuations v, we have $\hat{v}(\varphi) = \hat{v}(\psi)$. $\lhd$

It is easy to compute the truth definitions of the other connectives:

$$\begin{array}{rcl} \hat{v}(\varphi \wedge \psi) & = & \min(\hat{v}(\varphi), \hat{v}(\psi)). \\ \hat{v}(\varphi \to \psi) & = & \max(1 - \hat{v}(\varphi), \hat{v}(\psi)). \\ \hat{v}(\varphi \leftrightarrow \psi) & = & \begin{cases} 1 & \text{if } \hat{v}(\varphi) = \hat{v}(\psi)) \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Obviously, two formulas φ and ψ are semantically equivalent if for all valuations v, we have $\hat{v}(\varphi \leftrightarrow \psi) = 1$.

There are many derivation systems that provide a complete axiomatization for the notion of semantic consequence that we just defined. For our purposes it does not really matter which one we consider here, but it is convenient to fix one.

Definition 10.3 (axiomatics) An *axiom* is any formula instantiating one of the following schemas:

$(A1)$ $\varphi \vee \varphi \to \varphi$
$(A2)$ $\varphi \to \varphi \vee \psi$
$(A3)$ $\varphi \vee \psi \to \psi \vee \varphi$
$(A4)$ $(\varphi \to \psi) \to (\chi \vee \varphi \to \chi \vee \psi)$
$(A5)$ $\top$
$(A6)$ $\neg\bot$.

A *derivation* from a set Φ of formulas is a list of formulas such that every formula is either an axiom, or an element of Φ, or else it can be obtained from earlier members of the list by

an application of the rule of *detachment* or *modus ponens*:
(MP) from φ and $\varphi \to \psi$, derive ψ.

A formula φ is *derivable from* Φ, notation: $\Phi \vdash_C \varphi$, if it occurs on such a list. A formula φ is called a *theorem*, notation: $\vdash_C \varphi$, if it is derivable from the empty set. Two formulas φ and ψ are called *(provably) equivalent modulo* Φ, notation: $\varphi \equiv_\Phi \psi$, if the formula $\varphi \leftrightarrow \psi$ is derivable from Φ. In case Φ is the empty set, we just say that φ and ψ are *(provably) equivalent*, notation: $\varphi \equiv_C \psi$. $\lhd$

Example 10.4 Consider the following derivation of the formula $p \to p$:

(1)	$p \to p \vee p$	(A2 with $\varphi = p$, $\psi = p$)
(2)	$p \vee p \to p$	(A1 with $\varphi = p$)
(3)	$(p \vee p \to p) \to ((p \to p \vee p) \to (p \to p))$	(A4 with $\varphi = p \vee p$, $\psi = p$, $\chi = \neg p$)
(4)	$(p \to p \vee p) \to (p \to p)$	(MP on (2) and (3))
(5)	$p \to p$	(MP on (1) and (4))

In understanding (3) it may be useful to observe that A4 with $\varphi = p \vee p$, $\psi = p$, $\chi = \neg p$ is formally of the form $(p \vee p \to p) \to ((\neg p \vee (p \vee p)) \to (\neg p \vee p))$. But since $\neg\alpha \vee \beta$ is abbreviated as $\alpha \to \beta$, we get the form (3).

Remark 10.5 Let a *substitution* be any function mapping formulas to formulas by uniformly replacing proposition letters by formulas. Note that unlike our definition of equational logic, there is no rule of substitution here saying that 'if φ is derivable, then so is $\sigma(\varphi)$ for any substitution σ'. The reason for this is adding such a rule would have the clearly undesirable consequence that any formula φ were derivable from the proposition letter p.

Nevertheless, thanks to the fact that any instance of an axiom schema is an axiom, we do have a weaker version of this rule , cf. the Proposition below.

Although it does not really matter which axiomatization for classical propositional logic we take, there are a few properties that we need later on.

Proposition 10.6 *For any set of formulas Φ and any pair of formulas φ and ψ we have the following equivalences:*

1. $\Phi \vdash_C \varphi \wedge \psi$ *iff* $\Phi \vdash_C \varphi$ *and* $\Phi \vdash_C \psi$.
2. $\Phi \vdash_C \varphi \to \psi$ *iff* $\Phi, \varphi \vdash_C \psi$.
3. *If* $\Phi \vdash_C \psi$ *then* $\{\sigma(\varphi) \mid \varphi \in \Phi\} \vdash_C \sigma(\psi)$ *for any substitution σ. In particular, the set of classical theorems is closed under substitutions.*

Logicians are interested in various properties of relations like $\models_C$ and $\vdash_C$; as an example, we mention the fact that $\models_C$ and $\vdash_C$ *coincide*.

Theorem 10.7 (Strong Soundness and Completeness) *For any set of formulas Φ and any formula φ:*

$$\Phi \vdash_C \varphi \text{ iff } \Phi \models_C \varphi.$$

We will prove this fundamental theorem by algebraic means. Let us therefore turn to the algebraization of propositional logic; we first confine ourselves to the level of tautologies and theorems.

11 Algebraizing semantics

The basic idea underlying the semantics of propositional logic is the observation that interpreting a formula under a valuation is very similar to calculating the value of an algebraic term. In fact, we may *view* propositional formulas as terms of an algebraic similarity type BA. As an interesting outcome of this perspective, we may associate with the language of propositional logic a natural *algebra of formulas.*

Definition 11.1 Let BA be the algebraic similarity type having two unary function symbols $\bot$ and $\top$, one unary function symbol $\neg$, and one binary function symbol $\vee$.

We let $\mathcal{L}$ denote the term algebra of this signature over the set V of variables; $\mathcal{L}$ is called the *formula algebra.*

Algebras of type BA are usually denoted as $\mathcal{A} = (A, 0, 1, -, +)$. As an abbreviation we use $\cdot$, as in $a \cdot b = -(-a + -b)$. $\lhd$

If we impose an algebraic structure on the set of truth values, then valuations are nothing but assignments of variables to elements of this algebra.

Definition 11.2 Let $\mathbf{2} = (\{0,1\}, 0, 1, -, +)$ be the *Boolean algebra of truth values*, where $-$ and $+$ are given by

$$\begin{aligned} -a &= 1 - a \\ a + b &= \max(a, b). \end{aligned}$$

$\lhd$

Then the following is obvious.

