COMPUTER AIDED INVESTIGATIONS OF RELATION ALGEBRAS

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I dedicate this dissertation to the memory of my father

Hermann Johannes Carl Jipsen

and to my mother

Ruth Jipsen.

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CHAPTER I

INTRODUCTION

The theory of relation algebras is a branch of algebraic logic and universal algebra that investigates the abstract properties of binary relations and their application to logic. Systems of binary relations have been studied in their own right since the mid-nineteenth century. Some of the earliest results are due to A. DeMorgan, and much work was done by R. S. Pierce and E. Schröder during the later half of that century.

The notion of an abstract relation algebra is due to A. Tarski, and some results about them were first published in the early 1940's (Tarski [**41**]). R. Lyndon [**50**] proved the existence of nonrepresentable relation algebras, and D. Monk [**64**] showed that the variety of representable relation algebras is not finitely based. The concept of a Boolean algebra with operators is due to B. Jónsson and A. Tarski [**51**][**52**].

The aims of this dissertation are twofold. We wish to show that it is useful to consider the theory of relation algebras as a special case of the theory of Boolean algebras with a residuated binary operator and a unit element (*ur*-algebra for short), and secondly we hope to demonstrate how a computer may be utilized to prove results in this area.

Our approach is an algebraic one, concentrating on the structure of the lattice of subvarieties of the variety of all *ur*-algebras, and on some interesting members of this lattice, including the varieties of all

- residuated Boolean monoids (denoted by RM)
- Euclidean *ur*-algebras (EUR)
- commutative *ur*-algebras (CUR)
- nonassociative relation algebras (NA)
- semiassociative relation algebras (SA)
- relation algebras (RA)
- symmetric relation algebras (SRA)

as well as the varieties generated by all

- *ur*-algebras in which the unit is an atom (AUR)
- integral *ur*-algebras (IUR)
- complex algebras of monoids
- complex algebras of groups (GRA)
- proper relation algebras (RRA).

Listed below are some of the questions we address about these varieties.

Q1 Are they discriminator varieties?

Q2 Do they have a decidable equational theory?

Q3 Are all subvarieties of finite height finitely generated?

Q4 Do all subvarieties of finite height have only finitely many covers? (Can we list these covers effectively?)

Q5 For those varieties defined in terms of equations, can we find 'minimal' subclasses that generate them?

Q6 For those varieties generated by a class of algebras, can we find 'minimal' equational bases for them? (In particular, are they finitely based?)

In many cases the answers are trivial, in other cases they follow from well-known classical results of the theory of relation algebras, and in several cases they are still open (which does not mean that they are difficult, since some of these questions have not been addressed before). The last two questions are somewhat vague. The word 'minimal' can be interpreted in many ways, or should perhaps even be replaced by a more subjective term like 'illuminating' or 'natural'.

Of course there are many other varieties that could be considered, and many more questions that one could ask about them. A 'metaquestion' that arises is why anyone would be interested in these results. Questions like the ones above attempt to assess how complicated the equational theory of a variety is. If we are interested in the equational theory of relation algebras, which is rather complicated, we can get some insight into why this is so, by looking at some simpler varieties containing or contained in RA.

Questions 3 and 4 measure to some degree how well-behaved the lattice of subvarieties of the given variety is. If the answer to both questions is yes, then we can in principle draw a picture of the bottom of the lattice up to whatever (finite) height we desire. If the answer is no to either question, then even the bottom part of the lattice is bound to be quite complicated.

Some of the varieties listed above are not of intrinsic interest, but they may provide a contrast to some closely related variety with different properties. So for example in the case of relation algebras the properties of being integral and of having an atom as the unit element are equivalent whereas IUR is a proper subalgebra of AUR.

We briefly indicate where some of the other varieties appear. The concepts of residuation and conjugation are of course central to the theory of relation algebras (see Chin and Tarski [**51**], Birkhoff [**67**]). The variety UR of all ur-algebras is defined in Jónsson and Tsinakis [**a**], and it is shown there that RA can be defined relative to UR by a very simple equation.

The varieties NA, SA and also the weakly associative relation algebras WA were studied by R. D. Maddux [**78**][**82**]. It is shown there that members of SA are related to certain reducts of 3-dimensional cylindric algebras, and that WA is generated by all relativizations of proper relation algebras with respect to symmetric reflexive relations.

The variety *SRA* of symmetric relation algebras is a very natural subvariety of *RA*. Its members have interesting combinatorial properties. R. D. Maddux [**81**] showed that every modular lattice can be embedded into the lattice of equivalence elements in a symmetric relation algebra.

The variety generated by all complex algebras of monoids is relevant to the study of automata over regular languages. It has not been studied in any detail. *GRA* is a natural variety to consider, given that groups are important mathematical objects. It is mentioned in Lyndon [**59**], and R. N. McKenzie [**66**] showed that it is not finitely based relative to the variety generated by all integral representable relation algebras.

RRA is of course the variety of all representable relation algebras, ultimately responsible for the existence of this branch of algebraic logic.

We now outline the contents of the subsequent chapters. Chapter 2 begins with a brief introduction to Boolean algebras with operators (BAOs for short), varieties, residuation and the notion of a discriminator term. We then take a closer look at some general results about (residuated) BAOs. We investigate their algebraic structure and give a normal form for the variety of all BAOs of a given type. The third section describes an algorithm that can be used to search for proofs or finite counter-examples to conjectures about BAOs. The algorithm we propose is of a more semantic nature than other automated theorem proving methods of logic, like Gentzen systems, semantic tableau or natural deduction. Several of the results in Chapters 3 and 4 have been proved using a computer program based on this algorithm. Some of the issues concerning the implementation of this algorithm, and a proof found by computer are included in the appendix.

Chapter 3 discusses the questions Q1–Q6 with regards to the varieties of *ur*-algebras listed above and other varieties derived from them. In the first section we give some general results about *ur*-algebras and answer Question 1 and some forms of 5 and 6. One of the more interesting results is that *CRM* (the variety of all commutative residuated Boolean monoids) is a discriminator variety, whereas *ERM* (the variety of all Euclidean residuated Boolean monoids) is not. In the next section we consider Question 2. As a consequence of the undecidability of the equational theory of modular lattices, we are able to show by an easy argument that the equational theory of *SRA* is also undecidable. On the other hand, adapting a standard argument of J. C. C. McKinsey, we show that *UR*, *CUR*, *IUR* and *AUR* have decidable equational theories. The third section addresses Questions 3 and 4. The answer to Question 3 is easy: We give an example (from Jipsen and Lukács [**a**]) of an infinite simple symmetric relation algebra that is a member of each of the varieties listed above and generates a subvariety of height 2 in Λ. With regards to Question 4, we show that there is a height 1 subvariety of *RM* that has infinitely many covers, but the question is still open for *RA*, and we have only a partial answer for some subvarieties of *SRA*. In the fourth section we list all parts of Questions 1–6 that are still open as well as some other problems we consider interesting.

Chapter 4 contains further results about relation algebras. In the first section we consider a particularly simple sequence of symmetric relation algebras and show that they generate an ascending chain of subvarieties of *GRA*. From the uniform structure of these algebras we can then conclude that subalgebra lattices of members of *GRA* do not satisfy any nontrivial lattice equations. The next section is concerned with relation algebras that are generated by equivalence elements. In Jónsson [**88**] it is shown that a single equivalence element generates a finite representable relation algebra. The case of what chains (or more generally trees) of equivalence elements generate has been treated in a more general setting by S. Givant [**a**] in a recent monograph. Here we show that in a simple symmetric relation algebra any subalgebra generated by two equivalence elements is finite. In the last section we show that there exist nonrepresentable absolute retracts in *SRA*. Andréka, Jónsson and Németi [**91**] point out that the absolute retracts in *RRA* are precisely the full relation algebras over finite base sets, and these algebras are also absolute retracts in the larger variety *RA* (even in *SA*), but it is not known whether there are others. The result about *SRA* makes this seem likely.

We wish to emphasize that this is not a complete account of the theory of *ur*-algebras. Two important concepts that are not considered here, are the structure of the free algebras (in the variety of all *ur*-algebras or in some subvarieties; Andréka, Jónsson and Németi [**91**] prove some general results about free algebras in discriminator varieties), and the notion of splitting algebras (introduced by R. N. McKenzie [**72**] for lattices and used by W. J. Blok [**78**][**80b**] to analyse the lattice of varieties of modal algebras). The reason for not including these concepts is that they have not been investigated in detail for *ur*-algebras, and that at this stage we have little to add to what is known about them. There certainly is opportunity for future research here.

CHAPTER II

BOOLEAN ALGEBRAS WITH OPERATORS

Preliminaries

In this section we define Boolean algebras with operators, BAOs for short, and show how they occur naturally as complex algebras of relational structures. The duality between complete and atomic BAOs with homomorphisms and relational structures with bounded morphisms is mentioned and canonical extensions are defined. Some of the properties preserved by canonical extensions are listed. We then recall some basic results about the algebraic structure of BAOs (congruence ideals, relative subalgebras, etc).

Thereafter we introduce the concepts of residuation and conjugation. We define residuated BAOs and show what effect residuation has on the algebraic structure of BAOs.

To show that BAOs and residuated BAOs are useful abstractions we mention several examples of algebraic structures that fit into this framework: modal logics/algebras, tense logics/algebras, relation algebras, r-algebras, multigroups, geometries, relevance logics.

We end with a brief introduction to discriminator algebras and varieties, and show how the discriminator is realized in BAOs.

Boolean algebras with operators. Let $\mathbf{A}_0 = (A, +, 0, \cdot, 1, ^-)$ be a Boolean algebra. A unary operation f on $\mathbf{A}_0$ is *additive* if $f(x+y) = f(x) + f(y)$ and *normal* if $f(0) = 0$. Now let f be an n-ary operation on $\mathbf{A}_0$ and let $\underline{a}$ be a sequence in A^n. For $i < n$ we define the $(\underline{a}, i)$*-translate of* f to be the unary operation

$$f_{\underline{a},i}(x) = f(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{n-1}).$$

An *operator* on $\mathbf{A}_0$ is an n-ary operation that is additive and normal in each argument, i.e., an operation for which all $(\underline{a}, i)$-translates are additive and normal. Note that 0-ary operations (constants) have no translates, so they are operators by default.

Let ρ be a function from some fixed index set $\mathcal{I}$ to the set of natural numbers ω. We think of the elements of $\mathcal{I}$ as operation symbols, and later also as relational symbols.

$\mathbf{A} = (\mathbf{A}_0, \mathcal{F})$ is a *Boolean algebra with operators, or BAO, of type* ρ if $\mathcal{F} = (f^{\mathbf{A}} : f \in \mathcal{I})$ is a sequence of operators on $\mathbf{A}_0$ and for each $f \in \mathcal{I}$ the rank of $f^{\mathbf{A}}$ is $\rho(f)$. The class of all BAOs of type ρ is denoted by BAO_ρ. $\mathbf{A}_0$ is the *Boolean reduct* of $\mathbf{A}$. A BAO is said to be *complete and atomic* if the Boolean reduct is complete and atomic and all the operators are completely additive, meaning they commute with arbitrary joins in each argument. The class of all complete and atomic BAOs of type ρ is denoted by BAO_ρ^{ca}.

When confusion is unlikely, we will often omit the superscript of an operator $f^{\mathbf{A}}$, thus blurring the distinction between an operation symbol and its interpretation. Also, in the interest of readability we let n represent the rank of whatever operation is being considered at the time. So we may write

$$\text{‘}f(x_0, \ldots, x_{n-1})\text{’} \quad \text{and} \quad \text{‘}f(\underline{x}) \text{ where } \underline{x} \in A^n\text{’}$$

with $n = \rho(f)$ understood.

The fundamental operations in a BAO $\mathbf{A}$ are the Boolean operations $(+, 0, \cdot, 1, ^-)$ and the operators $(f^{\mathbf{A}} : f \in \mathcal{I})$. The notion of a homomorphism between two Boolean algebras of the same type is the standard definition of a map that commutes with all these operations. Likewise the concepts of subalgebra, direct product, congruence and subdirectly irreducible algebra are those of universal algebra. A good introduction to these ideas can be found in McKenzie, McNulty and Taylor [**87**] or Burris and Sankappanavar [**81**]. For a class $\mathcal{K}$ of BAOs of type ρ we let $\mathbb{H}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$ and $\mathbb{P}(\mathcal{K})$ denote the class of all homomorphic images, all subalgebras and all direct products of members of $\mathcal{K}$ respectively. We think of the index set $\mathcal{I}$ together with the Boolean operation symbols as the language (collection of all operation symbols) of BAO_ρ. The set of all terms in this language is denoted by T_ρ. Let $\mathcal{E} \subseteq T_\rho \times T_\rho$ be a set of equations. An element (r, s) of $\mathcal{E}$ is usually written as $r = s$, and $r \le s$ represents the equation $r \cdot s = s$. We will use the following notation throughout:

$\mathrm{Mod}(\mathcal{E})$ = the class of all algebras that satisfy all equations in $\mathcal{E}$.

$\mathrm{Eq}(\mathcal{K})$ = the set of all equations satisfied by all algebras in $\mathcal{K}$.

$\mathcal{K}$ is a *variety* if $\mathcal{K} = \mathrm{Mod}(\mathcal{E})$ for some set of equations $\mathcal{E}$.

$\mathrm{Var}(\mathcal{K}) = \mathrm{Mod}\ \mathrm{Eq}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K})$ = *the variety generated by* $\mathcal{K}$.

$\mathrm{Si}(\mathcal{K})$ = the class of all subdirectly irreducible members of $\mathcal{K}$.

$\Lambda_{\mathcal{K}}$ = the collection of all subvarieties of $\mathcal{K}$.

$\Lambda_{\mathcal{K}}$ is closed under intersections, and if $\mathcal{K}$ is a variety then it is the largest element, so $\Lambda_{\mathcal{K}}$ is a lattice. The meet of two varieties is given by their intersection, and the join is the variety generated by their union.

To be an operator on a Boolean algebra is of course an equational property, hence BAO_ρ is a variety. If $\mathcal{I}$ is finite then BAO_ρ is finitely based, meaning it can be defined by finitely many equations. The variety $BAO_{(1)}$ with one unary operator, is usually referred to as the variety of *modal algebras* (the algebraic counterpart of modal logic).

A *relational structure of type* ρ is an ordered pair $\mathbf{U} = (U, \mathcal{R})$ such that $\mathbf{U}$ is a set and $\mathcal{R} = (R^{\mathbf{U}} : R \in \mathcal{I})$ is a sequence of relations on U of rank $\rho(R)+1$. The class of all relational structures of type ρ is denoted by RS_ρ.

Boolean algebras with operators provide a natural framework for the study of relational structures from an algebraic point of view. In fact there is a duality between BAO_ρ^{ca} and RS_ρ. Given $\mathbf{U} = (U, \mathcal{R}) \in RS_\rho$ we define the *complex algebra* $\mathbf{U}^+ = (U^+, \mathcal{R}^+)$ as follows: U^+ is the Boolean algebra of all subsets of $\mathbf{U}$ and $\mathcal{R}^+ = (R^+ : R \in \mathcal{I})$ is a sequence of operations on U^+, where each R^+ is derived from the corresponding relation by

$$R^+(X_0, \ldots, X_{n-1}) = \{y : (\underline{x}, y) \in R \text{ for some } \underline{x} \in X_0 \times \cdots \times X_{n-1}\}.$$

It is easy to check that $\mathbf{U}^+ \in BAO_\rho^{\mathrm{ca}}$. Conversely, if $\mathbf{A} = (\mathbf{A}_0, \mathcal{F}) \in BAO_\rho^{\mathrm{ca}}$ then the *atom structure* $\mathbf{A}_+$ is the pair $(\mathbf{A}_{0+}, \mathcal{F}_+)$, where $\mathbf{A}_{0+}$ is the set of all atoms of $\mathbf{A}_0$ and $\mathcal{F}_+ = (f_+ : f \in \mathcal{I})$ is a sequence of relations on $\mathbf{A}_{0+}$ defined by

$$(x_0, \ldots, x_n) \in f_+ \quad \text{iff} \quad f(x_0, \ldots, x_{n-1}) \ge x_n.$$

This is obviously a relational structure of type ρ, and it is straight forward to verify that $\mathbf{A} \cong (\mathbf{A}_+)^+$ and $\mathbf{U} \cong (\mathbf{U}^+)_+$.

Let $\mathbf{U}, \mathbf{V} \in RS_\rho$ and let h be a map from U to V. We define $h^+ : V^+ \to U^+$ to be the map that sends a subset of V to its inverse image under h. A *bounded morphism* is a map $h : \mathbf{U} \to \mathbf{V}$ such that h^+ is a BAO homomorphism from $\mathbf{V}^+$ to $\mathbf{U}^+$. This property is characterized by the condition: for all $R \in \mathcal{I}$

$$(\underline{z}, h(y)) \in R^{\mathbf{V}} \qquad \text{iff} \qquad \begin{array}{l} \text{there exists } \underline{x} \in U^n \text{ such that } (\underline{x}, y) \in R^{\mathbf{U}} \\ \text{and } h(x_i) = z_i \ \ (i < n = \rho(R)). \end{array}$$

For $\mathbf{A}, \mathbf{B} \in BAO_\rho^{\text{ca}}$ and a complete homomorphism $k : \mathbf{A} \to \mathbf{B}$ one can also define a bounded morphism $k_+ : \mathbf{B}_+ \to \mathbf{A}_+$ and establish a dual equivalence between the category RS_ρ with bounded morphisms and BAO_ρ^{ca} with complete homomorphisms. This and other dualities are treated in detail in Goldblatt [**89**]. Beyond the concept of a bounded morphism we will not make use of the categorical aspects of this duality.

Jónsson and Tarski [**51**] showed that a BAO $\mathbf{A} = (\mathbf{A}_0, \mathcal{F})$ can be extended to a complete and atomic BAO $\mathbf{A}^\sigma = (\mathbf{A}_0^\sigma, \mathcal{F}^\sigma)$ called the canonical extension of $\mathbf{A}$. For the Boolean reduct $\mathbf{A}_0$ the extension arises from the Stone duality and can be characterized algebraically by the following two properties.

(σ_1) For all distinct atoms a, a' in $\mathbf{A}_0^\sigma$ there exists $b \in A$ with $a \leq b$ and $a' \leq b^-$.

(σ_2) Every subset of A that joins to 1 in $\mathbf{A}_0^\sigma$ has a finite subset that also join to 1.

The operations in $\mathcal{F}^\sigma = (f^\sigma : f \in \mathcal{I})$ are defined on a sequence $\underline{a}$ of atoms of $\mathbf{A}_0^\sigma$ by

$$f^\sigma(\underline{a}) = \prod\{f(\underline{b}) : \underline{a} \leq \underline{b} \in A^{\rho(f)}\}$$

and extend to all other elements of $\mathbf{A}_0^\sigma$ in a completely additive manner, i.e.,

$$f^\sigma(\underline{x}) = \sum\{f^\sigma(\underline{a}) : \underline{a} \leq \underline{x} \text{ and } a_i \in \mathbf{A}^\sigma_{0+} \ (i < \rho(f))\}.$$

Jónsson and Tarski also show that this extension preserves some key properties of $\mathbf{A}$. In particular any implication of the form $r = 0 \ \Rightarrow \ s \leq t$, where r, s, t are terms that do not involve complementation, holds in $\mathbf{A}$ if and only if it holds in $\mathbf{A}^\sigma$. Other preservation results can be deduced from what has become known as Sahlqvist's Theorem in modal logic (see Sahlqvist [**75**], Sambin and Vaccaro [**89**], Venema [**92**]). The results in Jónsson and Tarski [**51**] apply to more general classes of operations on Boolean algebras. For a recent treatment we refer the reader to Jónsson [**91**] [**92**].

From the existence of canonical extensions and the previous remarks about the duality with relational structures it follows immediately that every member of BAO_ρ can be embedded in the complex algebra of a member of RS_ρ. This result is known as the *representation theorem for Boolean algebras with operators.*

For a subvariety $\mathcal{V}$ of BAO_ρ we let $\mathcal{V}^s$ be the class of all relational structures with the property that their complex algebras are in $\mathcal{V}$. For a subclass $\mathcal{K}$ of RS_ρ we define $\mathcal{K}^a$ to be the variety generated by all complex algebras of members of $\mathcal{K}$. In symbols:

$$\begin{aligned} \mathcal{V}^s &= \{\mathbf{U} \in RS_\rho : \mathbf{U}^+ \in \mathcal{V}\} \\ \mathcal{K}^a &= \text{Var}(\{\mathbf{U}^+ : \mathbf{U} \in \mathcal{K}\}). \end{aligned}$$

The maps $\mathcal{V} \mapsto \mathcal{V}^s$ and $\mathcal{K} \mapsto \mathcal{K}^a$ establish a Galois connection between the subvarieties of BAO_ρ and the subclasses of RS_ρ. In general $\mathcal{V}^{sa} \subseteq \mathcal{V}$ and $\mathcal{K} \subseteq \mathcal{K}^{as}$. A variety $\mathcal{V}$ is said to be *complete* if $\mathcal{V}^{sa} = \mathcal{V}$.

If $\mathbf{A} \in \mathcal{V}$ implies $\mathbf{A}^\sigma \in \mathcal{V}$ then $\mathcal{V}$ is *canonical*. It follows from the representation theorem for BAOs that every canonical variety is complete.

The algebraic structure of Boolean algebras with operators. Since Boolean algebras are term-definably equivalent to Boolean rings, Boolean congruence relations are determined by their 0-congruence classes or ideals. In particular, if J is an ideal of $\mathbf{A}_0$ then the corresponding Boolean congruence relation is given by

$$x \,\theta_J\, y \qquad \text{iff} \qquad x \oplus y \in J,$$

where $x \oplus y = xy^- + x^- y$ denotes the operation of symmetric difference.

An ideal of a BAO is a *congruence ideal* if it is the 0-congruence class of some congruence relation on the algebra. The following result gives an internal characterization of congruence ideals.

Lemma 2.1 *Let $\mathbf{A} \in BAO_\rho$. For a Boolean ideal J of $\mathbf{A}_0$ the following are equivalent:*

(i) *J is a congruence ideal of $\mathbf{A}$,*

(ii) *$x \in J$ implies $f_{\underline{1},i}(x) \in J$ for all $f \in \mathcal{I}$ and $i < \rho(f)$, where $\underline{1}$ is a sequence of 1's of length $\rho(f)$.*

Proof. If (i) holds then θ_J is a congruence on $\mathbf{A}$, so for all $x \in J$, $f \in \mathcal{I}$ and $i < \rho(f)$

$$x \;\theta_J\; 0 \quad \text{implies} \quad f_{\underline{1},i}(x) \;\theta_J\; f_{\underline{1},i}(0)$$

and since $f_{\underline{1},i}(0) = 0$, it follows that $f_{\underline{1},i}(x) \in J$.

Conversely, suppose (ii) holds. We have to show that θ_J has the substitution property for all $f \in \mathcal{I}$. Since θ_J is transitive, it suffices to show that for all $f \in \mathcal{I}$, all $i < \rho(f)$ and all $\underline{a} \in A^{\rho(f)}$

$$x \;\theta_J\; y \quad \text{implies} \quad f_{\underline{a},i}(x) \;\theta_J\; f_{\underline{a},i}(y).$$

So suppose $x \oplus y = xy^- + x^- y \in J$. By the additivity of f we have

$$f_{\underline{a},i}(x) f_{\underline{a},i}(y)^- = f_{\underline{a},i}(xy) f_{\underline{a},i}(y)^- + f_{\underline{a},i}(xy^-) f_{\underline{a},i}(y)^- \le 0 + f_{\underline{1},i}(xy^-) \in J.$$

Similarly $f_{\underline{a},i}(x)^- f_{\underline{a},i}(y) \in J$, whence the result follows. □

For an element a in a BAO $\mathbf{A}$ let $Aa = \{xa : x \in A\} = \{y \in A : y \le a\}$ be the principal ideal generated by a, and let $\mathbf{A}_0 a$ be the *relativized* Boolean algebra $(Aa, +, 0, \cdot, 1, {}^{-a})$ with relative complement $x^{-a} = x^- a$. The *relativized BAO* $\mathbf{A}a$ is defined to be $(\mathbf{A}_0 a, (f_a : f \in \mathcal{I}))$ where $f_a(\underline{x}) = f(\underline{x})a$ for any sequence $\underline{x} \in (Aa)^{\rho(f)}$.

A *congruence element* in a BAO $\mathbf{A}$ is an element $a \in A$ for which the principal ideal Aa is a congruence ideal. By Lemma 2.1, a is a congruence element if and only if

$$f_{\underline{1},i}(a) \le a \quad \text{for all } f \in \mathcal{I} \text{ and } i < \rho(f).$$

Note that the map $h(x) = xa$ is a Boolean homomorphism from $\mathbf{A}_0$ to $\mathbf{A}_0 a$, with kernel Aa^-. Therefore h is a BAO homomorphism from $\mathbf{A}$ to $\mathbf{A}a$ if and only if a^- is a congruence element. A universal algebraic result about factor congruences can now be stated in this context as follows:

Theorem 2.2 *A BAO* $\mathbf{A}$ *can be decomposed into a direct product of two nontrivial factors if and only if there exists* $a \in A$ *such that* $a \notin \{0, 1\}$ *and both* a *and* a^- *are congruence elements of* $\mathbf{A}$*. In this case* $\mathbf{A} \cong \mathbf{A}a \times \mathbf{A}a^-$.

Finally we mention some useful algebraic properties that hold in all BAOs. It is well-known that congruence lattices of lattices are distributive, and since BAOs have lattice reducts, the congruence lattice of a BAO is also distributive. This allows us to invoke several results from universal algebra. Most notably, Jónsson's Lemma implies $\mathrm{Si}(\mathrm{Var}(\mathcal{K})) \subseteq \mathbb{HSP}_u(\mathcal{K})$, hence every finitely generated subvariety of BAO_ρ has finite height in the lattice Λ_{BAO_ρ}. Also, for BAOs of finite type, it follows from Baker's finite basis theorem that finitely generated varieties can be defined by finitely many equations. Another standard result from universal algebra is that, for any variety $\mathcal{V}$, the lattice $\Lambda_{\mathcal{V}}$ is dually isomorphic to the lattice of fully invariant congruences of the ω-generated free algebra in $\mathcal{V}$. Hence Λ_{BAO_ρ} is a dually algebraic, distributive lattice.

As a result of the equivalence between Boolean algebras and Boolean rings, BAOs have permutable congruences, and from the characterization of congruence ideals above, it follows that BAOs have the congruence extension property.

Residuated operators. A unary operation on a Boolean algebra $\mathbf{A}_0$ is *residuated* if there exists a *residual* operation g such that for all $x, y \in A$

$$f(x) \leq y \qquad \text{iff} \qquad x \leq g(y).$$

Equivalently f is residuated if there exists a *conjugate* operation h such that for all $x, y \in A$

$$y \cdot f(x) = 0 \qquad \text{iff} \qquad x \cdot h(y) = 0.$$

If they exist, then g and h are unique, and they are related by the formulas $h(x) = g(x^-)^-$ and $g(x) = h(x^-)^-$. An operation f is *selfconjugate* if it is equal to its conjugate.

Note that the relation 'is a conjugate of' is symmetric. This is one of the reasons why it is more convenient to consider conjugates instead of residuals. However, the concept of residuation applies more generally to posets, and is also known as Galois connection or adjunction in category theory.

The following simple but important observations about residuated operations in Boolean algebras appeared in Jónsson and Tarski [**51**].

Lemma 2.3

(i) *Every residuated operation is normal and completely additive (i.e. commutes with all existing joins).*

(ii) f *and* h *are conjugate operations on* $\mathbf{A}_0$ *if and only if they are normal and for all* $x, y \in A$

$$y \cdot f(x) \leq f(x \cdot h(y)) \qquad \textit{and} \qquad x \cdot h(y) \leq h(y \cdot f(x)).$$

Two n-ary operations f and h on $\mathbf{A}$ are *conjugate in the* i^{th} *argument* if $f_{\underline{a},i}$ is conjugate to $h_{\underline{a},i}$ for all $\underline{a} \in A^n$.

Let $\mathbf{A} = (\mathbf{A}_0, \mathcal{F}) \in BAO_\rho$. We say that $\mathbf{A}$ is a *Boolean algebra with residuated operators* (*residuated BAO* for short) if for each nonconstant $f \in \mathcal{I}$ and all $i < \rho(f)$ there exist an $\rho(f)$-ary term t which is conjugate to f in the i^{th} argument. Note that by the above lemma this can be expressed by equations in the language of BAO_ρ. So, for a fixed choice of terms $t = t_{(f,i)}$ for each $f \in \mathcal{I}$ and $i < \rho(f)$, there exists a largest subvariety of BAO_ρ where these terms are the conjugates of the operators in the respective arguments. If the terms $t_{(f,i)}$ do not involve complementation then this variety is canonical.

Relation algebras are motivating examples for studying residuated BAOs. A *relation algebra* is an algebra

$$\mathbf{A} = (\mathbf{A}_0, \circ, ^\smile, e) \in BAO_{(2,1,0)}$$

such that

(i) $(A, \circ, e)$ is a monoid,

(ii) $x^\smile$ is selfconjugate,

(iii) $x \circ y^\smile$ is conjugate to $x \circ y$ in the first argument and

(iv) $x^\smile \circ y$ is conjugate to $x \circ y$ in the second argument.