Proposition 11.3 *The set of valuations coincides with the set of homomorphisms from $\mathcal{L}$ to* $\mathbf{2}$.

This perspective, together with the fact that classical propositional logic contains some nice constants and connectives, allows us to capture the notion of a tautology algebraically. And conversely, the truth of an equation in the algebra of truth values can be captured logically as the semantic equivalence of the left and the right hand side of the equation; this embodies the idea that equivalent formulas are different ways of stating the same *proposition.*

Theorem 11.4 (2 algebraizes classical validity) *Let φ and ψ be propositional formulas/ BA-terms. Then*

$$\models_C \varphi \quad \textit{iff} \quad \mathbf{2} \models \varphi \approx \top \tag{3}$$

$$\mathbf{2} \models \varphi \approx \psi \quad \textit{iff} \quad \models_C \varphi \leftrightarrow \psi. \tag{4}$$

Proof. Immediate by the definitions. QED

12 Algebraizing axiomatics

Now that we have encountered the idea that the concept of semantic equivalence between formulas can be captured algebraically, we see how this equational perspective works out for the *provably* equivalent formulas. Technically, we will prove that the relation $\equiv_C$ is a congruence on the formula algebra $\mathcal{L}$, analogous to Proposition 6.2. However, whereas equational logic is tailored towards making any relation $\equiv_E$ a congruence relation, in the present case, proving the Proposition below requires some tedious work.

Proposition 12.1 *The relation $\equiv_C$ is a congruence on the formula algebra $\mathcal{L}$.*

PROOF. We have to prove that $\equiv_C$ is a reflexive, symmetric and transitive relation that 'respects' the connectives $\neg$ and $\vee$. The latter means that if $\varphi \equiv_C \varphi'$ and $\psi \equiv_C \psi'$ then $\neg\varphi \equiv_C \neg\varphi'$ and $\varphi \vee \psi \equiv_C \varphi' \vee \psi'$.

For reflexivity, we need to prove that for any formula φ, the formula $\varphi \leftrightarrow \varphi$ is a theorem. In Example 10.4 we gave a proof for $\vdash_C p \to p$; it then follows from Proposition 10.6 that $\vdash_C (\varphi \to \varphi) \wedge (\varphi \to \varphi)$ for every formula φ, which is what we needed. The cases of symmetry and transitivity are left to the reader.

Now we consider negation, leaving disjunction again to the reader. We have to prove that for any pair of formulas φ and φ', if $\varphi \leftrightarrow \varphi'$ is a theorem, then so is $\neg\varphi \leftrightarrow \neg\varphi'$. This can be done by showing that

$$\varphi \leftrightarrow \varphi' \;\;\vdash_C\;\; \neg\varphi \leftrightarrow \neg\varphi'. \tag{5}$$

For, from this it follows that whenever $\varphi \leftrightarrow \varphi'$ is a theorem, then we obtain a derivation for $\neg\varphi \leftrightarrow \neg\varphi'$ by attaching the derivation of (5) to a derivation of $\varphi \leftrightarrow \varphi'$. Since we are not interested in the details of propositional proof theory, we leave it to the reader to prove (5) in his or her favorite proof system. QED

A fundamental role in algebraic logic is played by the Lindenbaum-Tarski algebra of a logic; this structure is defined as the quotient of the formula algebra over the congruence relation of derivable equivalence. Recall that we are working with a *fixed* set V of variables.

Definition 12.2 Let $\mathcal{Q}$ denote the quotient algebra $\mathcal{L}/{\equiv_C}$; $\mathcal{Q}$ is usually referred to as the *Lindenbaum-Tarski* algebra of propositional logic. $\lhd$

Spelled out, in the Lindenbaum-Tarski algebra we have the following:

$$\begin{array}{rcl} [\top] & = & 1 \\ [\bot] & = & 0 \\ -[\varphi] & = & [\neg\varphi] \\ [\varphi] + [\psi] & = & [\varphi \vee \psi] \\ [\varphi] \cdot [\psi] & = & [\varphi \wedge \psi]. \end{array}$$

In Part I of these Notes we saw that the term model encodes a mass of information about the corresponding variety of algebras for which it is free. It therefore seems a good idea to find out what kind of variety corresponds to the algebra $\mathcal{Q}$. We are entering the realm of *Boolean Algebras.*

13 Boolean Algebras

Definition 13.1 An algebra $\mathcal{A}$ of type BA is called a *Boolean algebra* if it satisfies the following identities (since identities are syntactic objects, we use $\vee$, $\wedge$, $\bot$ and $\top$ rather than $+$, $\cdot$, 0 and 1, respectively):

$(B0)$	$x \vee y \approx y \vee x$	$x \wedge y \approx y \wedge x$
$(B1)$	$x \vee (y \vee z) \approx (x \vee y) \vee z$	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
$(B2)$	$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
$(B3)$	$x \vee \bot \approx x$	$x \wedge \top \approx x$
$(B4)$	$x \vee \neg x \approx \top$	$x \wedge \neg x \approx \bot$

The class of Boolean algebras is denoted as BA.

The relation $\leq$ on a Boolean algebra A is defined as follows: $a \leq b$ if $a + b = b$, (or equivalently, if $a \cdot b = a$); this relation is called the *order* of $\mathcal{A}$. $\lhd$

Whenever we need some basic facts concerning the class BA in the sequel, we simply refer to 'basic Boolean arithmetic' trusting the reader to be able to supply the details. For instance, we will freely use the fact that for any Boolean algebra, $\leq$ is a partial order with 0 as smallest and 1 as biggest element.

Here are some well-known examples of Boolean algebras.

Example 13.2 The algebra $\mathbf{2}$ of truth values is a Boolean algebra, as can easily be checked.

Example 13.3 Let A be some set; the *power set* algebra of A is the algebra $\mathcal{P}(A) = (Sb(A), \varnothing, A, \sim_A, \cup)$, where $Sb(A)$ is the collection of subsets of A, $\sim_A(X)$ denotes the *complement* of X with respect to A, and $X \cup Y$ is the *union* of X and Y. (In the sequel we will simply write $\sim$ instead of $\sim_A$ when no confusion arises.) Any subalgebra of a power set algebra is called a *field of sets*. An example of a field of sets that is not isomorphic to a power set algebra is given by the collection of finite and co-finite subsets of an arbitrary infinite set.

It is easy to show that any field of sets is a Boolean algebra; the order in such an algebra corresponds to set inclusion: $X \cup Y = Y$ if and only if $X \subseteq Y$.

It is interesting to see how fields of sets are related to the algebra of truth values:

Proposition 13.4 *Every field of sets is isomorphic to a subdirect power of* $\mathbf{2}$*, and conversely.*

Proof. Let $\mathcal{B} = (B, \varnothing, A, \sim, \cup)$ be a field of subsets of A, and consider the following function χ mapping elements of B to 2-valued maps on A:

$$\chi(X)(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\chi(X)$ is the *characteristic function* of X.

It is easy to see that χ is a homomorphism from $\mathcal{B}$ to $\mathbf{2}^A$ — we leave the proof to the reader. Injectivity of χ follows from the fact that if $X \neq Y$ then there is an element $a \in A$ belonging to exactly one of X and Y. For this a we have that $\chi(X)(a) \neq \chi(Y)(a)$, whence $\chi(X) \neq \chi(Y)$. Finally, we have to prove that for each projection function $\pi_a : \mathbf{2}^A \to \mathbf{2}$, the map $\pi_a \circ \chi$ is surjective. But since no $a \in A$ belongs to the empty set $\varnothing$, and every $a \in A$ belongs to the top set A, we obtain for each $a \in A$ that $\pi_a(\chi(\varnothing)) = 0$ and $\pi_a(\chi(A)) = 1$.

Conversely, in order to show that subdirect powers of $\mathbf{2}$ are isomorphic to fields of sets, it suffices to prove that direct powers of $\mathbf{2}$ have this property. In fact, we will show that any direct power of $\mathbf{2}$ is isomorphic to a *power set algebra*; from this the result is immediate. Let $\mathbf{2}^A$ be some power of $\mathbf{2}$. Consider the map $\alpha : 2^A \to Sb(A)$ defined by

$$\alpha(f) = \{a \in A \mid f(a) = 1\}.$$

We leave it to the reader to verify that this map is indeed an isomorphism between $\mathbf{2}^A$ and $\mathcal{P}(A)$. QED

As a consequence, one can say that classical propositional logic is the logic of sets.

14 Boolean algebras algebraize classical theorems

In the Lindenbaum-Tarski algebra of classical propositional logic we now meet a new specimen of a Boolean algebra.

Theorem 14.1 *The Lindenbaum-Tarski algebra of classical propositional logic is a Boolean algebra.*

PROOF. Just as in the proof of Proposition 12.1, this requires making lots of derivations in your favourite deductive system for classical propositional logic. For example, in order to prove that $\mathcal{Q} \models x \vee \bot \approx x$ one has to prove that for any formula φ we have $\vdash_C \varphi \vee \bot \leftrightarrow \varphi$. QED

In fact, $\mathcal{Q}$ is a rather special Boolean algebra:

Theorem 14.2 *The Lindenbaum-Tarski algebra of classical propositional logic is (isomorphic to) the free Boolean algebra over the set of variables V:*

$$\mathcal{Q} \simeq \mathcal{F}_{BA}(\bar{V}).$$

PROOF. First note that $\mathcal{Q}$ is generated by the set $\bar{V} = \{[p] \mid p \in V\}$. This follows from the facts that V generates $\mathcal{L}$ and that $\mathcal{Q}$ is a quotient algebra of $\mathcal{L}$. We will show that in fact, $\mathcal{Q}$ is *freely* generated by $\bar{V}$. Fom this and Proposition 4.7 the Theorem is immediate.

Assume that $\mathcal{A} = (A, 0, 1, -, +)$ is an arbitrary Boolean algebra and that $\alpha : \bar{V} \to A$ is some map. We have to show that α can be extended to a *homomorphism* $\hat{\alpha} : \mathcal{Q} \to \mathcal{A}$.

Let β be the uniquely defined homomorphism from $\mathcal{L}$ to $\mathcal{A}$ such that $\beta(p) = \alpha([p])$. That is, we have

$$\begin{aligned} \beta(\neg\varphi) &= -\beta(\varphi), \\ \beta(\varphi \vee \psi) &= \beta(\varphi) + \beta(\psi). \end{aligned}$$

for all formulas φ and ψ. We claim that

$$\beta(\varphi) = 1 \text{ for all theorems } \varphi. \tag{6}$$

This can be proved by an induction on the length of a derivation of φ. First suppose that φ is an axiom; as an example, suppose that φ is an instantiation of $(A1)$, say φ is of the form $\psi \to (\psi \vee \psi)$, or $\neg\psi \vee (\psi \vee \psi)$ if we do not wish to use abbreviated connectives. Then we can calculate

$$\begin{aligned} \beta(\varphi) &= \beta(\neg\psi \vee (\psi \vee \psi)) \\ &= -\beta(\psi) + (\beta(\psi) + \beta(\psi)) \\ &= -\beta(\psi) + \beta(\psi) \\ &= 1, \end{aligned}$$

using some basic boolean arithmetic. The cases of the other axioms are similar.

If φ is obtained by an application of the rule of Modus Ponens, then there are formulas $\psi \to \varphi = \neg\psi \vee \varphi$ and ψ to which the inductive hypothesis applies. In other words, we have that $-\beta(\psi) + \beta(\varphi) = 1$ and $\beta(\psi) = 1$. But from this we may conclude that $-1 + \beta(\varphi) = 1$ whence $\beta(\varphi) = 1$. This proves (6).

But now we may infer that for all formulas φ and ψ:

$$\varphi \equiv_C \psi \text{ implies } \beta(\varphi) = \beta(\psi). \tag{7}$$

For suppose that $\varphi \equiv_C \psi$. Then by definition, $\varphi \leftrightarrow \psi$ is a theorem, so (7) is immediate by (6).

From this it follows that the map $\hat{\alpha} : Q \to A$ defined by

$$\hat{\alpha}([\varphi]) = \beta(\varphi)$$

is well-defined. We leave it for the reader to prove that $\hat{\alpha}$ is actually a homomorphism extending α, thus finishing the proof of the Theorem. QED

But the fact the Lindenbaum-Tarski algebra is the *free* Boolean algebra for countably many generators makes the connection between Boolean algebras and classical propositional logic quite tight. The following Theorem can be read as stating that the Boolean algebras form a proper class for algebraizing classical propositional logic, at least on the level of formulas.