Of course this is not the original definition of Tarski [**41**], but it does show that relation algebras are included under the definition of residuated BAO. (Actually (ii) is an easy consequence of (iii), (iv) and the existence of an identity element for the relative multiplication $\circ$.) If (i) is weakened to

(i)$'$ $(A, \circ, e)$ is a groupoid with identity

then we obtain a definition for *nonassociative relation algebras*, as introduced by R. D. Maddux [**82**]. The varieties of all relation algebras and all nonassociative relation algebras are denoted by RA and NA respectively.

The classical examples of relation algebras are the *full relation algebras* $\mathbf{Re}(X) = ((X^2)^+, \circ, ^\smile, e)$ over a *base set* X. Here $\circ$ is the set theoretic relation composition, $^\smile$ gives the converse of a relation, and e is the identity relation on X. The variety generated by these algebras is the variety RRA of *representable relation algebras.*

A (*binary*) *residuated Boolean algebra*, or *r-algebra*, is a Boolean algebra with three residuated binary operations, $\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft)$, where $\triangleright$ and $\triangleleft$ are the *right* and *left* conjugates of $\circ$, which means that the conditions

$$x(y \circ z) = 0, \quad y(x \triangleleft z) = 0 \quad \text{and} \quad z(y \triangleright x) = 0$$

are equivalent. By Lemma 2.3(ii) this can be expressed by finitely many equations that do not involve complementation, hence the class of all r-algebras is a finitely based canonical subvariety of $BAO_{(2,2,2)}$.

A *unital* r-algebra, or *ur-algebra*, is of the form $(\mathbf{A}_0, \circ, \triangleright, \triangleleft, e)$, where e is a unit element with respect to $\circ$ (i.e. $e \circ x = x \circ e = x$ for all $x \in A$). A *residuated Boolean monoid*, or *rm-algebra*, is a ur-algebra in which the operation $\circ$ is associative.

The *right* and *left* residuals $\backslash$ and $/$ of $\circ$ are defined by $x\backslash y = (x \triangleright y^-)^-$ and $x/y = (x^- \triangleleft y)^-$. They satisfy the requirements of residuals in the sense that

$$x \circ y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y.$$

When writing r-algebra formulas, we will use the convention that unary operations have the highest priority, followed by the Boolean meet $(\cdot)$, then the r-operations $(\circ, \triangleright, \triangleleft, \backslash, /)$ and then Boolean join $(+)$. When confusion is unlikely, $x{\cdot}y$ will be written as xy. The notation x^n is defined inductively by

$$x^0 = e \quad \text{and} \quad x^n = x^{n-1} \circ x \quad \text{for } n > 0.$$

A motivation for studying r-algebras is that they are natural generalizations of relation algebras: for any relation algebra $\mathbf{A} = (\mathbf{A}_0, \circ, ^{\smile}, e)$ we obtain an rm-algebra $\mathbf{A}' = (\mathbf{A}_0, \circ, \triangleright, \triangleleft, e)$ if we define $x \triangleright y = x^{\smile} \circ y$ and $x \triangleleft y = x \circ y^{\smile}$. If we instead use a nonassociative relation algebra then we obtain a ur-algebra. Conversely it is shown in Jónsson and Tsinakis [**a**] that if a ur-algebra satisfies the equations $x \triangleright y = (x \triangleright e) \circ y$ and $x \triangleleft y = x \circ (y \triangleright e)$ then it is (term-definably equivalent to) a nonassociative relation algebra, with $x^{\smile} = x \triangleright e$. This allows us to consider NA and RA as subvarieties of the variety of all ur-algebras (denoted by UR). We therefore also use the symbols $\circ$ and e instead of the more traditional ; and 1' for the relative multiplication and identity element of relation algebra.

Other areas where r-algebras and ur-algebras occur are in the theory of automata as complex algebras over monoids, in axiomatic treatments of the betweeness relation in geometry, in the algebraic study of relevance logics (with classical negation), and in the study of complex algebras of ternary relational structures in general. Residuated BAOs have also been studied in a more category theoretical setting in Ghilardi and Meloni [**90**].

Another widely studied class of residuated BAOs are *tense algebras*, defined as Boolean algebras with two unary operators that are conjugates of each other. In the third section of Chapter III we show that reflexive tense algebras are term definably equivalent to certain r-algebras. However we do not discuss tense algebras in general.

We now look at how r-algebras can be obtained from ternary relational structures. Let $\mathbf{U} = (U, R) \in RS_{(3)}$, and note that the definition of the complex algebra $\mathbf{U}^+ = (U^+, R^+)$ treats the last coordinate of R-tuples in a special way. We can just as well define two other operations

$$\begin{aligned} R_1^+(X, Z) &= \{y \in U : (x, y, z) \in R \text{ for some } x \in X, z \in Z\} \quad \text{and} \\ R_0^+(Z, Y) &= \{x \in U : (x, y, z) \in R \text{ for some } y \in Y, z \in Z\}. \end{aligned}$$

The *residuated complex algebra* of $\mathbf{U}$, denoted by $\mathbf{U}^{\oplus}$, is defined as (U^+, R^+, R_1^+, R_0^+) and, when it is clear from the context which relation R we are working with, then we denote R^+, R_1^+, R_0^+ by $\circ, \triangleright, \triangleleft$ respectively. Clearly $\mathbf{U}^{\oplus}$ is a residuated BAO and thus an r-algebra. The concept of a residuated complex algebra can also be defined for relational structures of arbitrary type, but this is not needed here.

From the representation theorem for BAOs it follows that every r-algebra can be embedded in the residuated complex algebra of a ternary relational structure.

The discriminator. The theory of discriminator algebras and varieties has been investigated extensively, and provides us with a wealth of information and techniques applicable to specific examples of such algebras and varieties. Although we will concern ourselves only with BAOs, the concept of a discriminator applies to algebras in general. Therefore we define it first in that context and then show how it simplifies for BAOs.

A *discriminator algebra* is a nontrivial algebra $\mathbf{A}$ for which there exists a ternary term t (in the language of $\mathbf{A}$), called a *discriminator term*, such that for all $x, y, z \in A$

$$t^{\mathbf{A}}(x,x,z) = z \qquad \text{and} \qquad t^{\mathbf{A}}(x,y,z) = x \quad \text{if } x \neq y.$$

The most striking consequence of the existence of such a term is that $\mathbf{A}$ must be *simple*, i.e., $\mathbf{A}$ admits only two congruences, namely the identity relation and the universal relation. This is because any congruence θ other than the identity relation identifies at least two distinct elements of $\mathbf{A}$, say $a \neq b$, so by the substitution property

$$a = t^{\mathbf{A}}(a,b,c) \; \theta \; t^{\mathbf{A}}(a,a,c) = c$$

for any element $c \in A$, and now it follows from transitivity that θ must be the universal relation.

In general it is not true that every simple algebra is a discriminator algebra, though we will show below that for atomic r-algebras (in fact for Boolean algebras with at least one atom and finitely many residuated operators) this is indeed the case.

A *discriminator variety* is a variety generated by a class of (similar) algebras which are discriminator algebras with respect to the same term t. Discriminator varieties have nice structural properties. Here we list three of those that we use frequently.

(I) The concepts of an algebra being simple, subdirectly irreducible or directly indecomposable are equivalent in discriminator varieties.

(II) For a variety $\mathcal{V}$ and a term t in the language of $\mathcal{V}$, the statement 't is a discriminator term in the subdirectly irreducible members of $\mathcal{V}$' can be characterized by equations.

(III) In a discriminator variety $\mathcal{V}$ any universal sentence can be translated into an equation such that the sentence holds in all simple members of $\mathcal{V}$ if and only if the corresponding equation holds in $\mathcal{V}$.

A good survey of discriminator algebras in general can be found in the monograph of Werner [**78**], and more recent results relevant to algebraic logic are contained in Blok and Pigozzi [**89**]. Here we only note that for an algebra $\mathbf{A}$ that has a Boolean algebra reduct $\mathbf{A}_0 = (A, +, 0, \cdot, 1, ^-)$, $\mathbf{A}$ is a discriminator algebra if and only if there exists a unary term c, called a *unary discriminator* such that

$$c^{\mathbf{A}}(0) = 0 \quad \text{and} \quad c^{\mathbf{A}}(x) = 1 \text{ if } x \neq 0.$$

This follows from the observation that in a Boolean algebra c and t are interdefinable:

$$c(x) = t(0,x,1)^- \quad \text{and} \quad t(x,y,z) = x{\cdot}c(x \oplus y) + z{\cdot}c(x \oplus y)^-.$$

We use the symbol c because $c^{\mathbf{A}}$ is a monadic closure operator on $\mathbf{A}_0$. For example any simple relation algebra $(\mathbf{A}_0, \circ, ^{\smile}, e)$ is a discriminator algebra with unary discriminator term $c(x) = 1 \circ x \circ 1$. In the first section of Chapter III we address the question which generalizations of relation algebras are discriminator algebras.

We now recall an explicit description (for BAOs) of the translation from universal sentences to equations, mentioned in (III) above. Let $\mathcal{V}$ be a discriminator variety of BAOs with unary discriminator $c(x)$ and let σ be a universal sentence in the language of $\mathcal{V}$. Equivalently we can view σ as a universally quantified open formula and we may assume that it is built up from atomic formulas (i.e. equations of terms) using only conjunction and negation. Steps (A)-(C) below inductively define a term σ^* of $\mathcal{V}$ such that

$$\mathcal{V} \models \sigma^* = 1 \quad \text{if and only if} \quad \mathrm{Si}(\mathcal{V}) \models \sigma.$$

(A) If σ is an atomic formula $s = t$, let $\sigma^* = (s \oplus t)^-$,

(B) if σ is a conjunction of two open formulas φ and ψ, let $\sigma^* = \varphi^* \cdot \psi^*$ and

(C) if σ is the negation of an open formula φ, let $\sigma^* = c(\varphi^{*-})$.

The following theorem is a reformulation for BAOs of R. N. McKenzie's characterization of discriminator varieties (cf. (II) above, McKenzie [**75**]).

Theorem 2.4 *Let $\mathcal{V}$ be a subvariety of BAO_ρ and let c be a unary term of $\mathcal{V}$. The following are equivalent:*

(i) *c is a unary discriminator in all subdirectly irreducible members of $\mathcal{V}$*

(ii) *$\mathcal{V}$ satisfies the equations $c(0) = 0$, $x \leq c(x)$,*

$$f_{\underline{1},i}(c(x)) \leq c(x) \quad \text{and} \quad f_{\underline{1},i}(c(x)^-) \leq c(x)^- \tag{1}$$

for each operator $f \in \mathcal{I}$ and $i < \rho(f)$.

Proof. (i) $\Rightarrow$ (ii) By definition $c^{\mathbf{A}}(0) = 0$, $c^{\mathbf{A}}(x) = 1$ if $x \neq 0$, and $f^{\mathbf{A}}_{\underline{1},i}(0) = 0$ for each $f \in \mathcal{I}$ and $i < \rho(f)$, so the equations hold in $\mathcal{V}$.

(ii) $\Rightarrow$ (i) Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathcal{V}$. By Lemma 2.1 the equations (1) imply that $c^{\mathbf{A}}(x)$ and $c^{\mathbf{A}}(x)^-$ are congruence elements for every $x \in A$. Since any subdirectly irreducible member of $\mathcal{V}$ is directly indecomposible, Theorem 2.2 implies that $c^{\mathbf{A}}(x)$ is either 0 or 1. For $x \neq 0$ the equation $x \leq c(x)$ implies that $c^{\mathbf{A}}(x) \neq 0$, so $c^{\mathbf{A}}(x) = 1$. □

Observe that in a residuated BAO the complement of a congruence element is also a congruence element, since if t is a conjugate term for $f \in \mathcal{I}$ in the i^{th} argument then $t_{\underline{1},i}$ is normal and hence $t_{\underline{1},i}(a) \leq a$ for any congruence element a, which is equivalent to $t_{\underline{1},i}(a)a^- = 0$, $f_{\underline{1},i}(a^-)a = 0$ and finally $f_{\underline{1},i}(a^-) \leq a^-$. Therefore half the equations (1) are redundant in this case. Also, the existence of a nontrivial congruence element in a residuated BAO implies that the algebra is decomposable. Since every ideal in a finite Boolean algebra is necessarily principal, it follows that for finite residuated BAOs the properties of being indecomposable, subdirectly irreducible and simple are equivalent.

For a BAO $\mathbf{A} = (\mathbf{A}_0, \mathcal{F})$ of finite type (i.e. $\mathcal{I}$ finite), we define a term τ by

$$\tau(x) = \sum\{f_{\underline{1},i}(x) : f \in \mathcal{I}, 0 < \rho(f), i < \rho(f)\}.$$

So for example in a *ur*-algebra $\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft, e)$

$$\tau^{\mathbf{A}}(x) = 1 \circ x + x \circ 1 + 1 \triangleright x + x \triangleleft 1 + 1 \triangleleft x + x \triangleright 1.$$

It now follows from Lemma 2.1 that $\mathbf{A}$ and the modal algebra $(\mathbf{A}_0, \tau^A)$ have identical congruence lattices.

With this notation we can also summarize the equations (1) as

$$\tau(c(x)) \leq c(x) \quad \text{and} \quad \tau(c(x)^-) \leq c(x)^-.$$

Note that for r-algebras $\tau(x)$ is selfconjugate and so the two equations are equivalent.

Theorem 2.5 *Let $\mathbf{A} \in BAO_\rho$ be of finite type. If $\mathbf{A}$ is simple and contains at least one atom, and if $\tau(x)$ is selfconjugate then $\mathbf{A}$ is a discriminator algebra with unary discriminator $c(x) = \tau^m(x)$ for some $m \in \omega$.*

Proof. For any atom $a \in A$, the congruence ideal generated by a is the join of all principal ideals $A\tau^m(a)$, $m \in \omega$. If $\mathbf{A}$ is simple, then this join must be A, which is a compact congruence ideal of $\mathbf{A}$. Therefore there exists $m_a \in \omega$ such that $\tau^{m_a}(a) = 1$.

Now for any nonzero $x \in A$, $\tau^{m_a}(a)x \neq 0$, hence $\tau^{m_a}(x)a \neq 0$ and $a \leq \tau^{m_a}(x)$, since τ is selfconjugate and a is an atom. Consequently $1 = \tau^{m_a}(a) \leq \tau^{2m_a}(x)$ for all nonzero $x \in A$ and therefore $c(x) = \tau^{2m_a}(x)$ is a unary discriminator. $\square$

In light of the remarks after Theorem 2.4 we also have the following result.

Corollary 2.6 *Every finite subdirectly irreducible residuated BAO of finite type is a discriminator algebra with unary discriminator $c(x) = \tau^m(x)$ for some $m \in \omega$.*

General Results

Normal forms and decidability. We now take a closer look at the notion of a normal form for BAOs and its relationship to decidability. For a precise definition of an algorithm or a computable function, or the closely related concept of partial recursive function, we refer the reader to Davis [**58**] or Rogers [**67**]. For our purpose it will suffice to think of a computable function as a function f that can be implemented as a computer program such that for each input x the program computes the output $f(x)$ in finitely many steps. A set X is *decidable* if the characteristic function

$$\chi(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

is computable.

A variety $\mathcal{V}$ has a *normal form* if there is a computable function that maps every term $t \in T_\rho$ to a corresponding normal form term $\tilde{t} \in T_\rho$ such that for all $s, t \in T_\rho$,

$$s = t \in \mathrm{Eq}(\mathcal{V}) \quad \text{iff} \quad \tilde{s} = \tilde{t}.$$

Here the equality symbol is used in two different ways. On the left it denotes a formal equality, and we could just as well have written $(s, t) \in \mathrm{Eq}(\mathcal{V})$. On the right it denotes syntactic equality, i.e., $\tilde{s}$ and $\tilde{t}$ must be the same term. For terms in the language of BAOs we usually weaken this notion of syntactic equality to equality modulo the Boolean algebra axioms. This is permissible since Boolean algebras terms have a normal form.

We say $\mathcal{V}$ has a *decidable equational theory* or, more compactly, $\mathcal{V}$ is *decidable* if $\mathrm{Eq}(\mathcal{V})$ is a decidable subset of $T_\rho \times T_\rho$. Clearly every variety that has a normal form is decidable. The converse is not true in general, but we show below that it does hold for varieties of BAOs with finitely many operators.

For the rest of this section we consider the variety BAO_ρ with operator symbols from a *finite* set $\mathcal{I}$. Let X be a fixed finite set of variables and for $x \in X$ and $s, t, t_0, \ldots, t_{n-1} \in T_\rho$ define the degree of a term by

$$\deg(x) = 0, \qquad \deg(t^-) = \deg(t), \qquad \deg(s + t) = \max\{\deg(s), \deg(t)\}$$

$$\deg(f(t_0, \ldots, t_{n-1})) = 1 + \max\{\deg(t_0), \ldots, \deg(t_{n-1})\}.$$

$T_n(X)$ denotes the set of terms of degree n. The set $R_n(X)$ of *reduced terms of degree* n is defined by

$$R_0(X) = \mathbf{F}_{BA}(X)_+, \qquad R_n(X) = \mathbf{F}_{BA}(X \cup \mathcal{I}(R_{n-1}(X)))_+$$

where $\mathbf{F}_{BA}(X)_+$ is the set of atoms of the free Boolean algebra over X and, for a set of terms R,

$$\mathcal{I}(R) = \{f(t_0, \ldots, t_{n-1}) : f \in \mathcal{I} \text{ and } t_0, \ldots, t_{n-1} \in R\}.$$

Finally, let $R(X) = \bigcup_{n=0}^{\infty} R_n(X)$. Strictly speaking the elements of $R(X)$ are equivalence classes of terms modulo the Boolean algebra axioms, but since these axioms are well understood we will not distinguish between a class and any representative member. The following two results were proved for modal algebras in Fine [**75**], and the proofs hold for BAOs in general. Together they give a normal form for BAO_ρ.

Theorem 2.7 *Any term $t \in T_\rho(X)$ of degree $\leq n$ is equivalent in BAO_ρ to a (possibly empty) disjunction of reduced terms of degree n.*

Proof. We will consider an empty disjunction to be equivalent to 0. The proof proceeds by induction on the degree of t. If $\deg(t) = 0$ then t is a Boolean term, hence by the disjunctive normal form theorem for Boolean algebras, t is a disjunction of terms in $R_0(X)$.

Suppose now $\deg(t) = n > 0$. Then t is a Boolean combination of variables and terms $f(t_0, \ldots, t_{m-1})$ for some $t_i \in T_\rho(X)$ with $\deg(t_i) < n$, $f \in \mathcal{I}$, $(i < m = \rho(f))$. By the induction hypothesis each t_i is equivalent to a disjunction of terms in $R_{n-1}(X)$. Since the operators are normal and additive, $f(t_0, \ldots, t_{m-1})$ is equivalent to a disjunction of terms $f(s_0, \ldots, s_{m-1})$, where each $s_i \in R_{n-1}(X)$. Hence t is equivalent to a disjunction of terms from $X \cup \mathcal{I}(R_{n-1}(X))$ and therefore equivalent to a disjunction of terms in $R_n(X)$. $\square$

The (possibly empty) disjunction of reduced terms equivalent to t will be referred to as the disjunctive normal form, and is denoted by $\tilde{t}$. Let $\mathcal{V} = BAO_\rho$. For any $t \in T_\rho(X)$, $t = \tilde{t}$ is an identity of $\mathcal{V}$, so

$$\tilde{s} = \tilde{t} \quad \text{implies} \quad s = t \in \mathrm{Eq}(\mathcal{V}).$$

We still have to prove the reverse implication. Since $s = t$ is equivalent to $s \oplus t = 0$, and since $\tilde{0} = 0$, it suffices to show that if $t = 0 \in \mathrm{Eq}(\mathcal{V})$ then $\tilde{t} = 0$. Equivalently, it is sufficient to show that for all $t \in R_n(X)$, $t = 0$ is not an equation of $\mathcal{V}$. This is done by defining an algebra $\mathbf{A}_n \in \mathcal{V}$ in which the equation $t = 0$ fails. Let the set of atoms of $\mathbf{A}_n$ be $\bigcup_{i=0}^n R_i(X)$ and define the operators on $\mathbf{A}_{n+}$ by

$$f^{\mathbf{A}_n}(t_0, \ldots, t_{m-1}) = \sum\{s \in \mathbf{A}_{n+} : f(t_0, \ldots, t_{m-1}) \text{ is a conjugant of } s\},$$

and extend them additively to $\mathbf{A}_n$. Let $h : T_\rho(X) \to \mathbf{A}_n$ be the homomorphism that extends the assignment

$$h(x) = \sum\{s \in \mathbf{A}_{n+} : x \text{ is a conjugant of } s\}.$$

Theorem 2.8 *For any $t \in \mathbf{A}_{n+}$ we have $h(t) \geq t$ (in $\mathbf{A}_n$) and hence the equation $t = 0$ fails in $\mathbf{A}_n$.*

Proof. We show, by induction on the degree of t, that for any conjugant r of t, $h(r) \geq t$, whence the result follows. If $\deg(t) = 0$ then any conjugant is of the form x or x^- for some $x \in X$. In the first case we have $h(x) \geq t$ by definition, and in the second case x is not a conjugant of t, so $h(x) \not\geq t$ and therefore $h(x^-) \geq t$.

Now suppose $\deg(t) = n > 0$ and let r be a conjugant if t. If r is either x or x^- we proceed as before. If r is of the form $f(t_0, \ldots, t_{m-1})$ for some $f \in \mathcal{I}$ and $t_i \in R_{n-1}(X)$ ($i < m = \rho(f)$), then the induction hypothesis implies $h(t_i) \geq t_i$, hence

$$h(r) = f^{\mathbf{A}_n}(h(t_0), \ldots, h(t_{m-1})) \geq f^{\mathbf{A}_n}(t_0, \ldots, t_{m-1}) \geq t.$$

Finally, suppose r is of the form $f(t_0, \ldots, t_{m-1})^-$ for some $f \in \mathcal{I}$ and $t_i \in R_{n-1}(X)$ ($i < m = \rho(f)$). We want to conclude that $t \leq h(r) = f^{\mathbf{A}_n}(h(t_0), \ldots, h(t_{m-1}))^-$. Suppose to the contrary that the atom t is below $h(r)^- = f^{\mathbf{A}_n}(h(t_0), \ldots, h(t_{m-1}))$. Then there exist $s_i \in \mathbf{A}_{n+}$ with $s_i \leq h(t_i)$ and $t \leq f^{\mathbf{A}_n}(s_0, \ldots, s_{m-1})$. Note that $f(s_0, \ldots, s_{m-1})$ is a conjugant of t, while $f(t_0, \ldots, t_{m-1})$ is not. Therefore $s_i \neq t_i$ for some $i < m$, so s_i, t_i disagree for some conjugant, say q is a conjugant of s_i and q^- is a conjugant of t_i. Then $h(s_i) \leq h(q)$ and $h(t_i) \leq h(q)^-$. Since $\deg(s_i) < n$ we also have $s_i \leq h(s_i)$. But then $s_i \not\leq h(t_i)$, contradicting the choice of the s_i. $\square$

So $\tilde{t}$ is indeed a normal form of t in $\mathcal{V}$. Note that while this disjunctive normal form is useful for proving syntactical results about BAOs, it does not have much computational value, since the length of $\tilde{t}$ is a doubly exponential function of the degree of t and the size of X and $\mathcal{I}$. For example the term $xf(y)$ has a disjunctive normal form

$$xyf(xy)f(x^-y)f(xy^-)f(x^-y^-) + 23 \text{ similar terms.}$$

Of course we could contract it back to $xf(y)$, but even for Boolean algebras not every normal form has a unique shortest representative (e.g. Quine [**59**] shows that $xy^- + x^-y + yz^- + y^-z$

is equivalent to two minimal length terms $x^-y+xz^-+y^-z$ and $xy^-+x^-z+yz^-$). In practice we write terms and equations in whatever form is most convenient at the time. Naturally we prefer to write $x \le f(x)$ rather than $xf(x)^- = 0$ or $xf(x)^-f(x^-) + xf(x)^-f(x^-)^- = 0$.

Corollary 2.9 *Let BAO_ρ be of finite type. Then any decidable subvariety of BAO_ρ has a normal form.*

Proof. Let $\mathcal{V}$ be a decidable subvariety of BAO_ρ. Any term t is equivalent to its disjunctive normal form $\tilde{t}$. To obtain a normal form $\hat{t}$ for $\mathcal{V}$, we simply delete the disjuncts s of $\tilde{t}$ for which the equation $s = 0$ holds in $\mathcal{V}$. Since $\mathcal{V}$ is assumed to be decidable, $\hat{t}$ is computable from $\tilde{t}$. □

The next theorem shows that with the help of some extra variables we can 'flatten out' any equation by rewriting it as a collection of implications of equations with degree ≤ 1. We first give a specific example to illustrate the idea.

Lemma 2.10 *In $BAO_{(2)}$ the following formulas are equivalent:*

(i) $(x \circ y) \circ z \le x \circ (y \circ z)$

(ii) $u \le v \circ z$ *and* $v \le x \circ y$ *and* $y \circ z \le w$ *imply* $u \le x \circ w$

(iii) $u(x \circ w) = 0$ *and* $y \circ z \le w$ *and* $v \le x \circ y$ *imply* $u(v \circ z) = 0$

Proof. (i)⇒(ii) If $u \le v \circ z$, $v \le x \circ y$ and $y \circ z \le w$ then

$$u \le (x \circ y) \circ z \le x \circ (y \circ z) \le x \circ w$$

where the middle inequality follows from (i).

(ii)⇒(i) Define $u = (x \circ y) \circ z$, $v = x \circ y$ and $w = y \circ z$. Then the assumptions of (ii) are satisfied, hence $u \le x \circ w = x \circ (y \circ z)$.

(i)⇔(iii) is similar. □

Theorem 2.11 *Let BAO_ρ be of finite type. For any reduced term $r \in R(X)$ the equation $r = 0$ is equivalent in BAO_ρ to an implication $\bigwedge_{i=0}^{k} s_i = 0 \Rightarrow t = 0$ where each $s_i \in R_1(X \cup Y)$, Y is a finite set of additional variables and t is of the form $xf(y_0, \ldots, y_{n-1})$ or $xf(y_0, \ldots, y_{n-1})^-$ for some $x, y_i \in Y$ and $f \in \mathcal{I}$.*

Proof. Suppose r has degree ≥ 1. First assume we can write r in the form $r'f(r_0, \ldots, r_{n-1})$. If x is a variable that does not occur in r then $r = 0$ is clearly equivalent to

$$x \le r' \quad \Rightarrow \quad xf(r_0, \ldots, r_{n-1}) = 0. \tag{2}$$

Let $y_0, \ldots, y_{n-1}$ be a sequence of variables not in r. Then the formula

$$x \le r' \quad \wedge \quad \bigwedge_{i=1}^{n} y_i \le r_i \quad \Rightarrow \quad t = 0,$$

where $t = xf(y_0, \ldots, y_{n-1})$, is equivalent to (2) since f is isotone. Similarly if we write r in the form $r'f(r_0, \ldots, r_{n-1})^-$ then $r = 0$ is equivalent to

$$x \leq r' \quad \wedge \quad \bigwedge_{i=0}^{n} r_i \leq y_i \quad \Rightarrow \quad t = 0,$$

where $t = xf(y_0, \ldots, y_{n-1})^-$. Now we rewrite the equations to the left of the implication in normal form, i.e., as a conjunction of equations of the form $q = 0$, $q \in R_n(X \cup \{x, y_0, \ldots, y_{n-1}\})$. To reduce the degree of the terms q we do a similar substitution as above: For conjugants $f(\underline{t})$ and $f'(\underline{t}')^-$ of q we again introduce sequences of new variables $\underline{x}$, $\underline{z}$ and replace $q = q'f(\underline{t})f'(\underline{t}')^- = 0$ by

$$q'f(\underline{x})f'(\underline{z})^- = 0 \quad \wedge \quad \bigwedge_{i=0}^{n} t_i \leq x_i \quad \wedge \quad \bigwedge_{i=0}^{n'} z_i \leq t'_i.$$

Performing this step repeatedly on all (nontrivial) conjugants of q and for all terms in the formula, we obtain the desired result. □

Varieties that are generated by their finite members. We now give some general sufficient conditions under which the set of equations that hold in a variety is the same as the set of equations that hold in all finite members of the variety. This situation is of interest because of the following well-known result.

Theorem 2.12 *If a variety $\mathcal{V}$ of algebras is finitely based and generated by its finite members then $\mathcal{V}$ has a decidable equational theory.*

A heuristic argument is based on the observation that one can effectively enumerate all finite members of $\mathcal{V}$ (up to isomorphism) and all equations provable from the finite basis of $\mathcal{V}$. Given an equation ε, one checks these two lists in turn until one discovers a finite algebra in which ε fails or a proof of ε from the basis. Since $\mathcal{V}$ is generated by its finite members, one of these two alternatives must occur after finitely many steps. In practise this is not a viable decision procedure, but in the next section we show how the enumeration can done somewhat more efficiently for Boolean algebras with operators.