Theorem 14.3 (BA algebraizes classical theoremhood) *Let φ and ψ be propositional formulas/BA-terms. Then*

$$\vdash_C \varphi \quad \textit{iff} \quad \mathsf{BA} \models \varphi \approx \top \tag{8}$$

$$\mathsf{BA} \models \varphi \approx \psi \quad \textit{iff} \quad \vdash_C \varphi \leftrightarrow \psi. \tag{9}$$

Proof. Suppose that φ is a theorem of classical propositional logic, and let $\mu : V \to \mathcal{Q}$ be an assignment. We will show that $\hat{\mu}(\varphi) = \hat{\mu}(\top)$. (Note that this does *not* follow *directly* from

$\vdash_C \varphi$: from the theoremhood of φ we may only infer that $[\varphi] \approx [\top]$, that is, $\nu(\varphi) = \nu(\top)$ for the natural homomorphism).

Let σ be a substitution such that $\mu(p) = [\sigma(p)]$ for every proposition letter p. It follows from Proposition 10.6 that $\vdash_C \sigma(\varphi)$, from which we can infer that $\vdash_C \sigma(\varphi) \leftrightarrow \top$. But then we have that $\sigma(\varphi) \equiv_C \top$ and we immediately obtain that $\hat{\mu}(\varphi) = [\sigma(\varphi)] = [\sigma(\top)] = \hat{\mu}(\top)$. Since μ was arbitrary, this shows that $\mathcal{Q} \models \varphi \approx \top$. But then from the fact that $\mathcal{Q}$ is the free Boolean algebra, this gives that $\mathsf{BA} \models \varphi \approx \top$.

The remainder of the proof is left to the reader. QED

15 Weak Completeness via Stone

Let us briefly summarize some of our findings until now. In Theorem 11.4 we captured tautologies as equations valid in the algebra **2**:

(3) $$\models_C \varphi \quad \text{iff} \quad \mathbf{2} \models \varphi \approx \top,$$

whereas in the previous section we found an algebraic semantics for the notion of classical theoremhood:

(8) $$\vdash_C \varphi \quad \text{iff} \quad \mathsf{BA} \models \varphi \approx \top,$$

But this means that any connection between $\models_C$ and $\vdash_C$ will show up algebraically. The crucial connection here is Stone's Representation Theorem.

Theorem 15.1 (Stone Representation Theorem) *Any Boolean algebra is isomorphic to a field of sets, and hence, to a subdirect power of* **2**. *As a consequence, the variety of Boolean algebras is generated by* **2**:

$$\mathsf{BA} \;=\; \mathbb{V}(\{\mathbf{2}\}).$$

For instance, we can now prove the weak completeness theorem for classical propositional logic using Stone's Representation Theorem.

Proposition 15.2 (Weak Soundness and Completeness) *For any formula φ, φ is valid iff it is a theorem.*

Proof. Immediate from (3) and (8), since it follows from Stone's Theorem that $Equ(\{\mathbf{2}\}) = Equ(\mathsf{BA})$. QED

Later on we will see how to use Stone's Representation Theorem in order to prove the Strong Completeness Theorem. We first turn to proving Theorem 15.1.

16 Stone's Representation Theorem

We will show that any Boolean algebra can be embedded in the power set algebra of the collection of *ultrafilters* of the algebra.

Definition 16.1 A *filter* of a Boolean algebra $\mathcal{A} = (A, 0, 1, \sim, +)$ is a subset $F \subseteq A$ satisfying
(F1) $1 \in F$
(F2) F is closed under intersection; that is, if $a, b \in F$ then $a \cdot b \in F$
(F3) F is upward closed; that is, if $a \in F$ and $a \leq b$ then $b \in F$.
A filter is *proper* if it does not contain the smallest element 0, or, equivalently, if $F \neq A$.
An *ultrafilter* is a proper filter satisfying
(F4) For every $a \in A$, either a or $-a$ belongs to F.
The collection of ultrafilters of $\mathcal{A}$ is denoted as $Uf(\mathcal{A})$. $\lhd$

Example 16.2 For any element a of a Boolean algebra $\mathcal{B}$, the set $a{\uparrow} = \{b \in a \mid a \leq b\}$ is a filter. The collection of co-finite subsets of an infinite set A is a filter of the power set algebra of A.

For any element a of a set A, the set $\pi_a = \{X \subseteq A \mid a \in X\}$ is an ultrafilter of $\mathcal{P}(\mathcal{A})$, called the *principal ultrafilter* generated by a. In the field of finite and co-finite subsets of a countable set A, the collection of co-finite subsets of A forms an ultrafilter.

Example 16.3 Since the collection of filters of a Boolean algebra is closed under taking intersections, we may speak of the *smallest* filter F_D containing a given set $D \subseteq A$. This filter can also be defined as the following set:

$$\{a \in A \mid \text{ there are } d_0, \ldots, d_n \in D \text{ such that } d_1 \cdot \ldots \cdot d_n \leq a\} \tag{10}$$

which explains that we will also refer to F_D as the filter *generated by* D. This filter is proper if D has the so-called *finite meet property*; that is, if there is no finite subset $\{d_0, \ldots, d_n\}$ of D such that $d_1 \cdot \ldots \cdot d_n = 0$.

For future reference, we gather some properties of ultrafilters; the proof of the next Proposition is left to the reader.

Proposition 16.4 *Let $\mathcal{A}$ be a Boolean algebra. Then*

1. *For any ultrafilter U of $\mathcal{A}$ and for every pair of elements $a, b \in A$ we have that $a + b \in U$ iff $a \in U$ or $b \in U$.*

2. *$Uf(\mathcal{A})$ coincides with the set of maximal proper filters on $\mathcal{A}$ ('maximal' is understood with respect to set inclusion).*

Proposition 16.5 (Ultrafilter Theorem) *Let $\mathcal{A}$ be a Boolean algebra, a an element of A and F a proper filter of $\mathcal{A}$ that does not contain a. Then there is an ultrafilter extending F that does not contain a.*

Proof. We first prove that every proper ultrafilter can be extended to an ultrafilter. Let G be a proper filter of $\mathcal{A}$, and consider the set X of all proper filters H extending G. Suppose that Y is a *chain* in X; that is, Y is a nonempty subset of X of which the elements are pairwise ordered by set inclusion. We leave it to the reader to verify that $\bigcup Y$ is a proper filter; obviously, $\bigcup Y$ extends G; so $\bigcup Y$ belongs to X itself. This shows that X is closed under taking unions of chains, whence it follows from Zorn's Lemma that X contains a maximal element U. We claim that U is an ultrafilter.

For suppose otherwise. Then there is some element $b \in A$ such that neither b nor $-b$ belong to U. Consider the filters H and H' generated by $U \cup \{b\}$ and $U \cup \{-b\}$, respectively. Since neither of these can belong to X, both must be improper; that is, $0 \in H$ and $0 \in H'$. But then by definition there are elements $u_1, \ldots, u_n$, $u'_1, \ldots, u'_m$ in U such that

$$u_1 \cdot \ldots \cdot u_n \cdot b \leq 0$$

and

$$u'_1 \cdot \ldots \cdot u'_m \cdot -b \leq 0.$$

From this it can easily be derived that

$$u_1 \cdot \ldots \cdot u_n \cdot u'_1 \cdot \ldots \cdot u'_m = 0,$$

contradicting the fact that U is a proper filter.

Now suppose that a and F are as in the statement of the Proposition. It is not hard to show that $F \cup \{-a\}$ is a set with the finite intersection property. In Example 16.3 we saw that there is a proper filter G extending F and containing $-a$. Now we use the first part of the proof to find an ultrafilter U extending G. But if U extends G it also extends F and if it does contain $-a$ it cannot contain a. QED

It follows from the previous proposition and the facts mentioned in Example 16.3 that any subset of a Boolean algebra can be extended to an ultrafilter if it has the finite meet property. We now have all the necessary material to prove Stone's Theorem.

Proof of Theorem 15.1. Fix a Boolean algebra $\mathcal{A} = (A, 0, 1, \sim, +)$. We will embed $\mathcal{A}$ in the power set of $\mathit{Uf}(\mathcal{A})$).

Consider the map $\rho : A \to \mathcal{P}(\mathit{Uf}(\mathcal{A}))$ defined as follows:

$$\rho(a) \;=\; \{U \in \mathit{Uf}(\mathcal{A}) \mid a \in U\}.$$

We first show that ρ is a homomorphism. As an example we treat disjunction.

$$\begin{aligned} \rho(a+b) &= \{U \in \mathit{Uf}(\mathcal{A}) \mid a+b \in U\} \\ &= \{U \in \mathit{Uf}(\mathcal{A}) \mid a \in U \text{ or } b \in U\} \\ &= \{U \in \mathit{Uf}(\mathcal{A}) \mid a \in U\} \cup \{U \in \mathit{Uf}(\mathcal{A}) \mid b \in U\} \\ &= \rho(a) \cup \rho(b). \end{aligned}$$

Here the crucial, second equality follows from Proposition 16.4.

It is left to prove that ρ is injective. Suppose that a and b are distinct elements of A. We may derive from this that neither $a \leq b$ nor $b \leq a$. Without loss of generality we may assume the second. But if $b \not\leq a$ then a does not belong to the filter $b\uparrow$ generated by $\{b\}$, so by Proposition 16.5 there is some ultrafilter U such that $b\uparrow \subseteq U$ and $a \notin U$. Obviously, $b\uparrow \subseteq U$ implies that $b \in U$. But then we have that $U \in \rho(b)$ and $U \notin \rho(a)$

This shows that $\mathcal{A}$ is isomorphic to a field of sets; it then follows by Proposition 13.4 that $\mathcal{A}$ is isomorphic to a subdirect power of $\mathbf{2}$. From this it is immediate that BA is the variety generated by the algebra $\mathbf{2}$. QED

Remark 16.6 The fact that every Boolean algebra is isomorphic to a subdirect power of $\mathbf{2}$ can also be proved more directly by observing that there is a 1–1 connection between ultrafilters of $\mathcal{A}$ and homomorphisms from $\mathcal{A}$ onto $\mathbf{2}$. Given an ultrafilter U of $\mathcal{A}$, define $\alpha_U : \mathcal{A} \to \mathbf{2}$ by

$$\alpha_U(a) = \begin{cases} 1 & \text{if } a \in U \\ 0 & \text{otherwise} \end{cases}$$

And conversely, given a homomorphism $\alpha : \mathcal{A} \to \mathbf{2}$, define the ultrafilter U_α by

$$U_\alpha \;=\; \alpha^{-1}(1) \;\; (= \;\{a \in \mathcal{A} \mid \alpha(a) = 1\}).$$

17 Algebraizing $\models_C$ and $\vdash_C$

We will now see what we can say algebraically about the classical consequence and derivability *relations*. Whereas tautologies and theorems can be captured by *equations* we now have to consider (infinitary) quasi-equations as well.

Definition 17.1 Given an algebraic similarity type Σ, an *(infinitary) quasi-equation* is a pair (E, η) where η is an equation in Σ, and E is a set of such equations; such a pair is usually denoted as $\bigwedge E \Rightarrow \eta$, or as $(\eta_1 \& \cdots \& \eta_n) \Rightarrow \eta$ if $E = \{\eta_1, \ldots, \eta_n\}$ is finite.

A quasi-equation q of the form $\bigwedge E \Rightarrow \eta$ is *valid* in an algebra $\mathcal{A}$ (of the appropriate type), notation: $\mathcal{A} \models q$ if for all assignments $\mu : T_\Sigma(X) \to A$ we have that $\mathcal{A} \models E[\mu]$ implies $\mathcal{A} \models \eta[\mu]$. A quasi-equation q is valid in a class of algebras K, notation: $\mathsf{K} \models q$ if q is valid in every algebra in K. $\lhd$

Remark 17.2 The notion of a quasi-equation $\bigwedge E \Rightarrow \eta$ being valid in a class K is sometimes referred to as 'η being a semantic consequence of E over K'. However, in our terminology (cf. Definition 4.8) η being a semantic consequence of E means that for every algebra $\mathcal{A}$ the following holds: if $\mathcal{A} \models E$ (that is, if $\mathcal{A} \models E[\mu]$ for *all* assignments μ), then $\mathcal{A} \models \eta$ (that is, $\mathcal{A} \models \eta[\mu]$ for *all* assignments μ). In order to investigate the truth of a quasi-equation we *fix* an assignment, and check whether if E holds under *this* assignment, so does η.

Obviously, the two notions are related. They are not the same however: the quasi-equation $x \approx z \Rightarrow x \approx y$ is not valid in non-trivial algebras, while we clearly have $x \approx z \models x \approx y$.