Returning to the question when varieties are generated by their finite members, it is clear that a variety $\mathcal{V} \subseteq BAO_\rho$ has this property if and only if every equation that fails in some member of $\mathrm{Si}(\mathcal{V})$, fails in some finite member of $\mathcal{V}$. Given an equation ε, and an algebra $\mathbf{A} \in \mathrm{Si}(\mathcal{V})$ in which ε fails, we let $a_0, a_1, \ldots, a_{n-1}$ be elements of $\mathbf{A}$ corresponding to the values of all subterms of ε under some assignment for which ε fails. Suppose we can find a finite Boolean subalgebra $\mathbf{B}_0$ of $\mathbf{A}_0$ that contains $a_0, a_1, \ldots, a_{n-1}$, and suppose further that we can induce operations $f^{\mathbf{B}}$ on $\mathbf{B}_0$ such that $\mathbf{B} = (\mathbf{B}_0, (f^{\mathbf{B}} : f \in \mathcal{I})) \in \mathcal{V}$ and for all $\underline{b} \in B$

$$f^{\mathbf{A}}(\underline{b}) \in B \quad \text{implies} \quad f^{\mathbf{B}}(\underline{b}) = f^{\mathbf{A}}(\underline{b}).$$

Then ε will fail in $\mathbf{B}$ when evaluated under the same assignment.

The Boolean algebra $\mathbf{B}_0$ can be chosen to be any finite subalgebra between $\mathbf{Sg}^{\mathbf{A}_0}(a_0, \ldots, a_{n-1})$ and $\mathbf{A}_0$, but we have to find uniform ways of inducing the operations $f^{\mathbf{B}}$ on $\mathbf{B}_0$ such that the algebra $\mathbf{B}$ inherits desirable properties of $\mathbf{A}$.

This approach was first taken by J. C. C. McKinsey for closure algebras. In modal logic a closely related technique is referred to as a filtration. For a unary closure operator f, McKinsey defined

$$f^{\mathbf{B}}(b) = \prod\{f^{\mathbf{A}}(c) : b \le c \in B \text{ and } f^{\mathbf{A}} \in B\}$$

which is again a closure operator. However, if $f^{\mathbf{A}}$ is residuated, this is not necessarily the case for $f^{\mathbf{B}}$. We wish to preserve residuation, so we consider first the following more abstract setting.

Let $\mathbf{A}_0$ and $\mathbf{B}_0$ be Boolean algebras, and suppose $h : \mathbf{B}_0 \to \mathbf{A}_0$ is a conjugated Boolean homomorphism. That means there exists a (unique) map $h^c : \mathbf{A}_0 \to \mathbf{B}_0$ such that $ah(b) = 0$ if and only if $bh^c(a) = 0$. Then $h^c(a)$ is calculated in $\mathbf{B}_0$ by

$$h^c(a) = \prod\{b^- \in B : ah(b) = 0\} = \prod\{b \in B : a \le h(b)\}.$$

Note that if $\mathbf{B}_0$ is finite then these meets always exist, hence all maps from finite Boolean algebras are conjugated. For an n-ary operation $f^{\mathbf{A}}$ on A, we define f^β on B by

$$f^\beta(b_0, \ldots, b_{n-1}) = h^c(f^{\mathbf{A}}(h(b_0), \ldots, h(b_{n-1})))$$

$$\text{and} \quad \mathbf{B}^\beta = (\mathbf{B}_0, (f^\beta : f \in \mathcal{I})).$$

We are interested in the case when $\mathbf{B}_0$ is a subalgebra of $\mathbf{A}_0$ and h is the inclusion map. In this situation we usually do not refer to h explicitly, but we still need a name for its conjugate h^c. In lattice theory this map is denoted by β_h so, when the need arises, we will use β for the conjugate of the inclusion map.

Lemma 2.13 *Let $\mathbf{A} \in BAO_\rho$ and let $\mathbf{B}_0$ be a finite Boolean subalgebra of $\mathbf{A}_0$. Then for all $f \in \mathcal{I}$ and $\underline{b} \in B^{\rho(f)}$*

(i) $f^{\mathbf{A}}(\underline{b}) \le f^\beta(\underline{b})$,

(ii) $f^{\mathbf{A}}(\underline{b}) \in B$ *implies* $f^\beta(\underline{b}) = f^{\mathbf{A}}(\underline{b})$,

(iii) $\mathbf{B}^\beta \in BAO_\rho$ *and*

(iv) *if f and g are two n-ary operations that are conjugate in the i^{th} argument then f^β and g^β are also conjugate in the i^{th} argument.*

Proof. (i) and (ii) follow from the characterization of the conjugate map above, and (iii) holds because conjugates are additive and normal. To prove (iv) we note that the composition of conjugated maps is again conjugated, i.e., the following statements are equivalent:

$$\begin{aligned}
&0 = xf^\beta_{\underline{b},i}(y)\\
&0 = xh^c(f^{\mathbf{A}}(h(b_0), \ldots, h(y), \ldots, h(b_{n-1})))\\
&0 = h(x)f^{\mathbf{A}}(h(b_0), \ldots, h(y), \ldots, h(b_{n-1}))\\
&0 = h(y)g^{\mathbf{A}}(h(b_0), \ldots, h(x), \ldots, h(b_{n-1}))\\
&0 = xh^c(g^{\mathbf{A}}(h(b_0), \ldots, h(x), \ldots, h(b_{n-1})))\\
&0 = yg^\beta_{\underline{b},i}(x)
\end{aligned}$$

□

We say that a subvariety $\mathcal{V}$ of BAO_ρ is *β-closed* if for every $\mathbf{A} \in \mathrm{Si}(\mathcal{V})$ and every finite set $S \subseteq A$ there is a finite Boolean subalgebra $\mathbf{B}_0$ of $\mathbf{A}_0$ such that $S \subseteq B$ and $\mathbf{B}^\beta \in \mathcal{V}$. The variety $\mathcal{V}$ is *strongly β-closed* if for every $\mathbf{A} \in \mathrm{Si}(\mathcal{V})$ and every finite subset S of A containing the constants of $\mathbf{A}$, the Boolean algebra $\mathbf{B}_0 = \mathbf{Sg}^{\mathbf{A}_0}(S)$ satisfies $\mathbf{B}^\beta \in \mathcal{V}$. Note that both the variety of all r-algebras and the variety of all ur-algebras are strongly β-closed. From the above discussion we now get the following result.

Theorem 2.14 *Let $\mathcal{V}$ be a subvariety of BAO_ρ. If $\mathcal{V}$ is β-closed then it is generated by its finite members.*

A model-construction/theorem-proving algorithm

In this section we describe an algorithm useful to construct BAOs that satisfy a set of universal first-order sentences. In some cases the algorithm also functions as a theorem-prover by showing that the given set of sentences is unsatisfiable.

Methods to algorithmically decide equations or first-order sentences have been studied extensively in logic. For relation algebraic equations R. D. Maddux [**83**] and E. Orlowska [**91**] give sequent calculi and semantic tableau methods. The algorithm described here developed out of joint research with E. Lukács. It does not analyse equations syntactically, but rather in the way they impose restrictions on their models. This allows us to adapt the algorithm easily to different equational (and also universal) theories of BAOs.

Let $\mathcal{L} = \{+, 0, \cdot, 1, ^-\} \cup \mathcal{I}$ be the language of BAO_ρ. For a finite Boolean algebra $\mathbf{A} = (A, +, 0, \cdot, 1, ^-)$,

- $\mathcal{L}_A$ denotes the expansion of $\mathcal{L}$ with all elements of A as constants,
- $\mathcal{E}_\mathbf{A}$ denotes the set of sentences which asserts that any model $\mathbf{B}$ of $\mathcal{E}_\mathbf{A}$ is a Boolean algebra with $\mathbf{A}$ a subalgebra of $\mathbf{B}$, and
- $\mathcal{E}$ denotes a set of equations of the form $t = 0$, where t is a term in the language $\mathcal{L}_A$.

We will also assume that in all models of $\mathcal{E} \cup \mathcal{E}_\mathbf{A}$ the operation symbols $f \in \mathcal{I}$ denote operations isotone in each argument, i.e., $\mathcal{E}$ contains or implies equations such as

$$f(x_1y_1, \ldots, x_ny_n)\cdot f(x_1, \ldots, x_n)^- = 0$$

for each $f \in \mathcal{I}$. For all applications we have in mind, $\mathcal{E} \cup \mathcal{E}_\mathbf{A}$ contains an equational basis of the variety BAO_ρ, so this condition is certainly satisfied. In most of these applications $\mathcal{E}$ also contains universal sentences in the language of $\mathcal{L}_A$, but to keep things simple we initially restrict ourselves to equations.

Now consider the set $\mathcal{C}_A$ of all atomic formulas of the form

$$(i) \qquad a \leq f(a_0, \ldots, a_{n-1})$$

$$(e) \qquad a\cdot f(a_0, \ldots, a_{n-1}) = 0$$

where $f \in \mathcal{I}$, $a, a_0, \ldots, a_{n-1} \in A$ and $n = \rho(f)$, referred to as *inclusion* and *exclusion* formulas respectively.

The problem we consider is the following: Given a subset $\mathcal{C}$ of $\mathcal{C}_A$, find a model of $\mathcal{D} = \mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ or show that no such model exists, i.e., show that $\mathcal{D}$ is inconsistent.

This problem is closely related to the embedding problem for partial algebras (cf. Evans [**80**]), since we may view the formulas in $\mathcal{C}$ as defining partial operations on the Boolean algebra $\mathbf{A}$, and we are looking for an extension in BAO_ρ that is a model of $\mathcal{E}$.

In general this is easily seen to be an undecidable problem since it is possible to reduce equational decision problems to this form. However there is a simple effective procedure that uses the information in $\mathcal{C}$ and the equations in $\mathcal{E}$ to derive further inclusion or exclusion formulas. If there exists a finite model of $\mathcal{D}$ then this procedure will find one of least cardinality. In particular this algorithm provides an effective decision procedure in varieties that are generated by there finite members. However, if $\mathcal{D}$ has only infinite models, then the algorithm will not terminate. We begin with some definitions and a few simple lemmas that are used later on to describe the algorithm.

For a given set $\mathcal{C} \subseteq \mathcal{C}_A$ and for each $f \in \mathcal{I}$ define two operations $f^\lambda_{\mathcal{C}}$ and $f^\mu_{\mathcal{C}}$ on the Boolean algebra $\mathbf{A}$ by

$$f^\lambda_{\mathcal{C}}(\underline{b}) = \sum\{a : (a \le f(\underline{b})) \in \mathcal{C}\}$$
$$f^\mu_{\mathcal{C}}(\underline{b}) = \prod\{a^- : (a \cdot f(\underline{b}) = 0) \in \mathcal{C}\},$$

where $\underline{b} \in A^{\rho(f)}$. If it is clear which set $\mathcal{C}$ is used in the definition, then we will simply write f^λ and f^μ for the above two operations. It follows from the definition that in any model $\mathbf{B}$ of $\mathcal{D}$

$$f^\lambda(\underline{a}) \le f^{\mathbf{B}}(\underline{a}) \le f^\mu(\underline{a})$$

for all $\underline{a} \in A^{\rho(f)}$. Thus f^λ and f^μ are lower and upper bounds of $f^{\mathbf{B}}$ restricted to $\mathbf{A}$ or, looked at in another way, if $f^\lambda(\underline{a}) \not\le f^\mu(\underline{a})$ for some sequence $\underline{a} \in A^{\rho(f)}$ then $\mathcal{D}$ is inconsistent. We also define lower and upper bounds on the values of term functions on $\mathbf{A}$ as follows:

(i) $a^\lambda = a^\mu = a$ and $x^\lambda = x^\mu = x$ for constants and variables,

(ii) if r, s are terms and $t = r + s$ then $t^\lambda = r^\lambda + s^\lambda$ and $t^\mu = r^\mu + s^\mu$,

(iii) if s is a term and $t = s^-$ then $t^\lambda = s^{\mu -}$ and $t^\mu = s^{\lambda -}$ and

(iv) if $s_0, \ldots, s_{n-1}$ are terms, $f \in \mathcal{I}$ is n-ary and $t = f(s_0, \ldots, s_{n-1})$ then $t^\lambda = f^\lambda(s^\lambda_0, \ldots, s^\lambda_{n-1})$ and $t^\mu = f^\mu(s^\mu_0, \ldots, s^\mu_{n-1})$.

Lemma 2.15 *Suppose* $\mathbf{B}$ *is a model of* $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$*. Then for any* m*-ary terms* $t, s_0, \ldots, s_{n-1}$ *of* $\mathcal{L}_A$ *and for all* $\underline{a} \in A^m \subseteq B^m$ *we have:*

(i) $t^\lambda(\underline{a}) \le t^{\mathbf{B}}(\underline{a}) \le t^\mu(\underline{a})$,

(ii) $(t = 0) \in \mathcal{E}$ *implies* $t^\lambda(\underline{a}) = 0$,

(iii) *if* $t \cdot f(s_0, \ldots, s_{n-1}) = 0$ *is in* $\mathcal{E}$ *then* $\mathbf{B} \models t^\lambda(\underline{a}) \cdot f(s^\lambda_0(\underline{a}), \ldots, s^\lambda_{n-1}(\underline{a})) = 0$, *and*

(iv) *if* $t \cdot f(s_0, \ldots, s_{n-1})^- = 0$ *is in* $\mathcal{E}$ *then* $\mathbf{B} \models t^\lambda(\underline{a}) \le f(s^\mu_0(\underline{a}), \ldots, s^\mu_n(\underline{a}))$.

Proof. By definition (i) holds if t is a constant or a variable or a member of $\mathcal{I}$. So suppose it holds for terms r and s. If $t = r + s$ then

$$t^\lambda(\underline{a}) = r^\lambda(\underline{a}) + s^\lambda(\underline{a}) \leq r^{\mathbf{B}}(\underline{a}) + s^{\mathbf{B}}(\underline{a}) = t^{\mathbf{B}}(\underline{a}) \leq r^\mu(\underline{a}) + s^\mu(\underline{a}) = t^\mu(\underline{a})$$

and if $t = s^-$ then

$$t^\lambda(\underline{a}) = s^\mu(\underline{a})^- \leq s^{\mathbf{B}}(\underline{a})^- = t^{\mathbf{B}}(\underline{a}) \leq s^\lambda(\underline{a})^- = t^\mu(\underline{a}).$$

Finally, if (i) holds for terms $s_0, \ldots, s_{n-1}$ and $t = f(s_0, \ldots, s_{n-1})$ then

$$f^\lambda(s_0^\lambda(\underline{a}), \ldots, s_{m-1}^\lambda(\underline{a})) \leq f^{\mathbf{B}}(s_0^{\mathbf{B}}(\underline{a}), \ldots, s_{n-1}^{\mathbf{B}}(\underline{a})) \leq f^\mu(s_0^\mu(\underline{a}), \ldots, s_{n-1}^\mu(\underline{a}))$$

follows from the assumption that f is isotone.

(ii) follows from (i), since if $\mathbf{B} \models t = 0$ then $t^\lambda(\underline{a}) \leq t^{\mathbf{B}}(\underline{a}) = 0$ for all $\underline{a} \in A^m$.

To prove (iii) and (iv) we note that if $t{\cdot}f(s_0, \ldots, s_{n-1}) = 0$ is in $\mathcal{E}$ then

$$t^\lambda(\underline{a}){\cdot}f^{\mathbf{B}}(s_0^\lambda(\underline{a}), \ldots, s_{n-1}^\lambda(\underline{a})) = 0$$

and similarly, if $t{\cdot}f(s_0, \ldots, s_{n-1})^- = 0$ is in $\mathcal{E}$ then

$$t^\lambda(\underline{a}) \leq t^{\mathbf{B}}(\underline{a}) \leq f^{\mathbf{B}}(s_0^{\mathbf{B}}(\underline{a}), \ldots, s_{n-1}^{\mathbf{B}}(\underline{a})) \leq f^{\mathbf{B}}(s_0^\mu(\underline{a}), \ldots, s_{n-1}^\mu(\underline{a})).$$

Hence the desired atomic formulae hold in $\mathbf{B}$. $\square$

The significance of conditions (iii) and (iv) in the preceding lemma is that they tell us how to use the equations of $\mathcal{E}$ and the partial information in $\mathcal{C}$ about the operations to find other inclusion and exclusion formulas of $\mathcal{C}_A$ that are true in every model of $\mathcal{E}$. We will use the notation $\mathcal{C} \to \varphi$ for any such formula $\varphi \in \mathcal{C}_A$ that can be obtained in this way.

Now, starting with $\mathcal{C}_0 = \mathcal{C}$ we define

$$\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{\varphi \in \mathcal{C}_A : \mathcal{C}_i \to \varphi\} \qquad \text{and} \qquad \mathcal{C}_\omega = \bigcup_{i \in \omega} \mathcal{C}_i.$$

Observe that if A, $\mathcal{I}$ and $\mathcal{E}$ are finite then $\mathcal{C}_\omega$ can be calculated in finitely many steps. It follows from the above lemma that if $\mathbf{B} \models \mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ then $\mathbf{B} \models \mathcal{C}_\omega$. However, it may turn out that $f^\lambda_{\mathcal{C}_\omega}(\underline{a}) \not\leq f^\mu_{\mathcal{C}_\omega}(\underline{a})$ for some $\underline{a} \in A^{\rho(f)}$. In that case $\mathcal{C}_\omega \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ is inconsistent, and consequently $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ is also inconsistent.

The next lemma tells us that if $\mathcal{C}$ contains enough information about the operations $(f^\lambda : f \in \mathcal{I})$ then it defines a model of $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$.

Lemma 2.16 *Suppose $f^\lambda_{\mathcal{C}} = f^\mu_{\mathcal{C}}$ for all $f \in \mathcal{I}$. If*

$$t^\lambda(\underline{a}) = 0 \textit{ for all } \underline{a} \in A^{\rho(t)} \textit{ and all equations } t = 0 \textit{ in } \mathcal{E} \tag{3}$$

then $\mathbf{A}^\lambda = (\mathbf{A}, (f^\lambda_{\mathcal{C}} : f \in \mathcal{I}))$ is a model of $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$.

Proof. Note that $\mathbf{A}^\lambda$ satisfies the atomic formulae in $\mathcal{C}$ whenever $f^\lambda_{\mathcal{C}} \leq f^\mu_{\mathcal{C}}$. The stronger assumption $f^\lambda_{\mathcal{C}} = f^\mu_{\mathcal{C}}$ implies that $t^{\mathbf{A}^\lambda} = t^\lambda$, so (3) is equivalent to $\mathbf{A}^\lambda \models \mathcal{E}$. Finally, since $\mathbf{A}$ is a Boolean algebra, we always have $\mathbf{A}^\lambda \models \mathcal{E}_{\mathbf{A}}$. $\square$

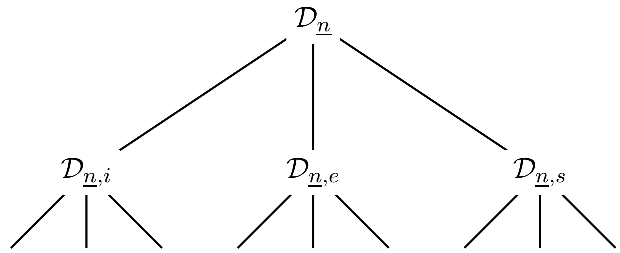

Figure 1: A ternary tree of formula sets

The most likely situation however is the following:

Lemma 2.17 *Suppose* $\mathbf{B} \models \mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ *and for some* $f \in \mathcal{I}$ *and* $\underline{a} \in A^{\rho(f)}$ *we have* $f^{\lambda}(\underline{a}) < f^{\mu}(\underline{a})$. *Let* a *be an atom of* $\mathbf{A}$ *such that* $a \leq f^{\mu}(\underline{a}) \cdot f^{\lambda}(\underline{a})^{-}$ *and define*

$$\mathcal{C}^i = \mathcal{C} \cup \{a \leq f(\underline{a})\},$$
$$\mathcal{C}^e = \mathcal{C} \cup \{a \cdot f(\underline{a}) = 0\} \text{ and}$$
$$\mathcal{C}^s = \mathcal{C} \cup \{a' \leq f(\underline{a})\} \cup \{a'' \cdot f(\underline{a}) = 0\},$$

where $\mathcal{C}^s \subseteq \mathcal{C}_{A^a}$ *and* $\mathbf{A}^a$ *is an extension of* $\mathbf{A}$ *in which the atom* a *is split into two new atoms* $a', a'' \notin A$ *such that* $a' + a'' = a$ *and all other atoms of* $\mathbf{A}$ *remain atoms of* $\mathbf{A}^a$. *Then* $\mathbf{B}$ *is a model of exactly one of* $\mathcal{C}^i$, $\mathcal{C}^e$ *or* $\mathcal{C}^s$.

Proof. Consider the element $a \in A \subseteq B$. If $a \leq f^{\mathbf{B}}(\underline{a})$ then $\mathbf{B} \models \mathcal{C}^i$ and if $a \cdot f^{\mathbf{B}}(\underline{a}) = 0$ then $\mathbf{B} \models \mathcal{C}^e$. However if neither of these cases apply then $a' = a \cdot f^{\mathbf{B}}(\underline{a})$ and $a'' = a \cdot f^{\mathbf{B}}(\underline{a})^{-}$ are two disjoint nonzero elements of B such that $a' + a'' = a$ and clearly $\mathbf{B} \models \mathcal{C}^s$. $\square$

The algorithm that searches for a model of $\mathcal{D} = \mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ proceeds by building a ternary tree of sets (nodes) $\mathcal{D}_{\underline{n}}$ of formulas (see Figure 1). Here $\underline{n}$ is a string of symbols from $\{i, e, s\}$ representing the path from the root of the tree to the node $\mathcal{D}_{\underline{n}}$ and thus $\underline{n}$ is unique for each node. The immediate descendents of $\mathcal{D}_{\underline{n}}$ are $\mathcal{D}_{\underline{n},i}$, $\mathcal{D}_{\underline{n},e}$ and $\mathcal{D}_{\underline{n},s}$. Each node $\mathcal{D}_{\underline{n}}$ has an associated Boolean algebra $\mathbf{A}_{\underline{n}}$, and $\mathcal{D}_{\underline{n}}$ is the union of three sets $\mathcal{C}_{\underline{n}}$, $\mathcal{E}$ and $\mathcal{E}_{A_{\underline{n}}}$. Starting with $\mathcal{D}$ as the root (labeled by the empty string), the following steps are repeated for each node $\mathcal{D}_{\underline{n}}$:

(1) compute $\mathcal{C}' = (\mathcal{C}_{\underline{n}})_{\omega}$;

(2) if $f^{\lambda}_{\mathcal{C}'}(\underline{a}) \not\leq f^{\mu}_{\mathcal{C}'}(\underline{a})$ for some $f \in \mathcal{I}$ and $\underline{a} \in A^{\rho(f)}$ then $\mathcal{D}_{\underline{n}}$ is inconsistent;

(3) if $f^{\lambda}_{\mathcal{C}'} = f^{\mu}_{\mathcal{C}'}$ for all $f \in \mathcal{I}$ then $\mathbf{A}^{\lambda}_{\underline{n}}$ is a model of $\mathcal{D}_{\underline{n}}$;

(4) if $0 \neq b = f^{\mu}_{\mathcal{C}'}(\underline{a}) \cdot f^{\lambda}_{\mathcal{C}'}(\underline{a})^{-}$ for some $f \in \mathcal{I}$ and $\underline{a} \in A^{\rho(f)}_{\underline{n}}$ then choose an atom $a \in A_{\underline{n}}$, such that $a \leq b$ and define the immediate descendents of $\mathcal{D}_{\underline{n}}$ by

$$\mathcal{C}_{\underline{n},i} = \mathcal{C}'^{i} \text{ and } \mathbf{A}_{\underline{n},i} = \mathbf{A}_{\underline{n}},$$
$$\mathcal{C}_{\underline{n},e} = \mathcal{C}'^{e} \text{ and } \mathbf{A}_{\underline{n},e} = \mathbf{A}_{\underline{n}},$$

$$\mathcal{C}_{\underline{n},s} = \mathcal{C}'^s \text{ and } \mathbf{A}_{\underline{n},s} = \mathbf{A}^a_{\underline{n}},$$

where the operations $^i, ^e, ^s$ are as in Lemma 2.17.

Since $\mathbf{A}_{\underline{n}}$ is finite, there are only finitely many choices at each stage in step (4), and we can in principle investigate each choice in turn. The next two results provide the model-construction and theorem-proving aspect of this algorithm respectively.

Theorem 2.18 *If $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_\mathbf{A}$ has a finite model then any tree constructed by steps* (1)–(4) *will contain a model of minimal cardinality.*

Proof. Let $\mathcal{T}$ be a tree constructed by steps (1)–(4). We claim that $\mathcal{T}$ effectively enumerates all finite minimal models of $\mathcal{D} = \mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_\mathbf{A}$ (i.e. models that have no proper subalgebras that model $\mathcal{D}$). Suppose $\mathbf{B}$ is a finite minimal model of $\mathcal{D}$. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}_0$. To check that $\mathbf{B}$ occurs in the tree, we start at the root of $\mathcal{T}$ and at each node we choose the descendent of which $\mathbf{B}$ is a model, using Lemma 2.17. Since each choice more closely approximates the finite algebra $\mathbf{B}$, step (3) will eventually be satisfied. □

Theorem 2.19 *If some tree constructed by steps* (1)–(4) *has inconsistent nodes at the end of each branch, then $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_\mathbf{A}$ is inconsistent.*

Proof. Let $\mathcal{T}$ be a tree that has inconsistent nodes at the end of each branch. Of course the assumption that each branch terminates implies that $\mathcal{T}$ is finite. Suppose to the contrary that $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_\mathbf{A}$ is consistent. Then it has a (possibly infinite) model $\mathbf{B}$. By Lemma 2.17 there has to be at least one branch of $\mathcal{T}$ such that $\mathbf{B}$ is a model of each node along this branch. This, however, contradicts the assumption that every branch ends in an inconsistent node. □

An implementation of this algorithm is discussed in the appendix. When working with relation algebras, we usually want to restrict our attention to the simple members of *RA*. Since they are characterized by a universal sentence ($x \neq 0 \Rightarrow 1 \circ x \circ 1 = 1$) it is useful to allow such sentences in $\mathcal{E}$. Step (2) of the algorithm must then be modified to check the consistency of $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_\mathbf{A}$ with respect to these sentences. This is done with the help of the following definition.

Let σ be a universal sentence in the language $\mathcal{L}_A$. Equivalently, we can view σ as an open formula, and we may assume that it is a disjunction of a conjunction of atomic formulas. For $\underline{a} \in \mathbf{A}^n$ we say that $\sigma(\underline{a})$ is *compatible* with $\mathcal{C} \cup \mathcal{E}_\mathbf{A}$ if

(i) σ is $t = 0$, and $t^\lambda(\underline{a}) = 0$,

(ii) σ is $t \neq 0$, and $t^\mu(\underline{a}) \neq 0$,

(iii) σ is $\varphi \wedge \psi$, and both $\varphi(\underline{a})$ and $\psi(\underline{a})$ are compatible with $\mathcal{C} \cup \mathcal{E}_\mathbf{A}$,

(iv) σ is $\varphi \vee \psi$, and either $\varphi(\underline{a})$ or $\psi(\underline{a})$ is compatible with $\mathcal{C} \cup \mathcal{E}_\mathbf{A}$.

The sentence σ is compatible with $\mathcal{C} \cup \mathcal{E}_\mathbf{A}$ if it is compatible for all $\underline{a} \in \mathbf{A}^n$, otherwise it is incompatible. The next result shows how compatibility is related to consistency.

Lemma 2.20 *If σ is a universal sentence and $\mathcal{C} \cup \{\sigma\} \cup \mathcal{E}_{\mathbf{A}}$ is consistent then σ is compatible with $\mathcal{C} \cup \mathcal{E}_{\mathbf{A}}$.*

Proof. Suppose $\mathbf{B}$ is a model of $\mathcal{C} \cup \{\sigma\} \cup \mathcal{E}_{\mathbf{A}}$. If σ is $t = 0$ then $t^{\lambda}(\underline{a}) = 0$ for all $\underline{a} \in \mathbf{A}^n$ by Lemma 2.15 (ii), and for $t \neq 0$ it follows from part (i) of the same lemma that $0 \neq t^{\mathbf{B}}(\underline{a}) \leq t^{\mu}(\underline{a})$ for all $\underline{a} \in \mathbf{A}^n$. If σ is a conjunction of two sentences, both sentences are consistent and hence compatible with $\mathcal{C} \cup \mathcal{E}_{\mathbf{A}}$. Finally, suppose σ is a disjunction of φ and ψ. Since $\mathbf{B}$ is a model of σ, either $\varphi(\underline{a})$ or $\psi(\underline{a})$ holds in $\mathbf{B}$ for all $\underline{a} \in \mathbf{A}^n$, hence either $\varphi(\underline{a})$ or $\psi(\underline{a})$ is compatible with $\mathcal{C} \cup \mathcal{E}_{\mathbf{A}}$. $\square$

We can therefore add the following step to the algorithm.