Remark 17.3 In passing we note that it is easy to prove the following completeness result concerning quasi-equations. Let E be some Σ-specification; consider the alternative version of equational logic (cf. Definition 3.2) in which the rule of (substitution) is not allowed. Then one can show that an equation η is derivable from E in this alternative sense if and only if the quasi-equation $\bigwedge E \Rightarrow \eta$ is valid in the class of all Σ-algebras.

The basic idea behind the use of quasi-equations in the algebraization of the classical consequence and derivability relation is that we encode the statement $\Phi \models_C \psi$ and $\Phi \vdash_C \psi$ by the quasi-equation

$$\bigwedge\{\varphi \approx \top \mid \varphi \in \Phi\} \Rightarrow \psi \approx \top.$$

We introduce some convenient notation.

Definition 17.4 Consider the following translation $(\cdot)^{\approx}$ mapping propositional formulas to BA-equations and the reverse translation $(\cdot)^{\leftrightarrow}$ mapping equations to formulas, given by

$$\begin{array}{rcl} \varphi^{\approx} & = & \varphi \leftrightarrow \top, \\ (\varphi \approx \psi)^{\leftrightarrow} & = & \varphi \leftrightarrow \psi. \end{array}$$

Given a set Φ of propositional formulas, we let $\Phi^{\approx}$ denote the set $\{\varphi^{\approx} \mid \varphi \in \Phi\}$, and given a set of Boolean equations E, we let $E^{\leftrightarrow}$ denote the set $\{\eta^{\leftrightarrow} \mid \eta \in E\}$. $\lhd$

We can now rather concisely state that $\{\mathbf{2}\}$ and BA are algebraizations of $\models_C$ and $\vdash_C$, respectively.

Theorem 17.5 ($\mathbf{2}$ algebraizes $\models_C$)

1. *Let Φ be a set of propositional formulas, and ψ a propositional formula. Then*

$$\Phi \models_C \psi \;\; \textit{iff} \;\; \mathbf{2} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}.$$

2. *Let E be a set of Boolean equations, and η a Boolean equation. Then*

$$\mathbf{2} \models \bigwedge E \Rightarrow \eta \;\; \textit{iff} \;\; E^{\leftrightarrow} \models_C \eta^{\leftrightarrow}.$$

3. *Furthermore, the translations $(\cdot)^{\approx}$ and $(\cdot)^{\leftrightarrow}$ are each other's inverse in the following sense:*

$$\varphi \mathrel{=\!\mid\!\models_C} (\varphi^{\approx})^{\leftrightarrow}.$$

PROOF. Immediate from the definitions. QED

Theorem 17.6 (BA algebraizes $\vdash_C$)

1. *Let Φ be a set of propositional formulas, and ψ a propositional formula. Then*

$$\Phi \vdash_C \psi \;\; \textit{iff} \;\; \mathsf{BA} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}.$$

2. Let E be a set of Boolean equations, and $\varphi \approx \psi$ a Boolean equation. Then

$$\mathsf{BA} \models \bigwedge E \Rightarrow \eta \;\; \textit{iff} \;\; E^{\leftrightarrow} \vdash_C \eta^{\leftrightarrow}.$$

3. Furthermore, the translations $(\cdot)^{\approx}$ and $(\cdot)^{\leftrightarrow}$ are each other's inverse in the following sense:

$$\varphi \;\dashv\vdash_C (\varphi^{\approx})^{\leftrightarrow}.$$

Proof. For the first part of the Theorem, fix a set Φ of propositional formulas. The direction from left to right can be proved by a straightforward induction on the length of derivations. For the other direction, observe that the relation $\equiv_\Phi$ of provable equivalence modulo Φ is a congruence on the formula algebra $\mathcal{L}$. The proof of this claim is roughly the same as the one for Proposition 12.1.

We then let $\mathcal{Q}^\Phi$ denote the quotient algebra $\mathcal{L}/{\equiv_\Phi}$, and prove that this algebra is a Boolean algebra — an easy way of proving this uses the fact that $\equiv_\Phi$ contains $\equiv_C$, whence $\mathcal{Q}^\Phi (= \mathcal{L}/{\equiv_\Phi})$ is a homomorphic image of $\mathcal{Q} = (\mathcal{L}/{\equiv_C})$.

Now consider the assignment $\mu : V \to Q^\Phi$ given by $\mu(p) = [p]$; i.e., $\hat{\mu}$ is the natural homomorphism from $\mathcal{L}$ to its quotient $\mathcal{Q}^\Phi$. It is immediate that for any formula ψ, we have that $\Phi \vdash_C \psi$ iff $\Phi \vdash_C \psi \leftrightarrow \top$ iff $\psi \equiv_\Phi \top$ iff $[\psi] = [\top]$ iff $\hat{\mu}(\psi) = \hat{\mu}(\top)$ iff $\mathcal{Q}^\Phi \models \psi \approx \top[\mu]$. But from

$$\Phi \vdash_C \psi \text{ iff } \mathcal{Q}^\Phi \models \psi^{\approx}[\nu].$$

it follows that if ψ is *not* derivabe from Φ, then the quasi-equation $\bigwedge \Phi^{\approx} \Rightarrow \psi \approx \top$ does not hold in $\mathcal{Q}^\Phi$. Since $\mathcal{Q}^\Phi$ is a Boolean algebra, this finishes the proof of the direction from right to left.

The remaining part of the proof is left to the reader. QED

The above two Theorems, together with Stone's Representation Theorem and a preservation result concerning quasi-equations, can be used to give an algebraic proof of the Strong Completeness Theorem.

Proof of Theorem 10.7. First we prove soundness: suppose that $\Phi \vdash_C \psi$. It follows from Theorem 17.6 that $\mathsf{BA} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}$. In particular, $\mathbf{2}$ is a Boolean algebra, so we find that $\mathbf{2} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}$. Then by Theorem 17.5 it follows that $\Phi \models_C \psi$.

Conversely, suppose that $\Phi \models_C \psi$. By Theorem 17.5 we obtain that $\mathbf{2} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}$. Now consider an arbitrary Boolean algebra $\mathcal{A}$. By Stone's Theorem, $\mathcal{A}$ is (isomorphic to) a subdirect power of $\mathbf{2}$. We leave it to the reader to verify that the validity of quasi-equations is preserved under taking subalgebras and products. From this it follows that $\mathcal{A} \models \bigwedge \Phi^{\approx} \Rightarrow \psi^{\approx}$. This proves the strong completeness direction of the Theorem. QED

18 Beth definability and surjective epis

In this last section of part 2 we discuss the algebraic pendant of another logical property, viz., Beth Definability. Since we will now compare various propositional languages, it is convenient to provide slight generalizations of some our definitions.