(2.1) If any universal sentence in $\mathcal{E}$ is incompatible with $\mathcal{C}' \cup \mathcal{E}_{\mathbf{A}_{\underline{n}}}$ then $\mathcal{D}_{\underline{n}}$ is inconsistent.

Further enhancements that make the algorithm more manageable (although they do not increase the class of problems to which it applies) are discussed in the appendix. The algorithm as described here has mainly been used to prove results about covers of the atoms in Λ_{SRA} like Theorem 3.43. Other results that were suggested or partially proved by this algorithm are Theorem 4.4 and 4.10.

An important aspect of this algorithm that has not yet been settled is the question of its completeness with respect to the deductive closure of $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$, i.e does the converse of Theorem 2.19 hold, or equivalently, if none of the trees constructed by steps (1)-(4) have inconsistent nodes at the end of all branches, does $\mathcal{C} \cup \mathcal{E} \cup \mathcal{E}_{\mathbf{A}}$ have a (possibly infinite) model? If $\mathrm{Mod}(\mathcal{E})$ is a variety that is generated by its finite members, then this follows from Theorem 2.18, but for an arbitrary finite set $\mathcal{E}$ of universal sentences, a general completeness theorem has not been established.

CHAPTER III

VARIETIES OF *UR*-ALGEBRAS

Varieties containing *SRA*

In this section we first give an overview of the varieties that we will be considering and prove some simple results about them. We discuss the relationship between ur-algebras and relation algebras (investigated in Jónsson and Tsinakis [**a**]) and a connection between varieties of r-algebras and varieties of ur-algebras (from Jipsen, Jónsson and Rafter [**a**]). We then show that all finite r-algebras, all integral r-algebras, ur-algebras with finitely many elements below the unit, and all commutative residuated monoids are discriminator algebras, provided they are subdirectly irreducible. These results are used to give equational bases for some varieties of ur-algebras. On the other hand we give an example of a Euclidean rm-algebra that is subdirectly irreducible but not simple, thereby showing that the variety of all Euclidean rm-algebras is not a discriminator variety. At the end of this section we prove that the varieties of all r-algebras, all integral r-algebras and all ur-algebras are generated by the residuated complex algebras of all partial groupoids, all groupoids and all partial groupoids with identities respectively.

Overview. The varieties we consider appear in Figures 2, 3 and 4. Their definitions are most readily obtained from Table 1. For example *ERM* is the variety of all Euclidean rm-algebras, defined relative to *UR* by the equations

$$(x \triangleright y) \circ z \leq x \triangleright (y \circ z) \quad \text{and} \quad (x \circ y) \circ z = x \circ (y \circ z).$$

The prefixes A, I and N are defined by universal sentences rather than equations. In that case $A\mathcal{V}$, $I\mathcal{V}$ and $N\mathcal{V}$ are the varieties generated by all members of $\mathrm{Si}(\mathcal{V})$ that satisfy the respective sentences.

Note that we treat (nonassociative) relation algebras as ur-algebras with the converse $(^{\smile})$ defined by the term function $x^{\smile} = x \triangleright e$. This allows us to view *NA* and *RA* as subvarieties of *UR*. The following result justifies this point of view.

Theorem 3.1 (Jónsson and Tsinakis [**a**]) *Let* $\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft, e)$ *be a ur-algebra and define* $x^{\smile} = x \triangleright e$. *Then the following are equivalent:*

(i) $(\mathbf{A}_0, \circ, ^{\smile}, e)$ *is a nonassociative relation algebra,*

(ii) $\mathbf{A}$ *satisfies the identities* $x^{\smile} \circ y = x \triangleright y$ *and* $x \circ y^{\smile} = x \triangleleft y$.

Figures 2, 3 and 4 are meet subsemilattices, but not sublattices of Λ_{UR}. The latter two figures present two different views into the interval between the variety *SRA* of all symmetric relation algebras and *UR*.

Table 1: Varieties of ur-algebras

Name	Description	Definition
UR	unital residuated Boolean groupoids $\mathbf{A} = (\mathbf{A}_0, \circ, \rhd, \lhd, e)$	$x \circ e = e = e \circ x$ and $x(y \circ z) = 0 \Leftrightarrow y(x \lhd z) = 0 \Leftrightarrow z(y \rhd x) = 0$
RM	residuated Boolean monoids	UR and $(x \circ y) \circ z = x \circ (y \circ z)$
NA	nonassociative relation algebras	UR and $x \rhd y = x^{\smile} \circ y$ and $x \lhd y = x \circ y^{\smile}$
WA	weakly associative relation algebras	NA and $xe \circ 1 = (xe \circ 1) \circ 1$
SA	semiassociative relation algebras	NA and $x \circ 1 = (x \circ 1) \circ 1$
RA	relation algebras	NA and $(x \circ y) \circ z = x \circ (y \circ z)$
RRA	representable relation algebras	$\mathrm{Var}(\{Re(\alpha) : \alpha \leq \omega\})$
GRA	group relation algebras	$\mathrm{Var}(\{G^+ : G \text{ is a group}\})$
$BGRA$	Boolean group relation algebras	$\mathrm{Var}(\{(\mathbb{Z}_2^\alpha)^+ : \alpha \leq \omega\})$
E	Euclidean	$(x \rhd y) \circ z \leq x \rhd (y \circ z)$
A	e is an atom	$ex = 0$ or $ex^- = 0$
I	integral	$x \circ y = 0$ implies $x = 0$ or $y = 0$
C	commutative	$x \circ y = y \circ x$
S	symmetric	$x \rhd y = x \circ y$
N	neat symmetric	symmetric and $x \leq x \circ x$ or $x(x \circ x) = 0$
T	totally symmetric	symmetric and $x \leq x \circ x$
Sa	subadditive	symmetric and $x \circ x^- y \leq x + y$

The lower half are prefixes for varieties in the upper half.
For a variety $\mathcal{V}$ and a prefix P, $P\mathcal{V} = \mathrm{Var}(\{\mathbf{A} \in \mathrm{Si}(\mathcal{V}) : \mathbf{A} \text{ has property } P\})$.
For NA, $x^{\smile}$ abbreviates $x \rhd e$.

Table 2: Implications between some ur-algebra properties

		I	$\Rightarrow$	A	
		S	$\Rightarrow$	C	
		S	$\Rightarrow$	NA	
E	and	A	$\Rightarrow$	I	(Theorem 3.7)
C	and	RM	$\Rightarrow$	A	(Theorem 3.15)
S	and	RM	$\Rightarrow$	SRA	
A	and	NA	$\Rightarrow$	WA	
I	and	NA	$\Rightarrow$	SA	
E	and	NA	$\Rightarrow$	RA	(Jónsson and Tsinakis [**a**])
A	and	SA	$\Rightarrow$	I	
C	and	SA	$\Rightarrow$	I	

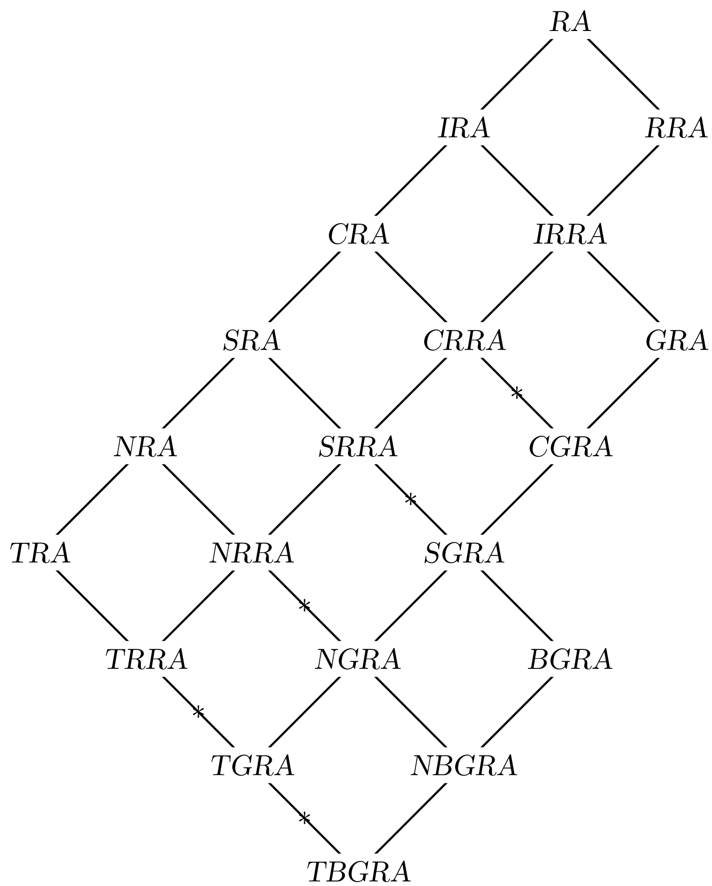

Figure 2: Some subvarieties of RA ordered by inclusion

(Prefixes I, C, S, N, T applied to $RA, RRA, GRA, BGRA$)
($*$ indicates varieties are not known to be distinct)

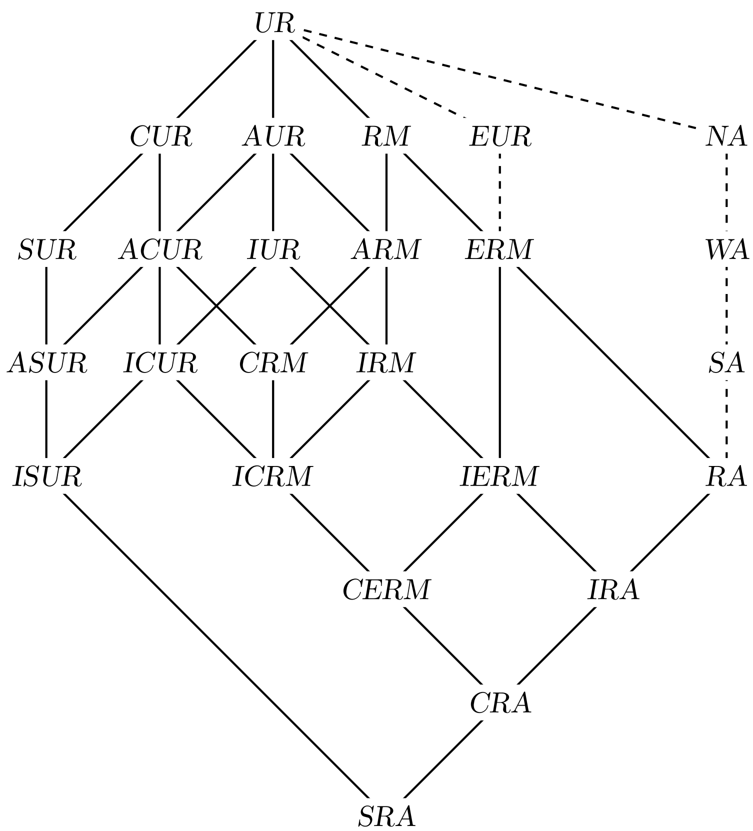

Figure 3: Some subvarieties of UR ordered by inclusion

(Prefixes A, I, C, S applied to UR, RM, ERM, RA)

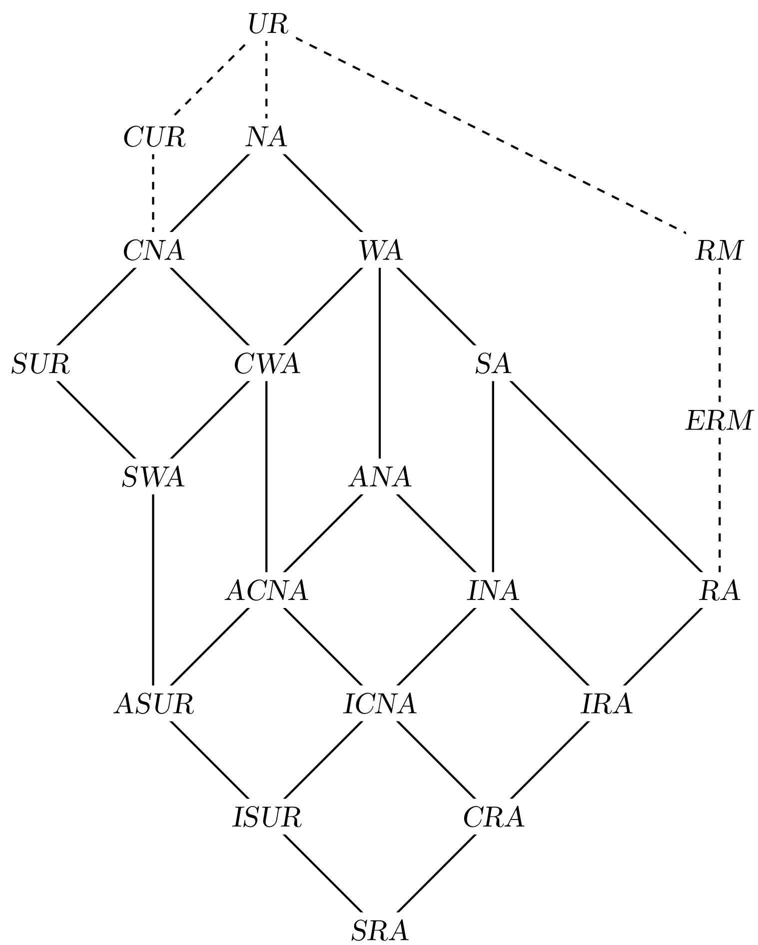

Figure 4: Some subvarieties of NA ordered by inclusion

(Prefixes A, I, C, S applied to NA, WA, SA, RA)

Jónsson [**82**] shows that Λ_{TGRA} has uncountably many subvarieties (that also satisfy the identity $x^3 = x^2$). Further results about the cardinality and structure of some intervals in Λ_{RRA} are proved by Andréka, Givant and Németi [**91**]. For example they show that the Boolean algebra of all subsets of ω is completely embedable into an interval of Λ_{RRA}.

Some of the varieties in Figure 2 are not known to be distinct. R. Lyndon [**59**] proved that *TRRA* is a proper subvariety of *TRA*, and D. Monk [**64**] extended this result by showing that *TRRA* is not even finitely based relative to *TRA*. (Both results are usually stated for *RRA* and *RA*, and they of course imply corresponding results for $\mathcal{P}$*RRA* and $\mathcal{P}$*RA*, where $\mathcal{P}$ is one of the properties I, C, S or N.) R. N. McKenzie [**66**] proved that *GRA* is not finitely based relative to *IRRA*. In the last section of Chapter IV we observe that *NBGRA* and *NGRA* are distinct. Table 2 lists some implications between ur-algebra properties that are also reflected in Figures 3 and 4.

The following result from Jipsen, Jónsson and Rafter [**a**], describes a connection between ur-algebras and their r-algebra reducts. In particular this result shows that the class of all r-algebras that can be embedded in reducts of ur-algebras is a variety defined by finitely many equations.

Theorem 3.2 *For any r-algebra* $\mathbf{A}$*, the following conditions are equivalent:*

(i) $\mathbf{A}$ *is embedable in an r-algebra with a unit.*

(ii) $\mathbf{A}^\sigma$ *has a unit.*

(iii) *For all $x, y, z \in A$,*

$$x \le x \circ (y/y)(z/z), \qquad x \le x \circ (y\backslash y),$$
$$x \le (y\backslash y)(z\backslash z) \circ x, \qquad x \le (y/y) \circ x.$$

We now outline a technique (from Jónsson and Tsinakis [**a**]) useful for deriving equivalent formulations of r-algebra identities.

For a conjugated operation f, denote the conjugate by f^c. It is easy to see that if f and g are conjugated operations then f^c, $f + g$ (pointwise join) and the composite operation fg (not the meet) are also conjugated with conjugates

$$f^{cc} = f, \qquad (f+g)^c = f^c + g^c \quad \text{and} \quad (fg)^c = g^c f^c.$$

The first two equations imply that

$$f \le g \qquad \text{iff} \qquad f^c \le g^c,$$

where $f \le g$ represents the identity $f(x) \le g(x)$ for all x. To apply this to ur-algebras we define the following translates and their conjugates

$$L_a(x) = a \circ x \qquad R_a(x) = x \circ a \qquad Q_a(x) = a \triangleleft x$$
$$L_a^c(x) = a \triangleright x \qquad R_a^c(x) = x \triangleleft a \qquad Q_a^c(x) = x \triangleright a.$$

An equation like $(x \circ y) \circ z \le x \circ (y \circ z)$ can now be written in three different ways as

$$R_z R_y \le R_{y \circ z} \qquad R_z L_x \le L_x R_z \qquad L_{x \circ y} \le L_x L_y$$

depending on whether we choose x, y or z as the variable for the translates. Taking conjugates, we obtain

$$R^c_y R^c_z \leq R^c_{y\circ z} \qquad L^c_x R^c_z \leq R^c_z L^c_x \qquad L^c_{x\circ y} \leq L^c_y L^c_x.$$

Translating back to the language of r-algebras we have proved the first part of the following result.

Lemma 3.3 *The identities in each group below are equivalent.*

$$\text{(i)} \qquad \begin{array}{l} (x \circ y) \circ z \leq x \circ (y \circ z) \\ (x \lhd z) \lhd y \leq x \lhd (y \circ z) \\ x \rhd (y \lhd z) \leq (x \rhd y) \lhd z \\ (x \circ y) \rhd z \leq y \rhd (x \rhd z) \end{array}$$

$$\text{(ii)} \qquad \begin{array}{l} (x \rhd y) \circ z \leq x \rhd (y \circ z) \\ y \lhd (x \lhd z) \leq (y \circ z) \lhd x \\ x \circ (y \lhd z) \leq (x \circ y) \lhd z \\ (x \rhd y) \rhd z \leq y \rhd (x \circ z) \end{array}$$

$$\text{(iii)} \qquad \begin{array}{l} (x \lhd y) \circ z \leq x \circ (y \rhd z) \\ (x \lhd z) \circ y \leq x \lhd (y \rhd z) \\ (y \lhd z) \rhd x \leq z \lhd (x \rhd y) \\ (x \lhd y) \rhd z \leq y \circ (x \rhd z) \end{array}$$

The identities in (ii) are equivalent forms of the Euclidean law

$$(x \rhd y)(u \lhd z) \neq 0 \quad \text{implies} \quad (x \circ u)(y \circ z) \neq 0$$

that is studied in a contemporary approach to geometry by Prenowitz [**61**]. The first identity of (iii) is derived from (half of) the relation algebra identity $(x \circ y^{\smile}) \circ z = x \circ (y^{\smile} \circ z)$. Reversing the inequality symbol gives another set of equations equivalent in groups of four. All of these equations hold in *RA*, and in *NA* they are all equivalent.

Discriminator *ur*-algebras. We now address the question which varieties in Figures 3 and 4 are discriminator varieties. A. Tarski showed that *RA* has a discriminator term $c(x) = 1 \circ x \circ 1$. R. D. Maddux [**78**] proves the same term c is also a discriminator term for *SA*. For the larger variety *WA*, however, Maddux shows that there are subdirectly irreducible members that are not simple, hence *WA* is not a discriminator variety.

An r-algebra is *integral* if for all elements x, y the condition $x \circ y = 0$ implies $x = 0$ or $y = 0$.

Lemma 3.4 *An r-algebra* $\mathbf{A}$ *is integral if and only if the term* $c(x) = x \rhd 1$ *is a unary discriminator for* $\mathbf{A}$.

Proof. Assume $\mathbf{A}$ integral and let x be a nonzero element of $\mathbf{A}$. Then the conditions

$$(x \triangleright 1)y = 0, \quad x \circ y = 0 \quad \text{and} \quad y = 0$$

are equivalent, whence $x \triangleright 1 = 1$. Conversely, if $x \triangleright 1$ is a unary discriminator then $x \neq 0$ and $y \neq 0$ imply $(x \triangleright 1)y \neq 0$, hence $x \circ y \neq 0$. □

Theorem 3.5 *Let IR be the variety generated by all integral r-algebras.*

(i) *IR is the largest variety of r-algebras for which $c(x) = x \triangleright 1$ is a unary discriminator in all its subdirectly irreducible members.*

(ii) *An equational basis for IR relative to the variety of all r-algebras is given by the equation $x + \tau(x \triangleright 1) \leq x \triangleright 1$.*

Proof. (i) is a consequence of Lemma 3.4, and (ii) follows from Theorem 2.4 once we observe that $0 \triangleright 1 = 0$ and that τ is selfconjugate for r-algebras. □

We now list some properties that hold in all ur-algebras.

Lemma 3.6 *For any x, y, u, v in a ur-algebra with $u, v \leq e$ we have*

(i) $x \leq y \circ u$ *implies* $x = x \circ u$,

(ii) $(x \circ u)y = (y \circ u)x$,

(iii) $x \triangleleft u = x \circ u$ *and* $u \triangleright x = u \circ x$

(iv) $u \circ v = uv$,

(v) $(x \circ u)v = (x \triangleright u)v = (x \triangleleft u)v = (u \circ x)v = (u \triangleright x)v = (u \triangleleft x)v = xuv$,

(vi) $(1 \circ u)x = x \circ u$ *and* $(u \circ 1)x = u \circ x$.

Proof. (i) By assumption, $x \leq y \circ u$ implies $x \leq y$, so $y = x + yx^-$. Now

$$x \leq y \circ u = (x + yx^-) \circ u = x \circ u + yx^- \circ u.$$

But $x(yx^- \circ u) \leq xyx^- = 0$, hence $x \leq x \circ u \leq x$.

(ii) Let $z = (y \circ u)x \leq y \circ u$. Then (i) implies that $z = z \circ u \leq x \circ u$. Since we also have $z \leq yx \leq y$, we get $z = (y \circ u)x \leq (x \circ u)y$. For the reverse direction we simply interchange x and y.

(iii) Conjugation and (ii) imply that the conditions $(x \triangleleft u)y \neq 0$, $(y \circ u)x \neq 0$ and $(x \circ u)y \neq 0$ are equivalent, hence $x \triangleleft u = x \circ u$.

(iv) First we note that $u = u \circ u$ since $u = u \circ e = u \circ u + u \circ eu^-$ and $u \circ eu^- \leq uu^- = 0$. Therefore $uv = uv \circ uv \leq u \circ v$. Conversely $u \circ v \leq (u \circ e)(e \circ v) \leq uv$.

(v) By additivity $(x \circ u)v = (xe \circ u)v + (xe^- \circ u)v = xuv$ since $(xe^- \circ u)v \leq e^- e = 0$. By (iii) we also have $(x \triangleleft u)v = xuv$. Now $(x \triangleright u)v = (xe \triangleright u)v + (xe^- \triangleright u)v = xeuv + 0 = xuv$, again using (iii).

(vi) A direct calculation gives $(1 \circ u)x = (x \circ u)x + (x^- \circ u)x = x \circ u$. □

Theorem 3.7 *Let $\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft, e) \in UR$.*

(i) *If* $\mathbf{A}$ *is integral then* e *is an atom.*

(ii) *If* e *is an atom and* $\mathbf{A}$ *is Euclidean then* $\mathbf{A}$ *is integral.*

Proof. (i) follows immediately from Lemma 3.6 (iv).

(ii) Suppose e is an atom and $\mathbf{A}$ is Euclidean. We show $x \triangleright 1$ is a unary discriminator term, whence $\mathbf{A}$ is integral by Lemma 3.4. For $x \neq 0$ we clearly have $x(x \circ e) = x \neq 0$ and, since e is an atom, it follows that $e \leq x \triangleright x$. By the Euclidean identity

$$1 = e \circ 1 \leq (x \triangleright x) \circ 1 \leq x \triangleright (x \circ 1) \leq x \triangleright 1.$$

□

The notions of domain and range of a relation generalize for elements in a *ur*-algebra in the following way. For $x \in A$ let

$$x^\delta = e(1 \triangleleft x) \quad \text{and} \quad x^\rho = e(x \triangleright 1)$$

be the domain and range of x respectively. With this definition, $u = ex^{\delta-}$ is the largest element below e for which $u \circ x = 0$ (see proof of (iii) below). We summarize some of the properties of these operations:

Lemma 3.8 *For any* x *in a* *ur-algebra*

(i) $x \leq e$ *implies* $x^\delta = x = x^\rho$,

(ii) $x^{\delta\delta} = x^\delta$ *and* $x^{\rho\rho} = x^\rho$,

(iii) $x^\delta \circ x = x = x \circ x^\rho$,

(iv) $x^\delta = e(x \triangleleft 1)$ *and* $x^\rho = e(1 \triangleright x)$.

Proof. (i) follows from Lemma 3.6 (v), and (ii) is an immediate consequence of (i). To prove (iii), we compute $x = e \circ x = x^\delta \circ x + x^{\delta-}e \circ x$, and $1(x^{\delta-}e \circ x) = 0$ since $x^{\delta-}e(1 \triangleleft x) = x^{\delta-}x^\delta = 0$. Finally, (iv) holds because the following statements are equivalent by Lemma 3.6 (iii):

$$yx^\delta = 0 \quad ye \circ x = 0 \quad ye \triangleright x = 0 \quad ye(x \triangleleft 1).$$

□

We now prove a general result about congruence lattices of modal algebras, and then apply it to *ur*-algebras in which the unit element is the join of finitely many atoms.

Theorem 3.9 *Let* $\mathbf{A} = (\mathbf{A}_0, f)$ *be a modal algebra and suppose that* A *satisfies the inclusion* $x \leq f(f^n(x)u)$ *for some (fixed)* $u \in \mathbf{A}$ *and* $n \in \omega$. *Then*

$$Con(\mathbf{A}) \cong Con(\mathbf{A}_0 u, g^{\mathbf{A}})$$

where $g(x) = f^{n+2}(x)u$.

Proof. We will show that the maps

$$F(J) = Ju \quad \text{and} \quad G(K) = Af(K) = \{x \in A : x \le f(y) \text{ for some } y \in K\}$$

map congruence ideals to congruence ideals and are inverses of each other. Since they are also order preserving, the result follows.

Note that the inclusion $x \le f(f^n(x)u)$ implies $f^2(x) \le f(g(x))$, and therefore

$$f^{n+1}(x) \le f(g^n(x)). \tag{4}$$

For any congruence ideal J of $\mathbf{A}$, $F(J)$ is clearly a Boolean ideal, and if $x \in Ju$ then $f^{n+2}(x) \in J$, hence $g(x) \in Ju$. Now consider a congruence ideal K of $(\mathbf{A}_0u, g^{\mathbf{A}})$. Again $G(K)$ is easily seen to be a Boolean ideal. So suppose $x \in Af(K)$, whence $x \le f(y)$ for some $y \in K$. Then

$$f(x) \le f^2(y) \le f(f^{n+2}(y)u) \le f(g(y))$$

and since $g(y) \in K$ it follows that $f(x) \in Af(K)$. This shows that F and G map congruence ideals to congruence ideals.

Next we show that $FG(K) = K$. Let $x \in FG(K) = Af(K)u$. Then $x \le f(y)u$ for some $y \in K$, hence

$$x \le f(f^n(x)u)u \le f^{n+1}(x)u \le f^{n+2}(y)u = g(y) \in K.$$

Conversely, for any $x \in K$, we have $x \le u$, hence $x \le f^{n+1}(x) \le f(g^n(x))$, where the last inclusion follows from (4) above.

Finally we have $GF(J) = J$ since $Af(Ju) \subseteq f(J) \subseteq J$, and for any $x \in J$, $x \le f(f^n(x)u) \in f(Ju)$, hence $x \in Af(Ju)$. □

Recall that for a BAO $\mathbf{A} = (\mathbf{A}_0, \mathcal{F})$ of finite type $\mathrm{Con}(\mathbf{A})$ is isomorphic to $\mathrm{Con}(\mathbf{A}_0, \tau^{\mathbf{A}})$, where τ is the join of all $(\underline{1}, i)$-translates of $f \in \mathcal{I}$, $i < \rho(f)$.

Corollary 3.10 *Suppose* $\mathbf{A}$ *is a ur-algebra, and let* $g(x) = \tau^3(x)e$. *Then*

$$Con(\mathbf{A}) \cong Con(\mathbf{A}_0 e, g^{\mathbf{A}}).$$

Proof. The result will follow from the previous theorem with $f(x) = \tau(x)$ and $n = 1$ once we establish the inclusion $x \le \tau(\tau(x)e)$. But this follows from Lemma 3.8 (iii) since $x = x^\delta \circ x \le (1 \lhd x)e \circ 1 \le \tau(\tau(x)e)$. □

Corollary 3.11 *Any subdirectly irreducible ur-algebra* $\mathbf{A}$ *in which the identity element* e *is the join of finitely many atoms is a discriminator algebra.*

Proof. In this case the algebra $(\mathbf{A}_0 e, g^{\mathbf{A}})$ from the preceding corollary is a finite subdirectly irreducible selfconjugated modal algebra. By Corollary 2.6 it is a discriminator algebra and therefore simple. But then $\mathbf{A}$ is also simple by Corollary 3.10, whence Theorem 2.5 implies that it is a discriminator algebra. □

If e is an atom of $\mathbf{A}$ then one can easily show by a direct calculation that $c(x) = (1 \triangleright x) \circ 1$ is a unary discriminator. Note also that e is an atom of $\mathbf{A}$ if and only if $\mathbf{A}$ satisfies the universal sentence

$$\text{for all } x \in A \text{ either } ex = 0 \text{ or } ex^- = 0.$$

Using steps (A)-(C) at the end of the first section of Chapter II to translate this into an equation and combining it with Theorem 2.4 we obtain the following result.