Definition 18.1 Given a set P of propositional variables, let L_P denote the set of formulas using variables in P. Instead of $L_{P\cup\{q\}}$ we write L_{Pq}.

It is not difficult to show that all of our earlier definitions and results apply to this more general setting. We fix some notation here: $\mathcal{L}_P$ denotes the formula algebra for the language L_P, $\mathcal{Q}_P$ denotes the Lindenbaum-Tarski algebra, and for a set of formulas Φ, $\mathcal{Q}_P^\Phi$ denotes the algebra $\mathcal{L}_P$, divided over the congruence $\equiv_\Phi$. $\lhd$

Note that in our definition of $\mathcal{Q}_P^\Phi$ we did not assume any connection between P and the proposition letters occurring in Φ.

In the remainder of this section, we fix a set P of propositional variables, together with three distinct proposition letters q, q' and r not belonging to P. When we write $\Delta(P,r)$ for a set of formulas, we assume that the variables occurring in this set belong to $P\cup\{r\}$; after introducing a set $\Delta(P,r)$ we let $\Delta(P,q)$ denote the set $\{\delta[q/r] \mid \delta\in\Delta(P,r)\}$.

Definition 18.2 Let $\Delta(P,r)$ be a set of formulas. We say that $\Delta(P,r)$ *implicitly defines* the proposition letter r if

$$\Delta(P,q),\Delta(P,q')\vdash_C q\leftrightarrow q'.$$

$\Delta(P,r)$ *explicitly defines* r if there is some formula φ, called an *explicit definition* of r, that does *not* use r, such that

$$\Delta(P,r)\vdash_C \varphi\leftrightarrow r.$$

We say that a relation $\vdash$ has *Beth's definability property* if whenever $\Delta(P,r)$ implicitly defines r, it also explicitly does so. $\lhd$

In this section we will give an algebraic proof of the following Theorem.

Theorem 18.3 (Beth Definability) *The derivability relation $\vdash_C$ of classical propositional logic has Beth's definability property.*

In order to prove the Theorem we need some algebraic (in fact, category theoretic) definitions.

Definition 18.4 (epis) Let Σ be an algebraic similarity type, and K a class of Σ-algebras. A homomorphism $\beta:\mathcal{A}\to\mathcal{B}$ is an *epi* if for every pair of homomorphisms $\gamma,\gamma':\mathcal{B}\to\mathcal{C}$ (with $\mathcal{C}$ in K), we have

$$\gamma\circ\beta=\gamma'\circ\beta \text{ only if } \gamma=\gamma'.$$

It is easy to see that surjective homomorphisms are epis; if conversely, all epis between algebras in K are surjective, we say that *epi's are surjective in* K, or simply that K has ES. $\lhd$

Remark 18.5 In practice, it is often easier to use an alternative criterion. Let Σ be an algebraic similarity type, and K a class of Σ-algebras. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are algebras in K. We say that $\mathcal{A}$ is an *epic subalgebra* of $\mathcal{B}$ if every two homomorphisms $\gamma,\gamma':\mathcal{B}\to\mathcal{C}$ (with $\mathcal{C}$ in K) that agree on A must be identical. That is, if $\gamma\restriction_A=\gamma'\restriction_A$ implies $\gamma=\gamma'$. (Here $\gamma\restriction_A$ denotes the restriction of γ to A.) In words, $\mathcal{A}$ is an epic subalgebra of $\mathcal{B}$ if the behaviour of a homomorphism $\gamma:\mathcal{B}\to\mathcal{C}$ is completely determined by its behaviour on A.

We leave it to the reader to verify that a class K has ES if and only it has no proper epic subalgebras.

Example 18.6 A *bounded distributive lattice* is an algebra $(A, 0, 1, +, \cdot)$ satisfying the equations (B0)–(B3). The class BDL of bounded distributive lattices does not have the property ES. To see this, consider the BDL $\mathcal{B} = (Sb(\{a,b\}), \varnothing, \{a,b\}, \cup, \cap)$ and its subalgebra $\mathcal{A}$ with universe $A = \{\varnothing, \{a\}, \{a,b\}\}$. We claim that $\mathcal{A}$ is an epic subalgebra of $\mathcal{B}$.

For, suppose that γ and γ' are both homomorphisms from $\mathcal{B}$ to some $\mathcal{C}$ in BDL, and that γ and γ' agree on A. It follows that both $\gamma(\{b\})$ and $\gamma'(\{b\})$ are complements of $\gamma(\{a\})$. (We call c a complement of d if $c + d = 1$ and $c \cdot d = 0$.) But then $\gamma(\{b\})$ and $\gamma'(\{b\})$ must be identical, since complements are unique in any bounded distributive lattice.

For a proof of the latter fact, suppose that both c and c' are complements of d in some bounded distributive lattice. Then we have that

$$c = c + 0 = c + (c' \cdot d) = (c + c') \cdot (c + d) = (c + c') \cdot 1 = c + c',$$

and with a symmetric argument,

$$c' = c' + c.$$

From this it follows that $c = c'$.

Without proof we mention the following fact.

Fact 18.7 *The class of Boolean algebras has ES.*

From this we can now easily derive that classical propositional logic satisfies Beth Definability.

Proof of Theorem 18.3. Suppose that $\Delta(P, r)$ is an implicit definition of r; that is, assume that $\Delta(P, q), \Delta(P, q') \vdash_C q \leftrightarrow q'$. We will prove the existence of some $\varphi \in L_P$ such that $\Delta(P, r) \vdash_C \varphi \leftrightarrow r$, thus showing that $\Delta(P, r)$ is also an explicit definition of r.

Let Δ abbreviate the set $\Delta(P, r)$. We will compare the algebras $\mathcal{Q}_P^\Delta$ and $\mathcal{Q}_{Pr}^\Delta$. We leave it to the reader to verify that the map $\iota : \mathcal{Q}_P^\Delta \to \mathcal{Q}_{Pr}^\Delta$ given by $\iota([\varphi]) = [\varphi]$ is an embedding. We claim that ι is in fact an epi.

To see this, suppose that γ and γ' are homomorphisms from $\mathcal{Q}_{Pr}^\Delta$ to some Boolean algebra $\mathcal{C}$ such that $\gamma \circ \iota = \gamma' \circ \iota$. From this it easily follows that for all formulas $\varphi \in L_P$ we have that

$$\gamma([\varphi]) = \gamma'([\varphi]). \tag{11}$$

Consider the algebras $\mathcal{L}_{Pqq'}$ and $\mathcal{L}_{Pr}$, and the substitution $\sigma : \mathcal{L}_{Pqq'} \to \mathcal{L}_{Pr}$ given by

$$\sigma(x) = \begin{cases} x & \text{if } x \in P \\ r & \text{if } x = q \text{ or } x = q', \end{cases}$$

and let $\nu : \mathcal{L}_{Pr} \to \mathcal{Q}_{Pr}^\Delta$ be the natural homomorphism given by

$$\nu(\varphi) = [\varphi].$$

Since for any $\delta \in \Delta$ we have that $\delta \equiv_\Delta \top$, this gives

$$\nu(\delta) = [\top] \text{ for any } \delta \in \Delta. \tag{12}$$

We are interested in the assignment $\mu : \mathcal{L}_{Pqq'} \to \mathcal{C}$ given by

$$\mu(x) = \begin{cases} \gamma(\nu(\sigma(x))) & \text{if } x \in P \cup \{q\} \\ \gamma'(\nu(\sigma(x))) & \text{if } x \in P \cup \{q'\} \end{cases}$$

Note that μ is well-defined by (11); the unorthodox shape of its definition facilitates the proof of the following claim:

(13) $$\mathcal{C} \models (\Delta(P,q) \cup \Delta(P,q'))^{\approx} \; [\mu].$$

This is not so hard to see: for any $\delta \in \Delta(P,q)$ we have that $\sigma(\delta) \in \Delta$, whence by (12)

$$\hat{\mu}(\delta) = \gamma(\nu(\sigma(\delta))) = \gamma([\top]) = 1,$$

and likewise for $\delta \in \Delta(P,q')$. But from Theorem 17.6 and our assumption that $q \leftrightarrow q'$ is derivable from $\Delta(P,q) \cup \Delta(P,q')$ it follows that

$$\mathcal{C} \models (\Delta(P,q) \cup \Delta(P,q'))^{\approx} \Rightarrow (q \leftrightarrow q' \approx \top) \; [\mu].$$

Hence, we may infer from (13) that $\hat{\mu}(q \leftrightarrow q') = 1$, or, equivalently,

$$\hat{\mu}(q) = \hat{\mu}(q').$$

But then by definition of μ we find that

$$\gamma([r]) = \gamma(\nu(\sigma(q))) = \hat{\mu}(q) = \hat{\mu}(q') = \gamma'(\nu(\sigma(q))) = \gamma'([r]).$$

But since $\mathcal{Q}^{\Delta}_{Pr}$ is *generated* by the set $\{[p] \mid p \in P\} \cup \{[r]\}$, from this we may infer that γ and γ' are identical. This shows that ι is an epi.

But if ι is an epi, it must be surjective by Fact 18.7. From this it follows that there is some formula $\varphi \in L_P$ such that $\iota([\varphi]) = [r]$. By definition of ι, this gives that $\varphi \equiv_{\Delta} r$, which be definition means that $\Delta \vdash_C \varphi \leftrightarrow r$, which is what we were after. QED

The attentive reader may have observed that, apart from ES, the proof of Theorem 18.3 does not use any particular fact about Boolean algebras. This already indicates that the connection between Beth Definability and surjectiveness of epis is not confined to the algebraization of classical propositional logic in the class of Boolean algebras.

In fact, one can show the following. If a derivability (or logical consequence) relation $\vdash$ between sets of formulas and formulas has an equivalent algebraic semantics, consisting of a class of algebras K that is related to $\vdash$ in the same way as BA algebraizes $\vdash_C$ (cf. Theorem 17.6), then we have the following equivalence:

$$\vdash \text{ has } \mathrm{B}^{\infty} \text{ iff } \mathsf{K} \text{ has ES},$$

where B^{∞} is some natural infinitary version of Beth's Definability Property. (The connection between Beth definability properties and surjectiveness of epis was established by I. Németi.) The above equivalence is used in particular as a tool for proving that a logic does *not* have Beth's definability property.

Example 18.8 Consider the *fragment* of classical propositional logic in which the only connectives that we have at our disposal are $\bot$, $\top$, $\vee$, $\wedge$ and $\leftrightarrow$.

(The last two connectives are now considered to be primitive.) It can be shown that the consequence relation for this fragment is algebraized by the class of algebras of the form $\mathcal{A} = (A, 0, 1, +, \cdot, \|)$ of which $(A, 0, 1, +, \cdot)$ is a bounded distributive lattice, and $\mathcal{A}$ satisfies $a = b$ iff $a \| b = 1$. Using a slight modification of the algebra in Example 18.6 we can show that not every epi between such algebras is surjective, from which it follows that the $\{\bot, \top, \vee, \wedge, \leftrightarrow\}$-fragment of classical propositional logic does not have (the infinitary version of) Beth's definability property. In fact, one can show that in this fragment one can define *negation* implicitly, but not explicitly.

Appendix: Literature

For readers that want to know more about these topics, here are a few suggestions for further reading.

A classic text book on Universal Algebra, giving considerable attention to logical questions and the connection with Model Theory, is the following.

- Stanley Burris & H.P. Sankappanavar, *A Course in Universal Algebra*, Springer Verlag, New York, 1981. (This book is now out of print, but its content is available from the WWW at `http://www.thoralf.uwaterloo.ca`)

Almost any introductory text book on logic introduces Boolean algebras, but the connection with logic is not always spelled out in detail. The following is a good exception.

- J. Donald Monk, *Mathematical Logic*, Springer Verlag, New York, 1976.

For a book that is devoted in particular to the algebraization of classical propositional logic, consult

- Paul Halmos & Steven Givant, *Logic as Algebra*, The Mathematical Association of America, 1998 (Dolciani Mathematical Expositions – No. 21)

And finally, for information about the general theory of algebraic logic, the reader is referred to

- W.J.Blok & Don Pigozzi, “Algebraizable logics”, *Memoirs of the American Mathematical Association*, January 1989, Volume 77, Number 396.