Theorem 3.12 *Let AUR be the variety generated by all ur-algebras in which e is an atom, and let* $c(x) = (1 \triangleright x) \circ 1$*. Then the following equations form a basis for AUR relative to the variety of all ur-algebras:*

$$x + \tau(c(x)) \le c(x) \qquad \text{and} \qquad c(c(ex)^- + c(ex^-)^-) = 1.$$

We now turn to rm-algebras, i.e. ur-algebras that are associative with respect to $\circ$. The main result (Corollary 3.15) is that the variety of commutative rm-algebras is a subvariety of AUR.

Lemma 3.13 *For any* x, u, v *in an* rm*-algebra with* $u, v \le e$ *we have*

(i) $(1 \circ u)(1 \circ v) = 1 \circ uv$ *and*

(ii) $(x \circ u)(y \circ v) = xy \circ uv$.

Proof. (i) $(1 \circ u)(1 \circ v) = 1 \circ v \circ u = 1 \circ uv$ by Lemma 3.6 (ii), (iv) and associativity.

(ii) Using (i) and Lemma 3.6 (vi) we calculate $(x \circ u)(y \circ v) = (1 \circ u)(1 \circ v)xy = (1 \circ uv)xy = xy \circ uv$. □

Note that the commutative law $x \circ y = y \circ x$ is equivalent to $x \triangleright y = y \triangleleft x$.

Theorem 3.14 *Let* $\mathbf{A} = (\mathbf{A}_0, \circ, \triangleright, \triangleleft, e)$ *be a commutative rm-algebra. Then* $1 \circ u$ *is a congruence element for any* $u \le e$.

Proof. Let $x = 1 \circ u$. Since $\mathbf{A}$ is commutative, it suffices to show that $1 \circ x \le x$, $1 \triangleright x \le x$ and $1 \triangleleft x \le x$. By associativity $1 \circ x = x$. By Lemma 3.6 (iii) $x = 1 \triangleleft u$, so

$$1 \triangleright x = 1 \triangleright (1 \triangleleft u) = (1 \triangleright 1) \triangleleft u = 1 \triangleleft u = x$$

using an equivalent form of the associative identity (Lemma 3.3 (i)). Finally $1 \triangleleft x = 1 \triangleleft (1 \circ u) = (1 \triangleleft u) \triangleleft 1$ also follows from Lemma 3.3 (i). By commutativity $(1 \triangleleft u) \triangleleft 1 = 1 \triangleright (1 \triangleleft u) = x$ as before. □

Corollary 3.15 *The variety CRM of all commutative rm-algebras is a discriminator variety, and in the simple members e is an atom.*

Proof. Suppose $\mathbf{A}$ is a commutative rm-algebra in which e is not an atom. Then there exists a nonzero element $u \le e$ such that $v = eu^-$ is also nonzero. By the preceding theorem $u \circ 1$ and $v \circ 1$ are congruence elements, and by Lemma 3.13 they are disjoint, so it follows from the additivity of $\circ$ that they are compliments of each other. Now $\mathbf{A}$ is directly decomposable by Theorem 2.2. Consequently e is necessarily an atom in any subdirectly irreducible commutative rm-algebra, and hence the variety of all commutative rm-algebras is a subvariety of AUR. By Corollary 3.11 AUR is a discriminator variety, and this property carries over to all its subvarieties. □

The previous result generalizes an unpublished result of C. Tsinakis, who proved that the variety *CERM* of commutative Euclidean rm-algebras is a discriminator variety.

Categories and Euclidean rm-algebras. Here we show that certain rm-algebras can be constructed as complex algebras of (small) categories. We then use this observation to show that, in spite of the many examples of discriminator algebras and varieties of residuated BAOs encountered so far, the variety of all Euclidean rm-algebras is *not* a discriminator variety.

We aim to construct an rm-algebra that is not a discriminator algebra, so we need to consider some specialized relational structures. A *partial semigroup* is a structure $(U, \circ)$ where $\circ$ is a partial binary operation on U such that whenever $a \circ b$ and $(a \circ b) \circ c$ are defined, then $b \circ c$ and $a \circ (b \circ c)$ are defined, and conversely, and $(a \circ b) \circ c = a \circ (b \circ c)$. Note that the residuated complex algebra of a partial semigroup is an associative r-algebra, with $\circ$ defined on subsets of U in the usual way by

$$X \circ Y = \{x \circ y : x \in X \text{ and } y \in Y\}.$$

A (small) *category* is a special kind of partial semigroup with a distinguished subset $E \subseteq U$ (of identity morphisms) such that $E \circ x = x \circ E = x$. Thus the residuated complex algebra of a category is a rm-algebra. But we get somewhat more since a category also has to satisfy the property: whenever $a \circ b$ and $b \circ c$ are defined, then $(a \circ b) \circ c$ (and $a \circ (b \circ c)$) is defined (see for example McKenzie, McNulty and Taylor [**87**]). This suggests considering the following property for r-algebras:

$$a \circ b \neq 0 \neq b \circ c \quad \text{implies} \quad (a \circ b) \circ c \neq 0 \neq a \circ (b \circ c).$$

An r-algebra that satisfies this property will be called *weakly Euclidean.*

As mentioned before, an r-algebra is *Euclidean* if it satisfies the inclusion $(a \triangleright b) \circ c \leq a \triangleright (b \circ c)$ or equivalently if

$$(a \triangleright x)(y \triangleleft c) \neq 0 \quad \text{implies} \quad (a \circ y)(x \circ c) \neq 0.$$

An r-algebra is *strongly Euclidean* if it satisfies the identity $(a \triangleright b) \circ c = a \triangleright (b \circ c)$. Every relation algebra is strongly Euclidean, and Jónsson and Tsinakis [**a**] show that conversely every strongly Euclidean ur-algebra is a relation algebra.

The following lemma establishes a connection between Euclidean and weakly Euclidean r-algebras.

Lemma 3.16 *Let* $\mathbf{A}$ *be an* r*-algebra.*

(i) *If* $\mathbf{A}$ *is Euclidean then it is weakly Euclidean.*

(ii) *Suppose* $\mathbf{A}$ *is atomic and weakly Euclidean. If* $(\mathbf{A}_{0+}, \circ)$ *is a partial semigroup then* $\mathbf{A}$ *is Euclidean.*

Proof. (i) Since the Euclidean inclusion contains no complementation, it is preserved under canonical extensions, so we may assume that $\mathbf{A}$ is atomic. Let a, b, c be atoms of $\mathbf{A}_0$. If $a \circ b \neq 0$ then $(a \triangleright (a \circ b))b \neq 0$, hence $(a \triangleright (a \circ b)) \geq b$, and similarly if $b \circ c \neq 0$ then $((b \circ c) \triangleleft c) \geq b$. Therefore $(a \triangleright (a \circ b))((b \circ c) \triangleleft c) \neq 0$ and, since $\mathbf{A}$ is Euclidean, $(a \circ (b \circ c))((a \circ b) \circ c) \neq 0$. So the weakly Euclidean property holds for all atoms of $\mathbf{A}$, and by additivity it extends to all of $\mathbf{A}$.

(ii) Suppose $(a \triangleright x)(y \triangleleft c) \neq 0$ for some atoms $a, x, y, c \in A$. Then there exists an atom $b \in A$ such that $b \leq a \triangleright x$ and $b \leq y \triangleleft c$, or equivalently $x \leq a \circ b$ and $y \leq b \circ c$. By assumption $\circ$ is a partial operation, so we actually have $x = a \circ b$ and $y = b \circ c$. From the associative law it follows that $x \circ c = a \circ b \circ c = a \circ y$. Now $\mathbf{A}$ is assumed to be weakly Euclidean, hence $0 \neq a \circ b \circ c = (a \circ y)(x \circ c)$. By additivity the Euclidean inclusion holds for all of $\mathbf{A}$. $\square$

Note that the complex algebra of a small category is weakly Euclidean. From part (ii) of the preceding lemma we can therefore deduce the following result.

Corollary 3.17 *The residuated complex algebra of any small category is a Euclidean rm-algebra.*

R. D. Maddux [**78**] gives an example of a weakly associative relation algebra that is subdirectly irreducible but not simple. However this example is not associative, which led B. Jónsson to ask the question whether every subdirectly irreducible rm-algebra is simple. The example below shows that this is not the case, even restricted to Euclidean rm-algebras, hence the variety of all Euclidean rm-algebras is not a discriminator variety.

Recall that any quasi-order can be viewed as a category (the elements are the objects and the pairs $a \leq b$ are the morphisms). The Euclidean rm-algebras that arise in this way are in fact relative subalgebras of full relation algebras, relativized with respect to the quasi-order relation.

Theorem 3.18 *The complex algebra of the partial order of an infinite fence, viewed as a category, is a subdirectly irreducible* (*Euclidean*) *rm-algebra that is not simple.*

Proof. Let $\mathbb{Z}$ be the set of integers and denote by U the partial order of an infinite fence over $\mathbb{Z}$, i.e.,

$$U = \{(m, n) \in \mathbb{Z}^2 : m = n \text{ or } (m \text{ is even and } |m - n| = 1)\}.$$

Let $\mathbf{A}$ be the residuated complex algebra obtained from the structure $(U, \circ, id_{\mathbb{Z}})$, where $\circ$ is the partial operation

$$(m, n) \circ (p, q) = (m, q) \quad \text{if } n = p \text{ and } (m, q) \in U$$

and $id_{\mathbb{Z}} = \{(n,n) : n \in \mathbb{Z}\}$. By Corollary 3.17, $\mathbf{A}$ is a Euclidean rm-algebra. Note that for any $(m,n) \in U$ we have

$$1 \lhd (m,n) = \begin{cases} \{(m,m)\} & \text{if } m \neq n \text{ or } m \text{ is even} \\ \{(m,m),(m-1,m),(m+1,m)\} & \text{otherwise} \end{cases}$$

$$(m,n) \rhd 1 = \begin{cases} \{(n,n)\} & \text{if } m \neq n \text{ or } n \text{ is odd} \\ \{(n,n),(n,n-1),(n,n+1)\} & \text{otherwise.} \end{cases}$$

If we now let a_1 be any atom (m,n) and define $a_{2i} = (a_{2i-1} \rhd 1) \rhd 1$ and $a_{2i+1} = 1 \lhd (1 \lhd a_{2i})$ for $i = 1, 2, 3, \ldots$, then the a_i form an unbounded increasing sequence which eventually exceeds every atom.

If we denote the ideal of all finite joins of atoms of $\mathbf{A}$ by J, then it follows that any nontrivial congruence ideal of $\mathbf{A}$ must contain J. On the other hand J is also a congruence ideal, hence it is the smallest nontrivial congruence ideal of $\mathbf{A}$ and therefore $\mathbf{A}$ is subdirectly irreducible but not simple. $\square$

Corollary 3.19 *The variety ERM is not a discriminator variety.*

Embedding r-algebras into complex algebras of partial algebras. A structure $\mathbf{U} = (U, \circ, E)$ is a partial groupoid *with identities* if $(U, \circ)$ is a partial groupoid, $E \subseteq U$ and $a \circ E = a = E \circ a$. If such a subset E exists, then it is unique, and the complex algebra $\mathbf{U}^+$ is a ur-algebra.

Theorem 3.20

(i) *Every r-algebra can be embedded in the complex algebra of a partial groupoid.*

(ii) *Every integral r-algebra can be embedded in the complex algebra of a groupoid.*

(iii) *Every ur-algebra can be embedded in the complex algebra of a partial groupoid with identities.*

Moreover, if the algebra is finite then the (partial) groupoid can also be taken as finite.

Proof. (i) Since the variety of all r-algebras is canonical, it suffices to consider a complete and atomic r-algebra $\mathbf{A} = (\mathbf{A}_0, \circ, \rhd, \lhd)$. Let V be the set of all atoms of $\mathbf{A}_0$ and let $U = V \times V$. We want to define a partial binary operation $\circ$ on U such that the projection $U \twoheadrightarrow V$ onto the first coordinate is a bounded morphism from $(U, \circ, \circ_2, \circ_1)$ to $\mathbf{A}_+$, where $\circ$ is considered as a ternary relation R, and R_2, R_1 are as in the first section of Chapter II. From the duality between structures and complete and atomic BAOs it then follows that $\mathbf{A}$ is embedded in $(U, \circ)^{\oplus}$.

Since every set can occur as the carrier set of a group, there exist operations $*$ and $^{-1}$ on V such that $(V, *, ^{-1})$ is a group. We define a partial binary operation $\circ$ on U by

$$(a,b) \circ (c,d) = (b * d, b) \quad \text{if } b * d \leq a \circ c.$$

By definition $(a,b) \circ (c,d) \geq (u,v)$ implies $a \circ c \geq u$, and the same holds for $\rhd$ and $\lhd$.

Suppose $a \circ c \geq u$. We have to show that for any $v \in V$ there exist $b, d \in V$ such that $(a, b) \circ (c, d) = (u, v)$. This is true since we may take $b = v$ and let $d = b^{-1} * u$. Then $u = b * d \leq a \circ c$, hence $(a, b) \circ (c, d) = (u, v)$.

Now suppose $a \triangleright u \geq c$. We have to show that for any $d \in V$ there exist $b, v \in V$ such that $(a, b) \triangleright (u, v) \geq (c, d)$, or equivalently $(a, b) \circ (c, d) = (u, v)$. In this case we choose $b = u * d^{-1} = v$.

Finally, if $u \triangleleft c \geq a$ then, given any $b \in V$ we take $v = b$ and $d = b^{-1} * u$ to get $(u, v) \triangleleft (c, d) \geq (a, b)$.

(ii) The proof is a slight modification of the one above, where we now have to extend the operation $\circ$ to the whole set U^2. Since we are assuming that the r-algebra $\mathbf{A}$ is integral, there exists a choice function $f : V^2 \to V$ such that $f(a, b) \leq a \circ b$. Now we define a (total) binary operation $\circ$ on U by

$$(a, b) \circ (c, d) = \begin{cases} (b * d, b) & \text{if } b * d \leq a \circ c \\ (f(a, c), b) & \text{otherwise.} \end{cases}$$

The verification that the projection map is a bounded morphism is the same as before.

(iii) The idea of the proof is again the same as in the first part, except that atoms below the identity need not be split. We let $V = \mathbf{A}_{0+}$ and define $U = (\mathbf{A}_0 e)_+ \cup ((\mathbf{A}_0 e^-)_+ \times V)$. The map $h : U \twoheadrightarrow \mathbf{A}_{0+}$ is now given by $h((a, b)) = a$ and, for $u \in (\mathbf{A}_0 e)_+$, $h(u) = u$. Again we let $*$ and $^{-1}$ be operations on V such that $(V, *, ^{-1})$ is a group. The definition of the partial operation $\circ$ on U has to be expanded in the appropriate way: for $(a, b), (c, d) \in ((\mathbf{A}_0 e^-)_+ \times V)$ and $u, v \in (\mathbf{A}_0 e)_+$

$$\begin{aligned} (a, b) \circ (c, d) &= \begin{cases} b * d & \text{if } b * d \leq (a \circ c)e \\ (b * d, b) & \text{if } b * d \leq (a \circ c)e^- \end{cases} \\ u \circ (c, d) &= (c, d) \qquad\quad \text{if } u \circ c \neq 0 \\ (a, b) \circ v &= (a, b) \qquad\quad \text{if } a \circ v \neq 0 \\ u \circ v &= uv \end{aligned}$$

The verification that h is a bounded morphism now splits into several cases, but is as straight forward as before and will be omitted. □

Let $\mathcal{G}$, $\mathcal{G}_P$ and $\mathcal{G}_{Pe}$ be the class of all groupoids, all partial groupoids and all partial groupoids with identities. Using the notation $\mathcal{K}^a = \mathrm{Var}(\{\mathbf{U}^+ : \mathbf{U} \in \mathcal{K}\})$ from Chapter II, the above result shows that the variety of all integral r-algebras is $IR = \mathcal{G}^a$, the variety of all r-algebras is $(\mathcal{G}_P)^a$ and the variety of all ur-algebras is $UR = (\mathcal{G}_{Pe})^a$.

Decidable and undecidable varieties

Here we address the question which varieties in Figures 2, 3 and 4 have decidable equational theories. Tarski showed that the equational theory of any variety between RRA and RA is undecidable. A proof of this result was first published in Tarski and Givant [**87**], although the result itself dates back to the 40's. Maddux [**78**] extends this result, proving that any variety between RRA and SA is undecidable. On the other hand, the larger varieties WA and NA are shown to be decidable in Németi [**87**]. A proof of this last result is also included below as an application of the notion of a (strongly) β-closed variety from the second section of Chapter II. In the same way we show that the other 'nonassociative' varieties in Figures 3 and 4 are decidable. In particular the variety of all integral semiassociative relation algebras is decidable, since it is equal to the variety of all integral ur-algebras. However the variety *IRA* of all integral relation algebras is not decidable. In fact, S. Givant proved that if $\mathcal{V}$ is any subvariety of *RA* that contains the complex algebra of S_ω (the permutation group on ω) then $\mathcal{V}$ has an undecidable equational theory. Consequently any variety between *GRA* and *RA* is undecidable. In this section we prove the undecidability of any variety $\mathcal{V}$ that satisfies $TRA \subseteq \mathcal{V} \subseteq CRA$. A recent result of Andréka, Givant and Németi [**a**] shows that any subvariety of *CRA* that contains the complex algebra of an infinite Boolean group (or infinitely many nonisomorphic finite Boolean groups) is undecidable. Hence any variety between *BGRA* and *CRA* is undecidable.

For a ur-algebra $\mathbf{A}$, let $L(A) = \{x \in A : e \leq x = x \circ x\}$ be the set of all transitive elements $\geq e$.

Theorem 3.21 *Let* $\mathbf{A}$ *be a commutative* rm*-algebra. Then* $L(\mathbf{A}) = (L(A), \circ, \cdot, e)$ *is a lattice with least element* e*.*

Proof. Let $x, y \in L(A)$. Then

$$e \leq e \circ e \leq x \circ y = x \circ x \circ y \circ y = x \circ y \circ x \circ y \quad \text{and}$$

$$e = ee \leq xy = xy \circ e \leq xy \circ xy \leq (x \circ x)(y \circ y) = xy$$

hence $L(A)$ is closed under the operations $\circ$ and meet. Since both operations are clearly commutative, associative and idempotent we only have to check that the absorption laws hold:

$$(x \circ y)x \leq x = (x \circ e)x \leq (x \circ y)x \quad \text{and}$$

$$xy \circ x \leq x \circ x = x = e \circ x \leq xy \circ x.$$

$\square$

Conversely, given a lattice $L = (L, \vee, \wedge, e)$ with least element e, it is just as easy to see that $(L, \vee, \{e\})^\oplus$ is a commutative rm-algebra. However the following result, due to Maddux [**81**], is much more useful. Recall that a lattice is *modular* if it satisfies the identity $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ (or its dual).

Theorem 3.22 *For a simple commutative relation algebra* $\mathbf{A}$*, let* $M(A) = \{x \in A : e \leq x = x^\smile \circ x\}$ *be the set of all equivalence elements of* $\mathbf{A}$ *that include* e*. Then* $M(\mathbf{A}) = (M(A), \circ, \cdot, e)$ *is a modular lattice with least element* e*. Conversely, given any modular lattice* $\mathbf{M} = (M, \vee, \wedge, e)$ *with least element* e*, let*

$$R = \{(x, y, z) \in M^3 : x \vee y = x \vee z = y \vee z\}$$

and define $A(\mathbf{M}) = (M, R, \{e\})^\oplus$*. Then* $A(\mathbf{M})$ *is a simple totally symmetric relation algebra and* $MA(\mathbf{M})$ *is isomorphic to the ideal lattice* $I(\mathbf{M})$ *of* $\mathbf{M}$*.*

Proof. As for $L(\mathbf{A})$, it is easy to see that $M(\mathbf{A})$ is a lattice. The modular law follows from conjugation (Lemma 2.3) since

$$x(y \circ xz) \leq y(x \lhd xz) \circ xz \leq y(x \circ x^{\smile}) \circ xz \leq xy \circ xz.$$

For the converse, note that $A(\mathbf{M})$ is certainly a totally symmetric ur-algebra. To prove the associative law, consider $u \leq (x \circ y) \circ z$ for $u, x, y, z \in M$. Then there exists $v \in M$ such that $u \leq v \circ z$ and $v \leq x \circ y$, whence

$$u \vee v = u \vee z = v \vee z \quad \text{and} \quad v \vee x = v \vee y = x \vee y.$$

To show that $u \leq x \circ (y \circ z)$, we need to find $w \in M$ such that $u \leq x \circ w$ and $w \leq y \circ z$, or equivalently

$$u \vee x = u \vee w = x \vee w \quad \text{and} \quad w \vee y = w \vee z = y \vee z.$$

Using the modular identity it is not hard to show that $w = (u \vee x) \wedge (y \vee z)$ satisfies the requirements. For example

$$\begin{aligned} u \vee w &= u \vee (u \vee x) \wedge (y \vee z) = (u \vee x) \wedge (u \vee y \vee z) \\ &= (u \vee x) \wedge (u \vee v \vee x \vee y \vee z) = (u \vee x). \end{aligned}$$

It remains to check that $I(\mathbf{M})$ and $MA(\mathbf{M})$ are isomorphic. Let J be an ideal of $\mathbf{M}$. For any $x \in J \circ J$, there exist $y, z \in J$ such that $x \vee y = x \vee z = y \vee z$, so $x \leq y \vee z \in J$ and therefore J is an equivalence element. On the other hand if $J \in MA(\mathbf{M})$, then $x, y \in J$ implies

$$x \vee y \in \{x\} \circ \{y\} \subseteq J \circ J = J$$

and $z \leq x$ implies $z \in \{x\} \circ \{x\} \subseteq J$, hence J is an ideal of $\mathbf{M}$. Finally we observe that in both $I(\mathbf{M})$ and $MA(\mathbf{M})$ the meet is given by intersection, so the lattices are indeed isomorphic. $\square$

We will use this result to interpret the equational theory of modular lattices into any variety $\mathcal{V}$ of relation algebras that satisfies $TRA \subseteq \mathcal{V} \subseteq CRA$. Since the equational theory of modular lattices is undecidable (Freese [**80**], Herrmann [**84**]) it follows that all varieties between TRA and CRA are undecidable. Originally this idea was used to show that SRA is undecidable. S. Givant pointed out that the same argument is valid for the whole interval of varieties. Also, since relation algebras are discriminator algebras, the above result can already be deduced from the undecidability of the universal horn theory of modular lattices (Hutchinson [**73**], Lipshitz [**74**]).

Previously A. Urquhart [**84**] proved by a more direct approach (using von Neumann coordinatization) that certain relevance logics are undecidable. His methods can also be used to deduce the above undecidability results. For example Urquhart outlines a proof of the undecidability of the variety of all distributive lattice-ordered commutative semigroups. Though his techniques require somewhat more work than our proof, they are clearly more general and could probably be used to show that several other varieties of ur-algebras are undecidable.

Let ε be a lattice identity $p \leq q$ in the variables $x_0, \ldots, x_{n-1}$. The terms p and q are translated into relation algebra terms $\tilde{p}$ and $\tilde{q}$ by replacing every $\vee$ by $\circ$ and every $\wedge$ by $\cdot$. Let σ_ε be the universal closure of the formula

$$e \leq x_0 = x_0^{\smile} \circ x_0 \ \text{ and } \ \ldots \ \text{ and } \ e \leq x_{n-1} = x_{n-1}^{\smile} \circ x_{n-1} \quad \Rightarrow \quad \tilde{p} \leq \tilde{q}.$$

Lemma 3.23 *A lattice identity ε holds in a modular lattice $\mathbf{M}$ if and only if the corresponding universal sentence σ_ε holds in $A(\mathbf{M})$.*

Proof. The antecedent of σ_ε limits the variables to equivalence elements of $A(\mathbf{M})$, i.e., to elements of $MA(\mathbf{M})$. Since a lattice and its ideal lattice satisfy the same lattice identities, the statement follows from the isomorphism between $I(\mathbf{M})$ and $MA(\mathbf{M})$. □

Theorem 3.24 *Let $\mathcal{V}$ be a subvariety of CRA containing TRA. Then the equational theory of $\mathcal{V}$ is undecidable.*

Proof. Let ε be a lattice equation. Using the fact that *CRA* is a discriminator variety we translate the universal sentence σ_ε into an equation σ_ε^* as outlined at the end of the first section of Chapter II. Then σ_ε^* holds in $\mathcal{V}$ iff σ_ε holds in $\mathrm{Si}(\mathcal{V})$ iff ε holds in the variety of all modular lattices (since every modular lattice occurs as $M(\mathbf{A})$ for some $\mathbf{A} \in \mathrm{Si}(\mathcal{V})$ by Theorem 3.22). The latter variety is undecidable, hence it follows that $\mathcal{V}$ is also undecidable. □

Recall the definition of a β-closed variety from the second section of Chapter II. We now show that several of the finitely based varieties in Figure 2 are β-closed, hence generated by their finite members and decidable.

Lemma 3.25 *Let σ be one of the following sentences:*

(i) *$\circ$ is commutative.*

(ii) *$\circ$ is symmetric ($x \circ y = x \triangleright y$).*

(iii) *$\circ$ is integral ($x \circ y = 0$ implies $x = 0$ or $y = 0$).*

(iv) *$\circ$ is totally symmetric ($x \leq x \circ x$).*

(v) *$x \neq 0$ implies $\tau^n(x) = 1$.*

(vi) *e is an atom ($ex = 0$ or $ex^- = 0$).*

Suppose $\mathbf{A}$ is a ur-algebra that satisfies σ and $\mathbf{B}_0$ is any finite Boolean subalgebra of $\mathbf{A}_0$ such that $e \in B$. Then $\mathbf{B}^\beta$ is a ur-algebra and satisfies σ.

Proof. The condition that $e \in B$ ensures that $\mathbf{B}^\beta$ is a *ur*-algebra. Recall that $x \circ^\beta y = \beta(x \circ y) = \prod\{b \in B : x \circ y \leq b\}$. Therefore (i) and (ii) are clearly preserved. The key observation for (iii)-(v) is that $x \circ^{\mathbf{A}} y \leq x \circ^\beta y$ holds true for all $x, y \in B$. Finally, (vi) is obvious. □

Theorem 3.26 *Let $\mathcal{V}$ be a β-closed variety, and let $\mathcal{V}'$ be the subvariety generated by all members of $\mathcal{V}$ that satisfy one of the conditions* (i)–(vi) *of the preceding lemma. Then $\mathcal{V}'$ is also β-closed.*

Proof. For sentences (i)–(v) this follows immediately from the above lemma. For (vi), let $\mathbf{A} \in \mathrm{Si}(\mathcal{V}')$ and suppose S is a finite subset of A. Since $\mathcal{V}$ is β-closed, there exists a finite Boolean subalgebra $\mathbf{B}_0$ of $\mathbf{A}_0$ that contains $S \cup \{e\}$ and satisfies $\mathbf{B}^\beta \in \mathcal{V}$, so obviously $e \in B$ is an atom and therefore $\mathbf{B}^\beta \in \mathcal{V}'$. $\square$

Corollary 3.27 *If any of the prefixes A, I, C, S, T (see Table 1) are applied to a variety that is β-closed, then the resulting variety is again β-closed. So in particular AUR, IUR, CUR, SUR, TUR are generated by their finite members, and since they are also finitely based, they are decidable.*

A *ur*-algebra is *weakly associative* if it satisfies the equation $xe \circ 1 = (xe \circ 1) \circ 1$. The variety *WA* of all weakly associative relation algebras is the subvariety of *NA* defined by this equation. The decidability of *NA* and *WA* was first proved by I. Németi [**87**]. Theorem 3.29 shows that for a strongly β-closed variety, the subvariety of all weakly associative members is again β-closed.

Lemma 3.28 *Suppose $\mathbf{A} \in UR$ and $\mathbf{B}_0$ is a finite Boolean subalgebra of $\mathbf{A}_0$.*

(i) *If $\mathbf{A}$ is weakly associative and $xe \circ^{\mathbf{A}} 1 \in B$ for all $x \in B$ then $\mathbf{B}^\beta$ is also weakly associative.*

(ii) *If $\mathbf{A} \in NA$ and $x \triangleright^{\mathbf{A}} e \in B$ for all $x \in B$ then $\mathbf{B}^\beta \in NA$.*

Proof. (i) By assumption

$$(xe \circ^\beta 1) \circ^\beta 1 = (xe \circ^{\mathbf{A}} 1) \circ^\beta 1 = \prod\{b \in B : (xe \circ^{\mathbf{A}} 1) \circ^{\mathbf{A}} 1 \leq b\} = xe \circ^\beta 1.$$

since $\mathbf{A}$ is weakly associative. The proof of (ii) is similar. $\square$

The assumption that B contains all elements of the form $xe \circ^{\mathbf{A}} 1$ for $x \in B$ is of course critical, and we will see below how it can always be satisfied for the weakly associative algebras in a strongly β-closed variety (as opposed to *RA* or *SA* that are not generated by their finite members).

Theorem 3.29 *Suppose $\mathcal{V}$ is a strongly β-closed variety of ur-algebras. Then the following varieties are β-closed:*

(i) *the subvariety $W\mathcal{V}$ of all weakly associative members of $\mathcal{V}$,*

(ii) $\mathcal{V} \cap NA$,

(iii) $\mathcal{V} \cap WA$.

Proof. (i) Suppose $\mathbf{A} \in \mathrm{Si}(W\mathcal{V})$. For any finite subset S of A that contains all constants of $\mathbf{A}$, there exists a finite Boolean subalgebra $\mathbf{B}_0$ of $\mathbf{A}_0$ such that $\mathbf{B}^\beta \in \mathcal{V}$. Let $S' = S \cup \{ex \circ^{\mathbf{A}} e^- : x \in B\}$. Since the elements in $S \setminus S'$ are all below e^-, the subalgebra $\mathbf{B}_0'$ generated by S' satisfies the condition of Lemma 3.28 (i), hence $\mathbf{B}'^\beta \in \mathcal{V}'$. This shows $W\mathcal{V}$ is β-closed.

(ii) Let $\mathbf{A} \in NA$, and consider S and $\mathbf{B}_0$ as in (i) above. The set S' is defined as $S \cup \{x \rhd^{\mathbf{A}} e : x \in B\}$. Now the subalgebra $\mathbf{B}_0'$ generated by S' satisfies the conditions of Lemma 3.28 because the map $x \mapsto x \rhd e$ is an endomorphism in any nonassociative ur-algebra. This was first proved in Maddux [**82**], and also in Jónsson and Tsinakis [**91**].

(iii) The proof of this is a combination of (i) and (ii). The set S' is defined as $S \cup \{xe \circ^{\mathbf{A}} e^- : x \in B\} \cup \{x \rhd^{\mathbf{A}} e : x \in B\}$. □

Varieties of finite height

In this section we focus our attention on the bottom of the lattice Λ_{UR}. The variety $\mathcal{O} = \mathrm{Mod}(0 = 1)$ of all one-element ur-algebras is the smallest element of this lattice. Any finitely generated variety of ur-algebras can be thought of as 'close' to the bottom of Λ_{UR} since it follows from congruence distributivity and Jónsson's Lemma that such varieties have only finitely many subvarieties. Furthermore, by Baker's finite basis theorem they are definable by finitely many equations. Since the intersection of an infinite descending chain of varieties cannot be finitely based, it follows that any variety that properly contains a finitely based variety $\mathcal{V}$, also contains a variety that covers $\mathcal{V}$ in Λ_{UR}. Thus we may endeavor to find all covers of a finitely based variety. Assuming that we are working our way up from the bottom, it is enough to find all join irreducible covers of a finitely generated variety, since the remaining covers are, by distributivity, joins of varieties of smaller height. The following simple lemma helps us recognize join irreducible covers.

Lemma 3.30 *Let $\mathcal{V}$ be a subvariety of a congruence distributive variety $\mathcal{W}$. For $\mathbf{A} \in \mathrm{Si}(\mathcal{W})$, the following are equivalent:*

(i) $\mathrm{Var}(\mathbf{A})$ *is a (join irreducible) cover of* $\mathcal{V}$ *in* $\Lambda_{\mathcal{W}}$.

(ii) $\mathcal{V} \subsetneqq \mathrm{Var}(\mathbf{A})$ *and for all* $\mathbf{B} \in \mathbb{HSP}_u(\mathbf{A}) \setminus \mathcal{V}$ *we have* $\mathbf{A} \in \mathbb{HSP}_u(\mathbf{B})$.

If $\mathbf{A}$ *is finite then* (ii) *can be simplified to:* $\mathcal{V} \subsetneqq \mathrm{Var}(\mathbf{A})$ *and for all* $\mathbf{B} \in \mathbb{HS}(\mathbf{A}) \setminus \mathcal{V}$ *we have* $\mathbf{A} \cong \mathbf{B}$.

Proof. (i)⇒(ii) Suppose $\mathrm{Var}(\mathbf{A})$ covers $\mathcal{V}$. Then $\mathcal{V}$ is obviously a proper subvariety of $\mathrm{Var}(\mathbf{A})$, and if $\mathbf{B} \in \mathbb{HSP}_u(\mathbf{A}) \setminus \mathcal{V}$ then $\mathcal{V} \subsetneqq \mathrm{Var}(\mathbf{B}) \subseteq \mathrm{Var}(\mathbf{A})$, whence $\mathrm{Var}(\mathbf{B}) = \mathrm{Var}(\mathbf{A})$. By Jónsson's Lemma $\mathbf{A} \in \mathbb{HSP}_u(\mathbf{B})$.

(ii)⇒(i) Suppose (ii) holds, let $\mathcal{V} \subsetneqq \mathcal{V}' \subseteq \mathrm{Var}(\mathbf{A})$ and consider $\mathbf{B} \in \mathrm{Si}(\mathcal{V}') \setminus \mathcal{V}$. Again it follows from Jónsson's Lemma that $\mathbf{B} \in \mathbb{HSP}_u(\mathbf{A})$, hence (ii) implies $\mathbf{A} \in \mathbb{HSP}_u(\mathbf{B})$ and therefore $\mathrm{Var}(\mathbf{A}) \subseteq \mathcal{V}'$. □

There are some obstacles to the approach of analysing the bottom of Λ_{UR} by looking for the covers small varieties. A finitely generated variety may have infinitely many covers, or some of the covers may not be finitely based. In the first case it is often impossible to describe all the covers (they may form a nonrecursive or an uncountable set) and in the second case the structure of Λ_{UR} above nonfinitely based varieties is no longer discrete, in the sense that such varieties are intersections of infinite descending chains of varieties.

It is easy to find infinitely many covers of $\mathcal{O}$ in Λ_{UR}. The example below shows this is also true in Λ_{CERM}. Let $\mathbf{M}_n$ be the monoid $(n, +_n, 0)$, where $n = \{0, 1, \ldots, n-1\}$ and $p +_n q = \min\{p + q, n - 1\}$ is truncated addition in n.

Theorem 3.31 *For each $n \in \omega$*

(i) $\mathbf{M}_n^{\oplus} \in CERM$ *and*

(ii) $\mathrm{Var}(\mathbf{M}_n^{\oplus})$ *is a cover of* $\mathcal{O}$.

Proof. (i) $\mathbf{M}_n^{\oplus}$ is obviously in CRM, and the Euclidean identity holds by Corollary 3.17.

(ii) $\mathbf{M}_n^{\oplus}$ is integral, hence simple. For $n > 0$, the term $e^-(e^- \circ e^-) = \{1\}$ generates the whole algebra, so $\mathbf{M}_n^{\oplus}$ has no proper subalgebras. By Lemma 3.30 it generates a variety that covers $\mathcal{O}$. □

However Jónsson and Tarski [**52**] proved that in Λ_{RA} there are exactly 3 atoms, generated by the algebras in Table 3. Andréka, Jónsson and Németi [**91**] point out that the proof given there requires only the semiassociative identity, so the same result holds in Λ_{SA}.

Theorem 3.32 *Let ε_1, ε_2 and ε_3 be the equations $e = 1$, $e^- \circ e^- = e$ and $e^- \circ e^- = 1$ respectively.*

(i) *Every $\mathbf{A} \in SA$ can be decomposed uniquely (up to isomorphism) into direct factors $\mathbf{A}_{\varepsilon_1}$, $\mathbf{A}_{\varepsilon_2}$ and $\mathbf{A}_{\varepsilon_3}$, such that $\mathbf{A}_{\varepsilon_i} \models \varepsilon_i$ $(i = 1, 2, 3)$.*

(ii) *For any $\mathbf{A} \in \mathrm{Si}(SA)$, the constants $\{0, e, e^-, 1\}$ form a subalgebra of $\mathbf{A}$. The three possible subalgebras are denoted by $\mathbf{A}_1$, $\mathbf{A}_2$ or $\mathbf{A}_3$ with the indices chosen so that $\mathbf{A}_i \models \varepsilon_i$ $(i = 1, 2, 3$, see Table 3).*

(iii) $\mathrm{Var}(\mathbf{A}_1) = \mathrm{Mod}(\varepsilon_1)$ *has no join irreducible covers in* Λ_{SA}.

(iv) $\mathrm{Var}(\mathbf{A}_2)$ *has exactly one join irreducible cover* $\mathrm{Mod}(\varepsilon_2) = \mathrm{Var}(\mathbf{Re}(2))$ *in* Λ_{SA}.

(v) $\mathrm{Var}(\mathbf{A}_3)$ *has infinitely many join irreducible covers in* Λ_{SA}.

As mentioned above, (i), (ii) and (iii) are essentially in Jónsson and Tarski [**52**]. The key result is of course the decomposition in (i), and then (ii) and (iii) are easy consequences. By Lemma 3.6 (iv), the members of $\mathrm{Mod}(\varepsilon_1)$ are Boolean algebras in disguise, with $xy = x \circ y = x \triangleright y = x \triangleleft y$.

(iv) was first proved by R. D. Maddux and, independently, with our implementation of the algorithm from the last section of Chapter II. Another proof can be found in Andréka, Jónsson and Németi [**91**].

(v) is due to E. Lukács. The examples she gives are in fact symmetric, so the same result holds in Λ_{ISUR}.

The following result, due to R. D. Maddux [**90**], shows that there are only finitely many covers generated by nonintegral semiassociative relation algebras.

Theorem 3.33 *If* $\mathbf{A} \in \mathrm{Si}(SA)$ *is nonintegral and has more than* 4 *elements then* $\mathbf{A}$ *has a subalgebra isomorphic to* $\mathbf{N}_1,\ \mathbf{N}_2,\ \mathbf{N}_3,\ \mathbf{N}_4,\ \mathbf{B}_2$ *or* $\mathbf{B}_4$ (*see Table* 4, 6).

For Λ_{RA} and Λ_{SRA} the problem of whether $\mathrm{Var}(\mathbf{A}_3)$ has finitely or infinitely many covers is still undecided. In Jipsen and Lukács [**a**] it is shown that there are only finitely many finitely generated covers that are subvarieties of the variety *TRA* of all *totally* symmetric relation algebras (defined relative to *SRA* by the equation $x \le x \circ x$). This result is extended here to the variety *NRA* of *neat* symmetric relation algebras, which is generated by all simple symmetric relation algebras that satisfy the universal sentence $x \le x \circ x$ or $x(x \circ x) = 0$.

A simple way to describe small relation algebras is as subalgebras of complex algebras of groups, if they have such a representation. In order to list the known relation algebras that generate covers of $\mathrm{Var}(\mathbf{A}_3)$, we also introduce the following notation. For a group $\mathbf{G}$, let

$$S(\mathbf{G}) = \{x + x^{\smile} : x \in \mathbf{G}^+\} \quad \text{and} \quad x^s = x + x^{\smile}.$$

It is easy to see that if $\mathbf{G}$ is abelian then $S(\mathbf{G})$ is in fact a subalgebra of $\mathbf{G}^+$. Table 4 is a list of the known simple symmetric relation algebras that generate varieties covering $\mathrm{Var}(\mathbf{A}_3)$. Andréka and Maddux [**88**] show that the finite representations given here are minimal, except in the case of $\mathbf{B}_7$, which they show has a smallest representation as a subalgebra of $S(\mathbb{Z}_3^2)$, with atoms $\{(0,1),(1,0)\}^s$, $\{(1,1),(1,2)\}^s$ and $\{(0,0)\}$. For completeness we also include a list of the 5 known integral nonsymmetric relation algebras that generate covers of $\mathrm{Var}(\mathbf{A}_3)$ (Table 5). Figure 5 shows the position of the join irreducible varieties generated by these algebras in Λ_{RA}.

Recall that an element u in a relation algebra $\mathbf{A}$ is an equivalence element if $u \circ u = u$ and $u^{\smile} = u$. We say that an equivalence element u is nontrivial if $e < u < 1$.

Theorem 3.34(Jónsson [**88**]) *If* $\mathbf{A}$ *is a relation algebra and* u *is an equivalence element in* $\mathbf{A}$ *then* u *generates a finite subalgebra of* $\mathbf{A}$. *If* $\mathbf{A}$ *is integral and* u *is a nontrivial equivalence element then* u *generates a subalgebra isomorphic to* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ *or* $\mathbf{B}_4$.

Lemma 3.35 *Let* $\mathbf{A}$ *be a relation algebra. For any* $a \in A$, $(a\backslash a)(a\backslash a)^{\smile}$ *is an equivalence element containing* e.

Proof. In any *ur*-algebra $e \le a\backslash a$, and in any *rm*-algebra $a\backslash a$ is a transitive element. In a relation algebra the intersection of a transitive element with its converse gives an equivalence element. □

In the subsequent results we frequently make use of the fact that every simple symmetric relation algebra is integral. The next four lemmas appeared in Jipsen and Lukács [**a**]. For completeness they are included here as well.

Lemma 3.36 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra. Then the following are equivalent:*

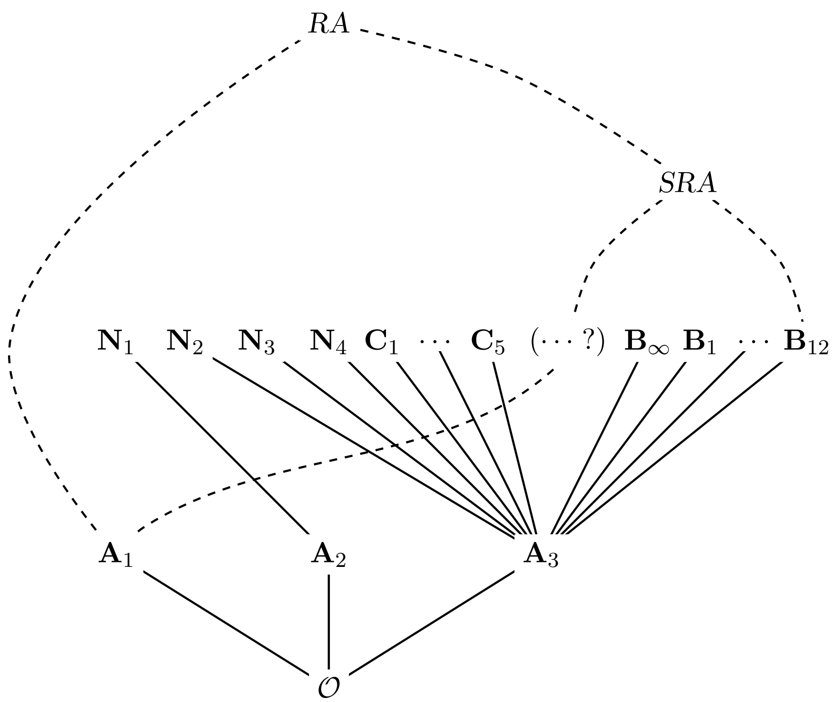

Figure 5: Algebras that generate join irreducible varieties at the bottom of Λ_{RA}

Table 3: Algebras that generate the three atoms of Λ_{SA}

Name - Representation	$e^- \circ e^-$
$\mathbf{A}_1 = \mathbb{Z}_1^+ = \mathbf{Re}(1)$	0
$\mathbf{A}_2 = \mathbb{Z}_2^+ < \mathbf{Re}(2)$	e
$\mathbf{A}_3 < \mathbb{Z}_3^+ < \mathbf{Re}(3)$	1

Table 4: Known symmetric algebras that generate covers of Var($\mathbf{A}_3$)

Name - Repr.	a	b	$a \circ a$	$b \circ b$	$a \circ b$
$\mathbf{B}_1 = S(\mathbb{Z}_4)$	$\{1\}^s$	$\{2\}$	$b+e$	e	a
$\mathbf{B}_2 < S(\mathbb{Z}_6)$	$\{1,3\}^s$	$\{2\}^s$	$b+e$	$b+e$	a
$\mathbf{B}_3 < S(\mathbb{Z}_6)$	$\{1,2\}^s$	$\{3\}$	1	e	a
$\mathbf{B}_4 < S(\mathbb{Z}_9)$	$\{1,2,4\}^s$	$\{3\}^s$	1	$b+e$	a
$\mathbf{B}_5 = S(\mathbb{Z}_5)$	$\{1\}^s$	$\{2\}^s$	$b+e$	$a+e$	e^-
$\mathbf{B}_6 < S(\mathbb{Z}_8)$	$\{2,3\}^s$	$\{1,4\}^s$	1	$a+e$	e^-
$\mathbf{B}_7 < S(\mathbb{Z}_{12})$	$\{1,2,5\}^s$	$\{3,4,6\}^s$	1	1	e^-

Name - Repr.	$a \circ a$	$b \circ b$	$c \circ c$	$a \circ b$	$a \circ c$	$b \circ c$
$\mathbf{B}_8 < S(\mathbb{Z})$	1	c^-	$a+e$	e^-	e^-	a
$\mathbf{B}_9 = S(\mathbb{Z}_7)$	$b+e$	$c+e$	$a+e$	$a+c$	$b+c$	$a+b$
$\mathbf{B}_{10}$ Nonrepr.	c^-	$c+e$	$a+e$	$a+c$	$b+c$	$a+b$
$\mathbf{B}_{11}$ Nonrepr.	c^-	a^-	$a+e$	$a+c$	$b+c$	$a+b$
$\mathbf{B}_{12}$ Nonrepr.	c^-	a^-	b^-	$a+c$	$b+c$	$a+b$

Name - Repr.	generator x	atoms a_n	$a_n \circ a_m$
$\mathbf{B}_\infty < S(\mathbb{Z} \times \mathbb{Z})$	$(\{(1,0),(0,1)\}^s)^2$	$x^n(x^{n-1})^-$	$\sum_{i=\lvert n-m\rvert}^{n+m} a_i$

$$x^s = x + x^\smile, \quad S(G) = \{x + x^\smile : x \in G^+\}$$

Table 5: Known nonsymmetric algebras that generate covers of Var($\mathbf{A}_3$)

Name - Repr.	a	$a \circ a$	$a \circ a^\smile$	$a^\smile \circ a$
$\mathbf{C}_1 < \mathbb{Z}_7^+$	$\{1,2,-3\}$	e^-	1	1
$\mathbf{C}_2 < \mathbb{Q}^+$	$\{q \in \mathbb{Q} : q > 0\}$	a	1	1
$\mathbf{C}_3 = \mathbb{Z}_3^+$	$\{1\}$	$a^\smile$	e	e

Name	$a \circ a$	$a \circ a^\smile$	$a^\smile \circ a$	$b \circ b$	$a \circ b$	$a^\smile \circ b$
$\mathbf{C}_4$	a	1	b^-	1	b	$a^\smile + b$

Name	$a \circ a$	$a \circ a^\smile$	$a^\smile \circ a$	$b \circ b$	$a \circ b$	$a^\smile \circ b$	$c \circ c$	$a \circ c$	$a^\smile \circ c$	$b \circ c$
$\mathbf{C}_5$	e^-	1	1	$(b+c)^-$	e^-	c^-e^-	b^-	b^-e^-	e^-	a

Table 6: Nonintegral algebras that generate varieties of height 2 in Λ_{SA}

Name - Repr.	e_1	e_2
$\mathbf{N}_1 = \mathbf{Re}(2)$	$\{(0,0)\}$	$\{(1,1)\}$
$\mathbf{N}_2 < \mathbf{Re}(3)$	$\{(0,0)\}$	$\{(1,1),(2,2)\}$
$\mathbf{N}_3 < \mathbf{Re}(4)$	$\{(0,0)\}$	$\{(1,1),(2,2),(3,3)\}$
$\mathbf{N}_4 < \mathbf{Re}(5)$	$\{(0,0),(1,1)\}$	$\{(2,2),(3,3),(4,4)\}$

e_1, e_2, c, $c^{\smile}$, d_1 and d_2 are atoms or zero, where
$e = e_1 + e_2$, $c = e_1 \circ 1 \circ e_2$ and $d_i = (e_i \circ 1 \circ e_i)e^-$

(i) $\mathbf{A}$ *has no nontrivial equivalence elements,*

(ii) *for any* $x \in A$, $0 < x < e^-$ *implies* $x \circ x^- e^- = e^-$,

(iii) $\mathbf{A}$ *has no subalgebra isomorphic to* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ *or* $\mathbf{B}_4$.

Proof. Assume (i) holds. Let $x \in A$ satisfy $0 < x < e^-$ and define $y = x^- e^-$. By Lemma 3.35 and symmetry, $x \backslash x = (x \circ x^-)^-$ is an equivalence element including e. Since $\mathbf{A}$ is integral, $x \circ x^- \neq 0$ hence $(x \circ x^-)^- \neq 1$. We are assuming that $\mathbf{A}$ has no nontrivial equivalence elements, so we have $(x \circ x^-)^- = e$, thus $x \circ x^- = e^-$. Now $e^- = x \circ x^- = x \circ (y + e) = x \circ y + x$ implies that $x \circ y \geq y$. Interchanging x and y in the above argument, we also get that $x \circ y \geq x$. Therefore $x \circ y \geq e^-$ and, since $xy = 0$, we have $(x \circ y)e = 0$ whence $x \circ y = e^-$.

Now suppose (i) fails and let $u \in A$ be a nontrivial equivalence element. Since $\mathbf{A}$ is integral, $e \leq u$. Then $0 < ue^- < e^-$, and $(ue^- \circ u)u^- \leq uu^- = 0$ implies $(ue^- \circ u^-)u = 0$, hence $ue^- \circ u^- \leq u^- < e^-$.

The equivalence of (i) and (iii) follows from Theorem 3.34. □

Lemma 3.37 *If* $\mathbf{A}$ *is a simple symmetric relation algebra that has no nontrivial equivalence elements then* $x \circ x + x^- \circ x^- = 1$ *for every element* $0 < x < e^-$.

Proof. Let $u = (x \circ x + x^- \circ x^-)^-$. Since $(x \circ x)u = 0$, it follows that $(x \circ u)x = 0$ and hence $x \circ u \leq x^-$. Now $(x^- \circ x \circ u)u \leq (x^- \circ x^-)u = 0$, which implies that $(u \circ u)(x \circ x^-) = 0$. By Lemma 3.36 we have $e^- \leq x \circ x^-$ and therefore $u \circ u \leq e$. As a result $u + e$ is an equivalence element and, since we are assuming that all equivalence elements are trivial, $u + e = 1$ or $u + e = e$. However $u \leq e^-$, so the first case implies $u = e^-$ and hence $e^- \circ e^- = u \circ u \leq e$. But then

$$x \circ 1 = x \circ e^- + x \circ e \leq e^- \circ e^- + x \leq e + x < 1$$

contradicts the assumption that $\mathbf{A}$ is simple. Therefore $u + e = e$, which implies $u = 0$ and hence $x \circ x + x^- \circ x^- = 1$. □

Lemma 3.38 *For any* x *in a simple symmetric relation algebra,* $x \circ x + x^- \circ x^- = 1$ *implies* $x \circ x = 1$ *or* $x^- \circ x^- = 1$.

Proof. Suppose $x^- \circ x^- \neq 1$ and let $z = (x^- \circ x^-)^-$. If $x \circ x + x^- \circ x^- = 1$ then we have $z \leq x \circ x$, hence $(x \circ x)z \neq 0$ and therefore $(z \circ x)x \neq 0$. It follows that

$$1 = (z \circ x)x \circ 1 = (z \circ x)x \circ x + (z \circ x)x \circ x^- \leq x \circ x + z \circ x \circ x^-.$$

Now $z(x^- \circ x^-) = 0$ implies $(z \circ x^-)x^- = 0$ whence $z \circ x^- \leq x$, so $1 \leq x \circ x + z \circ x \circ x^- \leq x \circ x$. □

Theorem 3.39 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra. If* $x \circ x < 1$ *and* $x^- e^- \circ x^- e^- < 1$ *for some* $0 < x < e^-$ *then* $\mathbf{A}$ *has a subalgebra isomorphic to* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$ *or* $\mathbf{A} \cong \mathbf{B}_5$.

Proof. Let $0 < x < e^-$, $y = x^- e^-$ and suppose $\mathbf{A}$ does not have a subalgebra isomorphic to $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$. We will show that if $x \circ x < 1$ and $y \circ y < 1$ then x and y are in fact atoms. The result then follows since $\mathbf{B}_5$ is the only 8 element simple symmetric relation algebra that satisfies the preceding conditions.

By Lemmas 3.37 and 3.38, we have that $1 = x^- \circ x^- = (y + e) \circ (y + e) = y \circ y + y + e = y \circ y + y$, and similarly $1 = x \circ x + x$. The meet of the two equations gives that $x \circ x + y \circ y = 1$ and we also obtain that $0 \neq (x \circ x)^- \leq x$ and $0 \neq (y \circ y)^- \leq y$. Let us assume that one of x and y, say x, is not an atom. Then there exist disjoint nonzero elements x_1, x_2 such that $x = x_1 + x_2$ and $x_1 \leq (x \circ x)^-$. This implies $x_1(x \circ x) = 0$ and consequently $x \circ x_1 \leq y + e$. In particular, $x_2 \circ x_1 \leq e^-(y + e) = y$ hence

$$x_1 \circ x_1 \leq x_1 \circ e \circ x_1 \leq x_1 \circ x_2 \circ x_2 \circ x_1 \leq y \circ y$$

and similarly $x_2 \circ x_2 \leq y \circ y$. Therefore $(y \circ y)^- \leq x \circ x = x_1 \circ x_1 + x_2 \circ x_2 + x_1 \circ x_2$ implies that $(y \circ y)^- \leq x_1 \circ x_2$. It follows that $(x_1 \circ x_2)(y \circ y)^- \neq 0$ and consequently $x_1((y \circ y)^- \circ x_2) \neq 0$. Let $u = x_1((y \circ y)^- \circ x_2)$. We will show that $u + e$ is a nontrivial equivalence element, thus reaching a contradiction.

Certainly $(u \circ u)x \leq (x_1 \circ x_1)x = 0$ by the choice of x_1, so $u \circ u \leq e + y$. But $(u \circ u)y \leq (((y \circ y)^- \circ x_2) \circ u)y \leq ((y \circ y)^- \circ x_2 \circ x_1)y \leq ((y \circ y)^- \circ y)y = 0$, since $(y \circ y)(y \circ y)^- = 0$. This shows that $u \circ u = e$ thus $(u + e)^2 = u + e$, so by symmetry, $u + e$ is indeed an equivalence element. □

Corollary 3.40 *If* $\mathbf{A}$ *is a simple symmetric relation algebra that has no subalgebra isomorphic to* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_5$ *then for all* $x \in A$

$$x \circ x < 1 \quad \textit{implies} \quad x^- e^- \circ x^- e^- = 1.$$

Theorem 3.41 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra and suppose* a, b, c *are disjoint nonzero elements that join to* e^-. *If* $0 = a(b \circ b) = b(c \circ c) = c(a \circ a)$ *then* $\mathbf{A}$ *has a subalgebra isomorphic to* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$ *or* $\mathbf{B}_{12}$.

Proof. Since $\mathbf{A}$ is simple and symmetric, we always have $e \leq x \circ x$ for $x \neq 0$ and $x \circ y \leq e^-$ for disjoint $x, y \in \mathbf{A}$. Suppose $\mathbf{A}$ has no subalgebra isomorphic to $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$. By

Lemma 3.36 $e^- \leq a \circ (b + c)$ and, since $a(a \circ c) = 0$ and $b(a \circ b) = 0$, we obtain $a \leq a \circ b$ and $b \leq a \circ c$. From associativity and $c(b \circ c) = 0$ it follows that

$$b \leq (a \circ b) \circ c = a \circ (b \circ c) \leq a \circ (a + b).$$

Since $b(a \circ b) = 0$ we conclude that $b \leq a \circ a$. By analogous arguments, permuting a, b, c cyclically, we obtain $c \leq b \circ a$, $c \leq b \circ b$, $a \leq c \circ b$ and $a \leq c \circ c$.

We now show that either $a \leq a \circ a$ or $a(a \circ a) = 0$. To that end, suppose $a = a_1 + a_2$, $a_1 \leq a \circ a$, $a_2(a \circ a) = 0$ and $a_1 \neq 0$. Then $a_2(a_1 \circ b^-) = 0$ and, by simplicity, $a_2 \leq a_1 \circ 1 = a_1 \circ b + a_1 \circ b^-$, whence $a_2 \leq a_1 \circ b$. On the other hand, from $a_1 \leq a \circ a$ and $a \circ b \leq a + c$ we get

$$a_2(a_1 \circ b) \leq a_2(a \circ a \circ b) \leq a_2(a \circ (a + c)) = 0.$$

So a_2 is both below and disjoint from $a_1 \circ b$ and therefore must equal 0. Again, relabelling a, b, c cyclically, we obtain that $b \leq b \circ b$ or $b(b \circ b) = 0$, and $c \leq c \circ c$ or $c(c \circ c) = 0$. Therefore $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$ or $\mathbf{B}_{12}$ (see Table 4). $\square$

A simple relation algebra $\mathbf{A}$ is *neat* if either $x \leq x \circ x$ or $x(x \circ x) = 0$ for all $x \in A$. The next result is an observation that will make the proof of Theorem 3.43 proceed somewhat more smoothly. It also gives an indication why small subalgebras are easier to find in neat symmetric relation algebras than in arbitrary symmetric relation algebras.

Lemma 3.42 *Let $\mathbf{A}$ be a simple neat symmetric relation algebra. If there exists an element $x < e^-$ such that $x^- \leq x \circ x < 1$ then $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_5$ or $\mathbf{B}_6$.*

Proof. Suppose $\mathbf{A}$ does not have a subalgebra isomorphic to $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$ and let $y = x^- e^-$. Then x and y are nonzero and Lemma 3.36 implies that $x \circ y = e^-$. By assumption $x \not\leq x \circ x$, so $x(x \circ x) = 0$ since $\mathbf{A}$ is neat. Lemma 3.39 implies that $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_5$ or $y \circ y = 1$. In the latter case however $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_6$. $\square$

We now prove the theorem from which our main result about finitely generated covers of $\mathrm{Var}(\mathbf{A}_3)$ follows. Note that for a finite *ur*-algebra $\mathbf{A}$ we can always find an element u such that u is minimal with respect to the condition $u \circ u = 1$.

Theorem 3.43 *Let $\mathbf{A}$ be a simple neat symmetric relation algebra with more than 4 elements. If there exists an element $u \in \mathbf{A}$ such that $u \circ u = 1$ and $x \circ x < 1$ for all $x < u$, then $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_7$, $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$ or $\mathbf{B}_{12}$.*

Proof. Suppose $\mathbf{A}$ and u are as in the statement of the theorem and assume $\mathbf{A}$ has no subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_7$. By Lemma 3.36 it follows that $x \circ x^- e^- = e^-$ for any nonzero $x < e^-$. Since we are assuming that $\mathbf{A}$ has more than 4 elements, Corollary 3.40 implies that $u < e^-$. Let $a = u^- e^-$, and observe that $a \leq a \circ a$ or $a(a \circ a) = 0$ because $\mathbf{A}$ is neat. If $u \leq a \circ a$ then $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_6$ or $\mathbf{B}_7$, so we conclude that $u(a \circ a)^- \neq 0$. Let $u = b + c$ where $b \leq a \circ a$ and $c(a \circ a) = 0$. Then $c \neq 0$ and, since $a(u \circ a) = ae^- \neq 0$, we also have $b \neq 0$. Note that $c(a \circ a) = 0$ and $a \leq e^- = (a + b) \circ c$ imply

$a \leq b \circ c$. We now prove a series of claims that show $\mathbf{A}$ must have a subalgebra isomorphic to $\mathbf{B}_7$, $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$, $\mathbf{B}_{12}$.

Claim: $a(c \circ c) \neq 0$. Suppose to the contrary that a is disjoint from $c \circ c$. Then $a(c \circ c) = 0$ and $c(a \circ a) = 0$, hence $(a + c)(a \circ c) = 0$ or equivalently $a \circ c \leq b$. Together with $a + c \leq e^- = (a + b) \circ c$, additivity implies $a + c \leq b \circ c$. Similarly $a + c \leq (a + b) \circ c$ implies $a + c \leq b \circ c$. Moreover, since $b < u$, we have $b \circ b < 1$ and so it follows from Corollary 3.40 that $(a + c) \circ (a + c) = 1$. Now $a \leq (a + c) \circ (a + c)$ and $a((a + c) \circ c) = 0$ imply

$$a \leq (a + c) \circ a \leq (b \circ c) \circ a = b \circ (c \circ a) \leq b \circ b.$$

Similarly $e^- = a \circ (b + c)$ implies $a + c \leq a \circ b$, and $c \leq (a + c) \circ (a + c)$ implies $c \leq c \circ (a + c)$, hence

$$c \leq c \circ (a + c) \leq c \circ (a \circ b) = (c \circ a) \circ b \leq b \circ b.$$

Finally, $e + b \leq a \circ a \leq (b \circ c) \circ a = b \circ (c \circ a) \leq b \circ b$. So now $b \circ b = 1$, contradicting the assumption about u.

Claim: $a \leq c \circ c$. Suppose $a = a_1 + a_2$, $a_1 \leq c \circ c$ and $a_2(c \circ c) = 0$. By the previous claim $a_1 \neq 0$. We show that the assumption $a_2 \neq 0$ leads to a contradiction. Since $\mathbf{A}$ is neat, either $a_2 \leq a_2 \circ a_2$ or $a_2(a_2 \circ a_2) = 0$. Suppose first that $a_2 \leq a_2 \circ a_2$, and note that $c \circ a_2 \leq b$. Therefore

$$e + a_2 \leq a_2 \circ a_2 \leq (b \circ c) \circ a_2 = b \circ (c \circ a_2) \leq b \circ b.$$

The assumption that $\mathbf{A}$ is neat further implies $c \leq (a_2 + c)^2$ since $(a_2 + c)(a_2 + c)^2 \neq 0$. On the other hand $c(a \circ a) = 0 = c(c \circ a_2)$, hence by additivity $c \leq c \circ c$. Also $c \leq e^- = a_2 \circ (u + a_1)$ and therefore $c \leq a_2 \circ b$. Now

$$a_1 + c \leq c \circ c \leq (b \circ a_2) \circ c = b \circ (a_2 \circ c) \leq b \circ b.$$

Since $b \circ b < 1$ and $\mathbf{A}$ is neat, this forces $b(b \circ b) = 0$. But from $a_2 \leq a_2 \circ a_2$ we may conclude that $a_2 + b \leq (a_2 + b)^2 = a_2 \circ a_2 + a_2 \circ b + b \circ b$ and therefore $b \leq a_2 \circ a_2 + a_2 \circ b = (a_2 + b) \circ a_2$. Furthermore

$$b \leq a \circ a \leq (b \circ c) \circ a = b \circ (c \circ a) \leq b \circ (b + c)$$

and $b(b \circ b) = 0$ imply $b \leq b \circ c$. Therefore

$$b \leq (a_2 + b) \circ a_2 \leq (b \circ c) \circ a_2 = b \circ (c \circ a_2) \leq b \circ b$$

which contradicts $b(b \circ b) = 0$.

So now we may assume that $a_2(a_2 \circ a_2) = 0$. Then $a_2(a_2 + c)^2 = 0$ and, since $\mathbf{A}$ is neat, $c(a_2 + c)^2 = 0$. Now let $b = b_1 + b_2$ where $b_1 \leq (a_2 + c)^2$ and $b_2(a_2 + c)^2 = 0$. From $a_2 \leq b \circ c$ we infer that $b(a_2 \circ c) \neq 0$ and hence $b_1 \neq 0$. If $b_2 = 0$ then $b = b_2 \leq (a_2 + c)^2$. But then $x = a_2 + c$ would satisfy the condition of Lemma 3.42, contradicting the assumption that $\mathbf{A}$ does not have a subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_7$. Therefore $b_2 \neq 0$.

Since $c(a \circ a) = 0$ and $(b_2 + a_2 + c)(a_2 + c)^2 = 0$ we have $b_1^-(a_2 \circ c) = 0$ and hence $a_2 \circ c \leq b_1$. This fact will be used several times below. Suppose first that $b_1(b_1 \circ b_1) = 0$. Then $e \leq c \circ c$ implies

$$b_1(a_2 \circ a_2) \leq b_1(a_2 \circ e \circ a_2) \leq b_1(a_2 \circ c \circ c \circ a_2) \leq b_1(b_1 \circ b_1) = 0$$

and therefore $b(a_2 \circ a_2) = 0$. Since $b \le a \circ a = a_1 \circ a_1 + a_1 \circ a_2 + a_2 \circ a_2$, we obtain

$$b = \le a_1 \circ a \le (c \circ c) \circ a = c \circ (c \circ a) \le c \circ (b + c).$$

Similarly, since we are assuming that $a_2 \neq 0$, $e \le a_2 \circ a_2$ implies

$$b_1(c \circ c) \le b_1(c \circ a_2 \circ a_2 \circ c) \le b_1(b_1 \circ b_1) = 0,$$

whence $b(c \circ c) = 0$. From the previous calculation $b \le c \circ b + c \circ c$, so we conclude that $b \le c \circ b$. Earlier we noted that $a \le c \circ b$ and $b_1^-(a_2 \circ c) = 0$, hence

$$b_1^- = b_1^-(a_2 \circ 1) \le a_2 \circ (a + b) \le a_2 \circ (c \circ b) = (a_2 \circ c) \circ b \le b \circ b.$$

Since $\mathbf{A}$ is neat, $b_2 \le b \circ b$ implies $b \le b \circ b$. But then $b \circ b = 1$, contradicting our assumption about u.

So we conclude that $b_1 \le b_1 \circ b_1$. From neatness we get $b_1 + c \le (b_1 + c)^2$ and, since $a_2(c \circ b_1^-) = 0$, it follows that $a_2 = a_2(c \circ 1) \le c \circ b_1$. By assumption $a_1 \le c \circ c$, so now we have $b_2^- \le (b_1 + c)^2$. Let $b_2 = r + s$, $r \le (b_1 + c)^2$ and $s(b_1 + c)^2 = 0$. If $r \neq 0$ then $b_1 + c + s < u$ and $s \le (b_1 + c + s)^2 = 1$ since $\mathbf{A}$ is neat. But this again contradicts our assumption about u, so $r = 0$ and $b_2(b_1 + c)^2 = 0$. Now $b_2 \le a \circ a$ and $b_2(a_2 \circ a_2) = 0$ imply $b_2 \le a_1 \circ a$, and from $a_1 \le c \circ c$ and $a(c \circ a) = 0$ we obtain $b_2 \le a_1 \circ a \le c \circ c \circ a \le c \circ (b + c)$. Since $b_2(c \circ (b_1 + c)) = 0$, it follows that $b_2 \le c \circ b_2$. Furthermore $b_2(a_1 \circ a_2) \le b_2(c \circ c \circ a_2) \le b_2(c \circ b_1) = 0$. Therefore $a_2(b_2 \circ a_1) = 0$ and together with $a_2(b_2 \circ (a_2 + c)) = 0$ and $a_2 \le e^- = b_2 \circ (b_1 + a + c)$ we get $a_2 \le b_2 \circ b_1$. Finally this leads to a contradiction since we now have

$$a_2 \le b_2 \circ b_1 \le (c \circ b_2) \circ b_1 = c \circ (b_2 \circ b_1) \le c \circ (a + b_2)$$

as well as $a_2(c \circ (a + b_2)) = 0$. This proves the claim that $a \le c \circ c$.
Claim: b is an atom. Suppose to the contrary that $b = b_1 + b_2$ with b_1 and b_2 both nonzero and disjoint. Note that $(a + b_1)^2 = 1$ by Corollary 3.40, and $a \le (b_2 + c)^2$. If $b_1 \le (b_2 + c)^2$ then neatness implies that $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_6$ or $\mathbf{B}_7$, contrary to our assumption. On the other hand $b_1(b_2 + c)^2 = 0$ implies

$$b_2 + c = (b_2 + c)e^- = (b_2 + c)(b_1 \circ (a + b_2 + c)) \le b_1 \circ a,$$

and together with $b_1 \le a \circ a \le (c \circ c) \circ a = c(c \circ a) \le c \circ (b + c)$ we have $b_1 \le c \circ b_1$. Now

$$b_2 + c \le b_1 \circ a \le (b_1 \circ c) \circ a = b_1 \circ (c \circ a) \le b_1 \circ (b + c)$$

and since $(b_2 + c)(b_1 \circ (b_2 + c)) = 0$ it follows that $b_2 + c \le b_1 \circ b_1$. But now $a \le c \circ c$, $b_1 \le c \circ b_1$ and $b_2 + c \le b_1 \circ b_1$ imply $(b_1 + c)^2 = 1$ which contradicts our assumption about u.

Therefore $b_1 = p + q$, $p \le (b_2 + c)^2$, $q(b_2 + c)^2 = 0$ and both p and q are nonzero. Suppose $q = r + s$ with $r \le (b_2 + c + p)^2$ and $s(b_2 + c + p)^2 = 0$. If $r \neq 0$ then $b_2 + c + p + s < u$ and, since $\mathbf{A}$ is neat, $(b_2 + c + p + s)^2 = 1$, again contradicting our assumption about u. Consequently $r = 0$ and hence $q(p + b_2 + c)^2 = 0$. But now we let $b_1' = q$, $b_2' = p + b_2$ and then obtain a contradiction as before for b_1 and b_2.

Claim: $b(c \circ c) = 0$ and $b \leq c \circ b$. Since b is an atom, the alternative would be that $b \leq c \circ c$ or $b(c \circ b) = 0$. However the second condition implies the first since

$$b \leq a \circ a \leq (c \circ c) \circ a = c \circ (c \circ a) \leq c \circ (b + c).$$

So $a + b \leq c \circ c$ and, since $c < u$ we have $c \circ c < 1$. But then c satisfies the conditions of Lemma 3.42, contrary to the assumption that $\mathbf{A}$ has no subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_6$.
Claim: $a(b \circ b) = 0$. Suppose to the contrary that $a(b \circ b) \neq 0$ If $a \leq b \circ b$ then $c \leq e^- = b \circ (a + c)$ and $c(b \circ c) = 0$ imply $c \leq b \circ a$, hence

$$c \leq b \circ a \leq (b \circ c) \circ a = b \circ (c \circ a) \leq b \circ (b + c)$$

and therefore $c \leq b \circ b$. But now $b^- \leq b \circ b$ and by assumption $b \circ b < 1$, so $x = b$ satisfies the conditions of Lemma 3.42. This contradicts the assumption that $\mathbf{A}$ has no subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_6$.

Now suppose $a = a_1 + a_2$, $a_1 \leq b \circ b$, $a_2(b \circ b) = 0$ and both a_1 and a_2 nonzero. We will show that $(a_1 + c)^2 = 1$ and $(a_2 + b)^2 = 1$. From $a \leq c \circ c$ and neatness of $\mathbf{A}$ we infer that $a + c \leq (a_1 + c)^2$. Moreover $a \leq e^- = c \circ (a + b)$ and $a(c \circ a) = 0$ imply $a \leq c \circ b$, hence $a_1(c \circ b) \neq 0$ and, since b is an atom, $b \leq c \circ a_1$. Therefore $(a_1 + c)^2 = 1$. Also $a \leq e^- = a \circ (b + c)$ and $a(a \circ c) = 0$ imply $a \leq a \circ b$, hence $a_2(a \circ b) \neq 0$ and $b \leq a2 \circ a$. Now

$$a \leq b \circ c \leq (a_2 \circ a) \circ c = a_2(a \circ c) \leq a_2 \circ (b + c)$$

and $a(a \circ c) = 0$ imply $a \leq a_2 \circ b$. Consequently $a_2(a_2 \circ b) \neq 0$ and, since b is an atom, $b \leq a_2 \circ a_2$. Finally, to see that $c \leq (a_2 + b)^2$, we check that $b \leq b \circ c$, $c \leq e^- = b \circ (a + c)$ and $c(b \circ c) = 0$ imply

$$c \leq b \circ a \leq (b \circ c) \circ a = b \circ (c \circ a) \leq b \circ (b + c)$$

and therefore $c \leq b \circ b$. Since we are assuming that $\mathbf{B}_7$ is not a subalgebra of $\mathbf{A}$, this is again a contradiction.
Claim: $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$ or $\mathbf{B}_{12}$. By definition a, b, c are disjoint nonzero elements that join to e^- and $c(a \circ a) = 0$. From the previous two claims $b(c \circ c) = 0$ and $a(b \circ b) = 0$ hence the present claim follows from Theorem 3.41. $\square$

Corollary 3.44 *The variety* $\mathrm{Var}(\mathbf{A}_3)$ *has 11 finitely generated join irreducible covers in* Λ_{NRA}, *generated by the algebras* $\mathbf{B}_1, \ldots, \mathbf{B}_7$, $\mathbf{B}_9$, $\mathbf{B}_{10}$, $\mathbf{B}_{11}$ *and* $\mathbf{B}_{12}$.

Note that $\mathbf{B}_8$ is the only finite minimal relation algebra in Table 4 that is not neat. The next lemma, and the theorem following are from Jipsen and Lukács [**a**].

Lemma 3.45 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra, and for* $u \leq e^-$ *define* $a = u^- e^-$, $b = u(u \circ u)$ *and* $c = u(u \circ u)^-$. *If* $a \circ a = 1$ *and* $a \leq (b \circ b)(c \circ c)(b \circ c)$ *then* a *generates a subalgebra isomorphic to* $\mathbf{B}_8$.

Proof. From the definition of b and c it follows that $u = b + c$ and $c(u \circ u) = 0$, hence $0 = c(b \circ b) = c(b \circ c) = c(c \circ c)$ and consequently $0 = b(b \circ c) = b(c \circ c)$. Also $b \leq u \circ u =$

$b \circ b + b \circ c + c \circ c$ and therefore $b \le b \circ b$. By assumption a is below each of $b \circ b$, $c \circ c$ and $b \circ c$, so we obtain $b \circ b = a + b + e$, $c \circ c = a + e$ and $b \circ c = a$.

It remains to show that $a \circ b = e^- = a \circ c$. By integrality we have $b = b \circ e \le b \circ (c \circ c) = (b \circ c) \circ c \le a \circ c$, and similarly $c \le a \circ b$. Now

$$\begin{aligned} &a = c \circ b \le c \circ (b \circ b) = (c \circ b) \circ b = a \circ b \\ &b \le c \circ a \le c \circ (b \circ b) = (c \circ b) \circ b = a \circ b \quad \text{and} \\ &a + c \le a \circ b \le (c \circ c) \circ b = c \circ (c \circ b) = c \circ a \end{aligned}$$

hence $a \circ b = e^- = a \circ c$. □

Theorem 3.46 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra, and suppose* a *is an atom of* $\mathbf{A}$ *such that* $a \le e^-$ *and* a *satisfies* $a \circ a = 1$. *Then either* $\mathbf{A} \cong \mathbf{A}_3$ *or* $\mathbf{A}$ *has a subalgebra isomorphic to* $\mathbf{B}_3$, $\mathbf{B}_4$, $\mathbf{B}_6$, $\mathbf{B}_7$ *or* $\mathbf{B}_8$.

Proof. If $\mathbf{A}$ has a nontrivial equivalence element, then Lemma 3.36 implies that $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{B}_3$ or $\mathbf{B}_4$. On the other hand, if $\mathbf{A}$ has no nontrivial equivalence element, then it follows that $x \circ x^- e^- = e^-$ for all $0 < x < e^-$. So if we let $u = a^- e^-$ then either $u = 0$, in which case $\mathbf{A} \cong \mathbf{A}_3$, or $u \le e^- = a \circ u$. In the latter case, since a is an atom, it follows that $a \le u \circ u$. If $u \circ u = a + e$ or $u \circ u = 1$ then we have a subalgebra isomorphic to $\mathbf{B}_6$ or $\mathbf{B}_7$ respectively. Hence we may assume that $u = b + c$, where $b, c \ne 0$, $b \le u \circ u$ and $c(u \circ u) = 0$. Note that $e^- = a + b + c$, and a, b, c are disjoint and nonzero, so $e^- = b \circ (a + c) = b \circ a + b \circ c$ by Lemma 3.36. Then $u(b \circ c) \le u(u \circ c) = 0$ implies that $b + c = u \le b \circ a$. Since a is an atom we have $a \le b \circ b$ and $a \le b \circ c$. Similarly $c \le e^- = c \circ (a + b) = c \circ a + c \circ b$ and $c(c \circ b) \le c(u \circ u) = 0$, hence $c \le c \circ a$ and therefore $a \le c \circ c$. Now we have satisfied all the assumptions of Lemma 3.45, so a generates a subalgebra isomorphic to $\mathbf{B}_8$. □

As another application of Lemma 3.45 above, the following result was proved by the program implementing the algorithm from the last section of Chapter II. This result is included to illustrate how the algorithm is applied. The proof, as found by the computer, is reasonably short and is reproduced in the appendix. The assumptions on the algebra $\mathbf{B}$ below are quite strong. The equation is equivalent to the statement that the unary operation $x \circ x$ is selfconjugate in the relative subalgebra $\mathbf{B}e^-$.

Theorem 3.47 *Let* $\mathbf{B}$ *be a finite simple symmetric relation algebra. If* $\mathbf{B}$ *has more than 4 elements and satisfies the equation* $xe^-((xe^- \circ xe^-)^- \circ (xe^- \circ xe^-)^-) = 0$ *then it has a subalgebra isomorphic to* $\mathbf{B}_5$, $\mathbf{B}_6$, $\mathbf{B}_7$ *or* $\mathbf{B}_8$.

A nonfinitely generated subvariety of height 2. W. Blok [**76**][**79**][**80**][**80a**] extensively investigated the lattice of subvarieties of modal algebras and proved that for closure algebras there are no nonfinitely generated subvarieties of finite height. However, already in the variety defined by the two equations $f^3(x) \le f^2(x)$ and $x \le f(x)$ Blok showed that there are uncountably many subvarieties of height 2.

We now give an example of an infinite simple symmetric relation algebra, denoted by $\mathbf{B}_\infty$, with the property that $\mathrm{Var}(\mathbf{B}_\infty)$ has height 2 in Λ_{RA}. At present this is the only member of RM known to us with this property.

We begin with two technical lemmas, essentially from Jipsen and Lukács [**a**], about subalgebras of residuated BAOs generated by special subsets, e.g. the set of all atoms, all elements of finite height closed under the operators or ideals closed under the operators.

Lemma 3.48 *Let* $\mathbf{A}$ *be a Boolean algebra and suppose* f *is a residuated operation on* $\mathbf{A}$ *with conjugate* f^c. *Then*

(i) *the equation* $f(x^-) = (f(x)^- + f(x^- f^c f(x)))f(1)$ *holds;*

(ii) *for any subset* X *of* A *that is closed under* $+, \cdot, f, f^c$ *and relative complementation (i.e.* $x, y \in X$ *imply* $xy^- \in X$*), if* $f(1)$ *and* $f^c(1)$ *are in* $\mathbf{B} = \mathbf{Sg}^{\mathbf{A}}(X)$ *then* $\mathbf{B}$ *is closed under* f *and* f^c.

Proof. (i) Let $y = x^- f^c f(x)$ and $z = f(x)f(y)^-$. Then

$$x^- f^c(z) \leq x^- f^c f(x) f^c(z) = y f^c(z) = 0$$

since $zf(y) = 0$. Therefore $zf(x^-) = 0$ or, equivalently, $f(x^-) \leq z^-$. On the other hand, since f is additive

$$f(x^-) + z = f(x^-) + f(x)f(y) + f(x)f(y)^- = f(x^-) + f(x) = f(1)$$

and, meeting on both sides with z^-, we get $f(x^-) = z^- f(1)$.

(ii) Under the assumptions on X, it is easy to check that $B = X \cup \{x^- : x \in X\}$, so we only have to show that for all $x \in X$ both $f(x^-)$ and $f^c(x^-)$ are in B. For $f(x^-)$ this follows from the equation in (i), since the elements y and z above are in X as well. The relationship of conjugation is symmetric, so the same argument can be used to show that B is closed under f^c. $\square$

Lemma 3.49 *Let* $\mathbf{A} = (\mathbf{A}_0, \circ, \breve{\ }, e) \in INA$ *and suppose* X *is a subset of* A *that is closed under* $+, \cdot, \circ, \breve{\ }$*, and relative complementation and contains* e. *If* X *has no largest element then*

(i) $\mathrm{Sg}^{\mathbf{A}_0}(X) = \mathrm{Sg}^{\mathbf{A}}(X)$;

(ii) *if* X' *is a subset of* $\mathbf{A}' \in INA$ *satisfying the above conditions and* $h : X \to X'$ *is a lattice isomorphism that commutes with* $\circ$ *and* $\breve{\ }$ *then* h *extends to an isomorphism between* $\mathbf{Sg}^{\mathbf{A}}(X)$ *and* $\mathbf{Sg}^{\mathbf{A}'}(X)$;

(iii) *if* X *contains no nontrivial equivalence elements then* $\mathbf{Sg}^{\mathbf{A}}(X)$ *has no nonconstant finite subalgebras.*

Proof. (i) Let $\mathbf{B} = \mathbf{Sg}^{\mathbf{A}_0}(X)$. Since X is closed under join and meet, $B = X \cup \{x^- : x \in X\}$. From the integrality of $\mathbf{A}$ it follows that $x \circ 1 = 1$ for any nonzero $x \in X$, so we can apply the preceding lemma with $f(y) = L_x(y) = x \circ y$ and conclude that B is closed under L_x. Therefore $x, y \in X$ implies $x \circ y^- \in B$ and similarly $x^- \circ y \in B$.

We now show that $x^- \circ y^- = 1$ for any $x, y \in X$. From the assumption that X has no largest element we obtain a nonzero $u \in X$ such that $u \leq x^-$. Then $1 = u \circ 1 = u \circ y + u \circ y^- \leq u \circ y + x^- \circ y^-$, so it suffices to show that $u \circ y \leq x^- \circ y^-$. Let $v = u \circ y$ and choose a nonzero $w \in X$ such that

$$w \leq (x + v \circ y^\smile)^- = x^-(v \circ y^\smile)^-.$$

Again this is possible since $x + v \circ y^\smile \in X$ and X has no largest element. Now $w \circ 1 = 1$ and $w(v \circ y^\smile) = 0$ implies $v(w \circ y) = 0$, hence $v \leq w \circ y^- \leq x^- \circ y^-$.

(ii) The extension $\tilde{h}$ of h is defined by $\tilde{h}(x) = h(x)$ and $\tilde{h}(x^-) = h(x)^-$ for $x \in X$. This is clearly a Boolean homomorphism from $\mathbf{Sg}^{\mathbf{A}_0}(X)$ to $\mathbf{Sg}^{\mathbf{A}'_0}(X)$ and $\tilde{h}$ commutes with $\circ$ and $^\smile$ since the equations

$$x \circ y^- = (x \circ y)^- + x \circ (x^-(x^\smile \circ (x \circ y))), \qquad x^- \circ y^- = 1 \quad \text{and} \quad x^{-\smile} = x^{\smile -}$$

hold for all nonzero $x, y \in X$ and in X'.

(iii) Assume $0, e$ are the only equivalence elements in X. Let $\mathbf{C}$ be a finite subalgebra of $\mathbf{Sg}^{\mathbf{A}}(X)$ and define a to be the join of the elements $C \cap X$. Then $a \circ a \in C \cap X$, hence $a \circ a \leq a$ and similarly $a^\smile \leq a$ and $e \leq a$. Therefore a is an equivalence element in X, and thus by assumption $a = e$. This implies that $\mathbf{C}$ is a subalgebra of constants. □

Corollary 3.50 *Let $\mathbf{A} \in INA$, let $Y \subseteq A$ be a set of infinitely many pairwise disjoint elements and suppose the set X of all finite joins of elements from Y is closed under $\circ, ^\smile$ and contains e. Then*

(i) $\mathrm{Sg}^{\mathbf{A}_0}(Y) = \mathrm{Sg}^{\mathbf{A}}(Y)$;

(ii) *if Y' is a subset of $\mathbf{A}' \in INA$ satisfying the above conditions and $h : Y \to Y'$ is a bijection such that the additive extension of h to X commutes with $\circ$ and $^\smile$ then h extends to an isomorphism between $\mathbf{Sg}^{\mathbf{A}}(Y)$ and $\mathbf{Sg}^{\mathbf{A}'}(Y')$;*

(iii) *If X contains no nontrivial equivalence elements then $\mathbf{Sg}^{\mathbf{A}}(Y)$ has no nonconstant finite subalgebras.*

Usually the notation x^n is only defined for $n \geq 0$. For the remaining part of this section it is convenient to define $x^n = 0$ if $n < 0$. So, for example, $x^0(x^{(-1)})^- = x^0 = e$.

Theorem 3.51 *Let $\mathbf{A} = (\mathbb{Z}_2^\omega, +, \underline{0})^+$ and let $x = \{u_i \in \mathbb{Z}_2^\omega : i < \omega\}$ where $u_i = (0, \ldots, 0, 1, 0 \ldots)$ with the 1 in the i^{th} position. Then $\mathbf{B} = \mathbf{Sg}^{\mathbf{A}}(\{x\})$ is an infinite one-generated atomic subalgebra of $\mathbf{A}$, with atoms $a_n = x^n(x^{n-1})^-$ and multiplication table*

$$a_m \circ a_n = \sum \{a_k : k = m - n + 2i \text{ and } i \leq n\} \qquad \text{for } m \leq n < \omega. \tag{5}$$

Proof. By definition $a_0 = e = \{\underline{0}\}$ and $a_1 = x$. For each n, x^n is the set of all sequences in $\mathbb{Z}_2^\omega$ with exactly $n - 2i$ nonzero entries ($i = 0, \ldots, \lfloor n/2 \rfloor$), hence a_n is the set of all sequences with exactly n nonzero entries. From this observation (5) follows by an easy induction over m and n. □

By Corollary 3.50 the multiplication table (5) completely determines the structure of $\mathbf{B}$, and since the set of all finite joins of the a_i has no nontrivial equivalence elements, $\mathbf{B}$ has no nonconstant finite subalgebras. However it has a proper infinite subalgebra, generated by $b = x^2$, and we denote this algebra by $\mathbf{B}_\infty$.

Theorem 3.52 *The algebra $\mathbf{B}_\infty = \mathbf{Sg}^{\mathbf{B}}(\{b\})$ has atoms $b_n = b^n(b^{n-1})^-$ and multiplication table*

$$b_m \circ b_n = \sum_{k=n-m}^{m+n} b_k = b^{m+n}(b^{n-m-1})^- \qquad \text{for } m \leq n < \omega.$$

Proof. Examining the multiplication table (5), we see that $x^2 = e + a_2$ and hence $b_n = a_{2n}$. The equations for b_n above are exactly how the atoms with even indices multiply, so by Corollary 3.50 these elements are the atoms of a subalgebra. □

It is now easy to check that if y is any element in (a subalgebra of) an ultrapower of $\mathbf{B}_\infty$ then x^2 generates a subalgebra isomorphic to $\mathbf{A}_3$ or $\mathbf{B}_\infty$, whence $\mathrm{Var}(\mathbf{B}_\infty)$ has height 2 in Λ (see Lemma 3.30). The following theorem proves somewhat more in that it deduces this result from a finite set of universal sentences that are satisfied in $\mathbf{B}_\infty$.

Theorem 3.53 *Suppose $\mathbf{A}$ is a simple symmetric relation algebra that satisfies the universal sentences*

(i) $x \leq x^2$

(ii) $e < x^2 < 1$ *implies* $x^2 < x^4 < 1$

(iii) $0 < y^2 < 1$ *implies* $(x^2 \circ y^2)/y^2 = x^2$

(iv) $e < z^2$ *and* $x^2 \leq y^2 < 1$ *imply*

$$(x \circ z)^2(x^2)^- \circ (y \circ z)^2(y^2)^- = (x \circ z \circ y \circ z)^2(x^2 \circ z^2 \circ y^{2-}).$$

Any element $a \in A$ that satisfies $e < a^2 < 1$ generates an infinite subalgebra isomorphic to $\mathbf{B}_\infty$.

Proof. We first show that $\mathbf{B}_\infty$ satisfies (i)–(iv). Suppose $b, b_0, b_1, \ldots$ are defined as in the preceding theorem. Any element $x \in B_\infty$ is either a finite or a cofinite join of atoms. In the latter case $x^2 = 1$, and otherwise $x \leq b^n$ for some n. If we assume n to be minimal, then $x^2 = b^{2n}$. Therefore (i) and (ii) hold.

(iii) Suppose $0 < y^2 < 1$. If $x^2 = 1$ then $1/y^2 = (1^- \circ y^2)^- = 1$, and if $x^2 = 0$ then $0/y^2 = (0^- \circ y^2)^- = 0$ since $\mathbf{B}_\infty$ is integral and $y^2 > 0$. If $0 < x^2 < 1$, then $x^2 = b^m$ for some m and $y^2 = b^n$ for some n. Therefore

$$x^2 \circ y^2/y^2 = ((b^m \circ b^n)^- \circ b^n)^- = ((b^m)^-)^- = x^2.$$

(iv) If $z^2 = 1$ then both sides of the equation are 1, and if $x = 0$ then both sides are 0, so we may assume that $0 < x^2, y^2, z^2 < 1$. Then $x^2 = b^m$, $y^2 = b^n$ and $z^2 = b^k$ for some $m, n, k \in \omega$ with $m \le n$. Therefore $(x \circ z)^2(x^2)^- = (x^2 \circ z^2)(x^2)^- = (b^{m+k})(b^m)^-$ and $(y \circ z)^2(y^2)^- = (b^{n+k})(b^n)^-$. Now the left hand side of the equation is $b^{m+n+2k}(b^{n-m-k-1})^-$, while the right hand side gives $b^{m+n+2k}(b^{m+k} \circ (b^n)^-)$. If $m + k \ge n$ then both sides equal b^{m+n+2k}, and if $m + k < n$ then $(b^{m+k} \circ (b^n)^-)^- = b^{n-m-k-1}$, so again they agree.

Now suppose $a \in A$ satisfies $e < a^2 < 1$ and let $b = a^2$. It follows from (i) and (ii) that

$$e < b < b^2 < b^3 < \cdots,$$

so the element b generates an infinite Boolean subalgebra with atoms $b_0 = e$ and $b_n = b^n(b^{n-1})^-$, and all finite and cofinite joins of these atoms. Using (iii) and (iv) we will show that this Boolean subalgebra is closed under relative multiplication, in fact

$$b_m \circ b_n = b^{m+n}(b^{n-m-1})^- \quad \text{for} \quad m \le n < \omega.$$

Since this is the multiplication table for $\mathbf{B}_\infty$, and since $\mathbf{B}_\infty$ satisfies the sentences (i)-(iv), this proves the result.

By associativity $b_n = (a^n)^2(a^{n-1})^{2-}$, so (iv) with $x = a^{m-1}$, $y = a^{n-1}$ and $z = a$ gives

$$\begin{aligned} b_m \circ b_n &= (a^m)^2(a^{m-1})^{2-} \circ (a^n)^2(a^{n-1})^{2-} \\ &= (a^m \circ a^n)^2((a^m)^2 \circ (a^{n-1})^{2-}) \\ &= b^{m+n}((a^{n-1})^2/(a^m)^2)^-. \end{aligned}$$

Therefore it suffices to show that $(a^{n-1})^2/(a^m)^2 = b^{n-m-1}$. For $m < n$ this follows from (iii):

$$(a^{n-1})^2/(a^m)^2 = ((a^{n-m-1})^2 \circ (a^m)^2)/(a^m)^2 = (a^{n-m-1})^2 = b^{n-m-1}.$$

If $m = n$ then $x = y$, and (ii) implies that $\mathbf{A}$ contains no nontrivial equivalence elements, hence by Lemma 3.36 $x^2 \circ x^{2-} = e^-$ whenever $0 < x^2 < 1$. Since $a^2 > e$ and $\mathbf{A}$ is simple, we have $x^2 \circ a^2 \circ y^{2-} = e^- \circ a^2 = 1$ as required. □

Relation algebras form a discriminator variety, so we can of course rewrite (i)-(iv) in the form of equations. The above theorem proves that the finitely based variety defined by these equations has a unique subvariety of height 2 generated by $\mathbf{B}_\infty$. We do not known if this variety and $\mathrm{Var}(\mathbf{B}_\infty)$ are distinct, or even if $\mathrm{Var}(\mathbf{B}_\infty)$ is finitely based.

Using the algorithm from the last section of Chapter II we have been able to prove that if a simple totally symmetric relation algebra $\mathbf{A}$ has no finite nonconstant subalgebras then there is an element $a \in \mathbf{A}$ such that $a^n < a^{n+1}$ for all n. (This does not follow directly from Theorem 3.43, and since our present proof is long and unintuitive it is not included here.) As a result, the subvariety generated by all simple members of *TRA* that satisfy (iii) and (iv) has exactly 4 join irreducible subvarieties of height 2 generated by $\mathbf{B}_4$, $\mathbf{B}_7$, $\mathbf{B}_{12}$ and $\mathbf{B}_\infty$ respectively.

Symmetric subadditive r-algebras. Since Λ_{SRA} is rather complicated, we now consider further restrictions on r-algebras. Recall that a tense algebra is a BAO with two unary operators f and f^c that are conjugates of each other. A tense algebra is *reflexive* if it satisfies $x \le f(x)$, and hence also $x \le f^c(x)$. The next two results prove that the variety of reflexive tense algebras is equivalent to a certain subvariety $\mathcal{V}$ of totally symmetric r-algebras. Therefore the lattice of subvarieties of $\mathcal{V}$ is isomorphic to the lattice of subvarieties of reflexive tense algebras.

Theorem 3.54 *Let $\mathbf{A} = (\mathbf{A}_0, f, f^c)$ be a tense algebra and define a binary operation $\circ$ on A by*

$$x \circ y = f(xy) + xf^c(y) + yf^c(x).$$

Then $(\mathbf{A}_0, \circ)$ is a symmetric r-algebra and satisfies the equation $x \circ x^- y \le x + y$. If $\mathbf{A}$ is reflexive then the operations f, f^c can be recovered from $\circ$ by the term functions

$$f(x) = x \circ x \qquad \text{and} \qquad f^c(x) = x + x \circ x^-.$$

Proof. Note that $z(x \circ y) = 0$ if and only if $zf(xy) + zxf^c(y) + zyf^c(x) = 0$ which is equivalent to

$$xyf^c(z) = 0 \quad \text{and} \quad zxf^c(y) = 0 \quad \text{and} \quad zyf^c(x) = 0.$$

Therefore $\circ$ is symmetric. To check that it satisfies the equation, we compute

$$x \circ x^- y = f(xx^- y) + xf^c(x^- y) + x^- yf^c(x) \le 0 + x + x^- y = x + y.$$

Now assume that $\mathbf{A}$ is reflexive. Then $x \le f(x)$ implies $x \circ x = f(x) + xf^c(x) = f(x)$, and furthermore

$$x + x \circ x^- = x + f(xx^-) + xf^c(x^-) + x^- f^c(x) = f^c(x)$$

where we used $x \le f^c(x)$ in the last step. $\square$

A symmetric r-algebra is *subadditive* if it satisfies the equation $x \circ x^- y \le x + y$.

Theorem 3.55 *Let $\mathbf{A} = (\mathbf{A}_0, \circ)$ be a symmetric subadditive r-algebra and define $f(x) = x + x \circ x$ and $f^c(x) = x + x \circ x^-$. Then $(\mathbf{A}_0, f, f^c)$ is a reflexive tense algebra, and if $\mathbf{A}$ is totally symmetric then $\circ$ can be recovered from f and f^c by $x \circ y = f(xy) + xf^c(y) + yf^c(x)$.*

Proof. We show that $yf(x) = 0$ iff $xf^c(y) = 0$, then $(\mathbf{A}_0, f, f^c)$ is a tense algebra and by definition it is reflexive. Suppose $y(x + x \circ x) = 0$. Then $xy = 0$ and $x \circ x \le y^-$, hence

$$y(x \circ y^-) = y(x \circ xy^-) + y(x \circ x^- y^-) \le yy^- + y(x + x^- y^-) = 0.$$

This shows that $x(y \circ y^-) = 0$ and therefore $x(y + y \circ y^-) = xf^c(y) = 0$. Conversely $xf^c(y) = 0$ implies $xy = 0$ and $x \circ y \le y$. It follows that $x(x \circ y) \le xy = 0$ and hence $yf(x) = 0$.

Suppose now that $\circ$ is totally symmetric ($x \le x \circ x$). Then

$$\begin{aligned} f(xy) + xf^c(y) + yf^c(x) &= xy + xy \circ xy + x(y + y \circ y^-) + y(x + x \circ x^-) \\ &= xy \circ xy + x(y \circ y^-) + y(x \circ x^-) \qquad (6) \\ \text{and} \quad x \circ y &= xy \circ xy + xy^- \circ xy + xy \circ x^- y + xy^- \circ x^- y. \end{aligned}$$

The first three terms are below the corresponding terms in (6), and by subadditivity $xy^- \circ x^-y \leq (x+y)(y^- \circ y)(x \circ x^-) \leq x(y \circ y^-) + y(x \circ x^-)$, hence $x \circ y$ is below (6). Conversely

$$\begin{aligned} x(y \circ y^-) &= xy(y \circ y^-) + xy^-(y \circ y^-) \\ &\leq xy \circ xy + xy^-(y \circ xy^- + y \circ x^-y^-) \\ &\leq xy \circ xy + xy^-(y \circ xy^-) + xy^-(y + x^-y^-) \\ & x \circ y \end{aligned}$$

and similarly $y(x \circ x^-) \leq x \circ y$. Therefore $x \circ y = f(xy) + xf^c(y) + yf^c(x)$. □

Corollary 3.56 *The variety of totally symmetric subadditive r-algebras is equivalent to the variety of reflexive tense algebras.*

At this point it is worthwhile to examine the atom structure of a complete and atomic member $\mathbf{A}$ of $\mathcal{V}$. The only restrictions on the relation $R = \circ_+ = \{(a,b,c) \in \mathbf{A}_{0+} : a \circ b \geq c\}$ are

(i) $(a,a,a) \in R$ for all $a \in \mathbf{A}_{0+}$

(ii) $(a,b,c) \in R$ implies $(a,c,b), (c,b,a) \in R$ and

(iii) $(a,b,c) \in R$ implies $a = b$ or $a = c$ or $b = c$.

(i) and (ii) express the fact that $\circ$ is totally symmetric, while (iii) is the rather severe restriction that R include no '3-cycles'. The binary relation f_+ of the tense algebra corresponding to $\mathbf{A}$ is coded up in R by

$$(a,b) \in f_+ \qquad \text{iff} \qquad (a,a,b) \in R.$$

To extend the interpretion of Theorem 3.54 to *ur*-algebras, we consider *unital* tense algebras, defined as algebras of the form $\mathbf{A} = (\mathbf{A}_0, f, f^c, e)$, where e is a constant and $\mathbf{A}$ satisfies the identities

$$f(xe) \leq x \qquad \text{and} \qquad f^c(e) = 1.$$

The first identity implies $ef^c(x) \leq x$, hence $f(xe) + xf^c(e) + ef^c(x) = x$. On the other hand $xe \circ xe \leq x$ and $e + e \circ e^- = 1$ hold in any *ur*-algebra. These observations prove the following result.

Corollary 3.57 *The variety TSaUR of all totally symmetric subadditive ur-algebras is equivalent to the variety of all reflexive unital tense algebras.*

The integral *ur*-algebras in *TSaUR* correspond to reflexive unital tense algebras that satisfy the condition $x \neq 0 \quad \Rightarrow \quad f(x) + f^c(x) = 1$. By Theorem 2.4 the variety generated by such unital tense algebras is defined relative to the variety of all reflexive unital tense algebras by the equation $\tau\tau(x) \leq \tau(x)$, where $\tau(x) = f(x) + f^c(x)$. Members of this variety are called *linear* since in the atom structure of a simple, complete and atomic member, the binary relation f_+ has to satisfy $a\, f_+ b$ or $b\, f_+ a$ for all atoms a, b.

The variety of subadditive relation algebras is denoted by *SaRA*. Maddux [**a**], and independently Tuza [**91**], show that the finite simple members of *SaRA* are characterized as integral symmetric *ur*-algebras in which the term function $f(x) = x + x \circ x$ is a closure operator, i.e., satisfies $x \leq f(x) = ff(x)$. We note that this is true for all members of *SaRA*, but a purely algebraic proof of the associative law, given that f is a closure operator, is tedious. Instead one may observe that subadditivity is equivalent to the implication $xy = 0 \;\Rightarrow\; x \circ y \leq x + y$. Since this form involves no complementation, *SaRA* is closed under canonical extensions and one can therefore use an argument as in Tuza [**91**] for the atom structure of any complete and atomic member of *SaRA*.

Using a construction of Comer [**83**], Maddux completely describes the structure of these algebras and shows that one can decide whether a finite simple member is representable over a set or a group, and that these two notions coincide. The variety *SaRA* has other interesting properties. In fact, *SaRA*

- is not locally finite,
- includes all symmetric 3-atom relation algebras,
- has infinitely many subvarieties,
- has infinitely many simple nonrepresentable members,
- has infinitely many simple representable members that are not finitely representable.

The last two items are from Maddux [**78**]. In conclusion we note that Corollary 3.57 and the above remarks imply that *TSaRA* is equivalent to the variety of unital tense linear closure algebras. From tense logic it is known that the variety of tense linear closure algebras is generated by its finite members, and the same result holds for unital tense algebras. (Alternatively, this can be proved using a slight modification of the $\mathbf{B}^\beta$ construction of Chapter II. Since f^β and $f^{c\beta}$ may not be closure operators, we have to replace them by their transitive closure computed in the finite Boolean algebra $\mathbf{B}_0$.) Therefore *TSaRA* is decidable.

Some open problems

Problem 3.58 *Are SWA or CWA discriminator varieties?*

Problem 3.59 *Let $\mathcal{V}$ be the variety generated by all residuated complex algebras of monoids. Is $\mathcal{V}$ finitely based?*

A positive answer to the following problem would give a finite basis.

Problem 3.60 *Can every member of IERM be embedded in the complex algebra of a monoid? (I.e. is $\mathcal{V} = \mathit{IERM}$?)*

Problem 3.61 *Are any of EUR, RM, ERM, ARM, IRM, CRM, ICRM, IERM, CERM decidable?*

Problem 3.62 *Are the varieties indicated by $*$ in Figure 2 distinct?*

Problem 3.63 *Are any of the varieties CGRA, SGRA, BGRA, TGRA,* $\mathrm{Var}(B_\infty)$, $\mathrm{Var}(\{A(\mathbf{M}) :$ $\mathbf{M}$ *is a modular lattice*$\})$ *finitely based?*

Problem 3.64 *Let* $\mathbf{A}$ *be a simple BAO of finite type and suppose* $\tau^{\mathbf{A}}$ *is selfconjugated. Is* $\mathbf{A}$ *a discriminator algebra? (I.e. can the assumption that* $\mathbf{A}$ *has an atom in Theorem 2.5 be removed?*

Problem 3.65 *Does* $\mathrm{Var}(\mathbf{A}_3)$ *have infinitely many covers in* Λ_{RA} *or* Λ_{SRA}*?*

Problem 3.66 *With regards to the algebras* $\mathbf{B}$ *and* $\mathbf{B}_\infty$ *of Theorem 3.52, is* $\mathrm{Var}(\mathbf{B})$ *a cover of* $\mathrm{Var}(\mathbf{B}_\infty)$ *in* Λ_{SRA}*?*

Problem 3.67 *Does* $\mathrm{Var}(\mathbf{A}_3)$ *have infinitely many finitely generated covers in* $\Lambda_{\mathrm{TSaIUR}}$*?*

Problem 3.68 *Does SaRA have uncountably many subvarieties?*

Problem 3.69 *Does SaRA have a decidable equational theory?*

CHAPTER IV

FURTHER RESULTS ABOUT RELATION ALGEBRAS

Representations of $\mathbf{E}_n(1,2,3)$

R. D. Maddux [**78**] defined 8 sequences of symmetric *ur*-algebras called $\mathbf{E}_n(X)$ for each subset X of $\{1,2,3\}$. We recall the definition here. Let $n = \{0,1,\dots,n-1\}$ and for $i = 1,2,3$ define

$$R_i = \{(a,b,c) \in (n \setminus \{0\})^3 : |\{a,b,c\}| = i\}.$$

Triples of R_i are called i-cycles. Also let

$$R_0 = \bigcup_{a \in n} \{(0,a,a),(a,0,a),(a,a,0)\}.$$

For each $X \subseteq \{1,2,3\}$ the relation R_X is $R_0 \cup \bigcup_{i \in X} R_i$ and the symmetric *ur*-algebra $(n, R_X, 0)^+$ is denoted by $\mathbf{E}_n(X)$. Maddux [**78**] also lists for what n and X the algebras $\mathbf{E}_n(X)$ are in *SRA*. In particular it is shown there that $\mathbf{E}_n(1,2,3)$ is in *RRA* for all n. Using a probabilistic argument, Maddux later showed that these algebras are in fact finitely representable and H. Andréka constructed explicit representations over finite base sets. We now recall a result of R. C. Lyndon [**59**] and deduce from it that $\mathbf{E}_n(1,2,3)$ can be embedded in the complex algebra of a finite abelian group.

Lyndon showed that a relation algebra can be constructed from the set of points in a projective space $\mathbf{P}$ and that this algebra is representable if and only if $\mathbf{P}$ is embedable into a projective space of dimension one more than the dimension of $\mathbf{P}$. The algebra $\mathbf{E}_n(1,3)$ is obtained by this construction from a projective line that has $n-1$ points, hence $\mathbf{E}_n(1,3)$ is representable if and only if there exists a projective plane of order $n-2$ (i.e. with $n-1$ points on each line). From the Bruck-Ryser Theorem on the nonexistence of certain projective planes, Lyndon deduced that $\mathbf{E}_n(1,3)$ is nonrepresentable for infinitely many n (the smallest value being 8). On the other hand, for every prime p there exists a projective plane of order p^n (constructed in the vector space $GF(p^n)^3$, where $GF(p^n)$ is the Galois field of order p^n), hence $\mathbf{E}_{p^n+2}(1,3)$ is representable. From this result Lyndon obtains the following explicit group representation for $\mathbf{E}_{p^n+2}(1,3)$.

Theorem 4.1 *Let* $\mathbf{F} = GF(p^n) = (\mathbb{Z}_p^n, +, -, \underline{0}, *, \underline{1})$ *and denote the elements of* F *by* $g_0, g_1, \dots, g_{p^n-1}$ *where* $g_0 = \underline{0}$ *and* $g_1 = \underline{1}$. *Define the set* L_i *to be the 'line of slope* i*' (without the origin) in* $\mathbf{F} \times \mathbf{F}$, *i.e.,*

$$L_i = \{h * (\underline{1}, g_i) : h \in F,\ h \neq \underline{0}\} \qquad \text{for } i < p^n,$$

and let L_{p^n} *be the vertical line* $\{h * (\underline{0}, \underline{1}) : h \in F, h \neq \underline{0}\}$. *Then* $\{\underline{0}\}, L_0, \dots, L_{p^n}$ *are the atoms of a symmetric subalgebra* $\mathbf{B}$ *of* $(F^2, +, \underline{0})^{\oplus}$ *and* $\mathbf{B}$ *is isomorphic to* $\mathbf{E}_{p^n+2}(1,3)$.

Proof. For each i, $L_i \cup \{\underline{0}\}$ is a subspace of the vector space $\mathbf{F}^2$ over $\mathbf{F}$, hence $L_i \circ L_i = L_i \cup \{\underline{0}\}$ (or $= \{\underline{0}\}$ if $F = \mathbb{Z}_2$) and $L_i^{\smile} = \{-1 * v : v \in L_i\} = L_i$. On the other hand, since $\mathbf{F}^2$ is 2-dimensional, any two vectors from distinct L_i and L_j span the whole space, whence

$$L_i \circ L_j = \bigcup\{L_k : k \leq p^n,\ k \neq i, j\}.$$

Therefore the relation $\circ_+$ is indeed isomorphic to $R_{\{1,3\}}$ under the map $L_i \mapsto i+1$ and $\{\underline{0}\} \mapsto 0$. $\square$

Thus $\mathbf{E}_{p^n+2}(1,3)$ is a member of *CGRA* and, in fact, representable in a finite group. The following observation shows how one can deduce the same result for $\mathbf{E}_n(1,2,3)$.

Theorem 4.2 $\mathbf{E}_n(1,2,3)$ *is a subalgebra of* $\mathbf{E}_m(1,3)$ *whenever* $m > 2n-1$, *hence* $\mathbf{E}_n(1,2,3)$ *is representable in a finite (Boolean) group for every* $n \in \omega$.

Proof. Note that $\mathbf{E}_n(1,2,3)$ is determined by the conditions $a \circ b = e^-$ and $a \circ a = 1$ for all distinct atoms $a, b \leq e^-$. Let $e, a_1, a_2, \ldots, a_m$ be the atoms of $\mathbf{E}_m(1,3)$, and define $b_i = a_{2i-1} + a_{2i}$ for $i = 1, \ldots, n-1$ and $b_n = a_{2n-1} + \ldots + a_m$. Then the elements $e, b_1, \ldots, b_n$ are atoms of a subalgebra isomorphic to $\mathbf{E}_n(1,2,3)$. $\square$

Clearly any set of m disjoint nonzero elements that join to e^- in $\mathbf{E}_n(1,2,3)$ are atoms of a subalgebra isomorphic to $\mathbf{E}_m(1,2,3)$. Hence the varieties $\mathcal{E}_n = \mathrm{Var}(\mathbf{E}_n(1,2,3))$ form an increasing covering chain of varieties in *TBGRA* with

$$\mathcal{E}_1 = \mathrm{Var}(\mathbf{A}_3) \prec \mathcal{E}_2 = \mathrm{Var}(\mathbf{B}_7) \prec \mathcal{E}_3 \prec \mathcal{E}_4 \prec \cdots.$$

Another consequence is that the subalgebra lattice of $\mathbf{E}_n(1,2,3)$ is isomorphic to the partition lattice of an n-element set. Since the finite partition lattices generate the variety of all lattices (a nontrivial result of Pudlak and Tuma [**80**]) we conclude that the subalgebra lattices of complex algebras of finite abelian groups do not satisfy any nontrivial lattice equations.

Relation algebras generated by equivalence elements

The theorem we prove in this section is a contribution to a collection of results about what kind of subalgebras are generated by equivalence elements in a relation algebra. The first result, from Jónsson and Tarski [**52**], shows that the special equivalence element e always generates a finite representable subalgebra of constants and, in a simple relation algebra, this subalgebra is isomorphic to $\mathbf{A}_1$, $\mathbf{A}_2$ or $\mathbf{A}_3$ (see Table 3 and Theorem 3.32). Following that, Jónsson [**88**] proves a similar result for the subalgebra generated by any single equivalence element, namely that this subalgebra is finite, representable and, in case it is integral, isomorphic to $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ or $\mathbf{B}_4$. One view of this result for an equivalence element a in a relation algebra $\mathbf{A}$ is to consider the *factor algebra* $a \circ \mathbf{A} \circ a = (a \circ A \circ a, \circ, ^{\smile}, a)$ in which a is now the identity element, so by the first result it generates a finite subalgebra (in $a \circ \mathbf{A} \circ a$). Also e generates a finite subalgebra in the relativized subalgebra $\mathbf{A}a$, and now one can prove that these two algebras essentially determine the structure of the subalgebra generated by a. The more general question, what is generated by the elements in a relativized subalgebra below an equivalence element, is treated in detail in a monograph of S. Givant [**a**]. Included there is the following result.

Theorem 4.3 *Let* $\mathbf{A}$ *be a relation algebra and* $X \subseteq A$ *a set of equivalence elements such that for all* $u, v \in X$

$$u \leq v \quad \text{or} \quad v \leq u \quad \text{or} \quad uv = 0.$$

Then $\mathbf{Sg}^{\mathbf{A}}(X)$ *is representable, and if* X *is finite then* $\mathbf{Sg}^{\mathbf{A}}(X)$ *is finitely representable (hence finite).*

So, for example, finite chains of equivalence elements generate finite representable relation algebras. However, neither representability nor finiteness are necessary properties of a relation algebra generated by an arbitrary finite set of equivalence elements. R. Lyndon's nonrepresentable symmetric relation algebras, described in the preceding section, are generated by equivalence elements (their atoms joined with e). With regards to finiteness, B. Jónsson observed that one can construct an infinite simple symmetric relation algebra generated by 4 equivalence elements from the free modular lattice on 4 generators using the correspondence between modular lattices and equivalence elements in a symmetric relation algebra (Maddux [**81**], included here as Theorem 3.22). H. Andréka and I. Németi give an example (described in Givant [**a**]) of two commuting equivalence elements in a relation algebra that generate an infinite subalgebra. The algebra in their example is not symmetric, and we now show that this is necessarily so.

Theorem 4.4 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra. Then any two equivalence elements in* $\mathbf{A}$ *generate a finite subalgebra of cardinality at most* 2^6*.*

Proof. Suppose u and v are nonzero equivalence elements of $\mathbf{A}$. Since we are assuming that $\mathbf{A}$ is simple and symmetric, e is an atom and hence below both u and v. Let

$$a = uve^-, \quad b = ua^-e^-, \quad c = va^-e^-, \quad d = (u \circ v)u^-v^- \quad \text{and} \quad w = (u + v + d)^-.$$

Since the case $u \leq v$ is covered by Theorem 4.3, we may assume that b and c are nonzero. We will show that they are atoms of $\mathbf{Sg}^{\mathbf{A}}(u, v)$, and that a, d, w are either zero or also atoms of $\mathbf{Sg}^{\mathbf{A}}(u, v)$. Since $1 = e + a + b + c + d + w$, it follows that $\mathrm{Sg}^{\mathbf{A}}(u, v)$ has cardinality at most 2^6.

We begin by showing that most of the relative products between a, b, c, d, w are already determined by the assumption that $u = e+a+b$ and $v = e+a+c$ are equivalence elements. In fact the (partial) operation table for $\circ$ with respect to these elements is given in Table 7.

Entries of the form $[x, y]$ in the table mean that the respective relative product has a value in the interval $[x, y] = \{z : x \leq z \leq y\}$. The second half of the proof involves showing that in such a case the relative product cannot 'split' any of a, b, c, d, w that are below yx^-.

Note that in the table $b \circ c = d$ (as we will prove below). Since b and c are assumed to be nonzero and $\mathbf{A}$ is integral, it follows that d is also nonzero. However a or w could be zero, in which case the corresponding row and column of Table 7 are to be ignored. Furthermore, our assumptions are invariant under interchanging b and c, so whenever we prove something about b, a corresponding result holds for c.

Table 7: Partial operation table for **A**

$\circ$	a	b	c	d	w
a	$[e, e+a]$	b	c	d	w
b		$[e+a, u]$	d	$[c, c+d]$	w
c			$[e+a, v]$	$[b, b+d]$	w
d				$[e+a, w^-]$	w
w					$[w^-, 1]$

Claim: If $a \neq 0$ then $e \leq a \circ a \leq e + a$. The first inequality holds because **A** is integral, and the second follows from the observation that $e + a = uv$ is an equivalence element.
Claim: If $a \neq 0$ then $a \circ b = b$ and $a \circ c = c$. By definition $a \circ b \leq u \circ u \leq u$ and, since $(e + a)(a \circ b) = 0$, we have $a \circ b \leq b$. By simplicity $x \circ 1 = 1$ for any nonzero x, hence $b \leq a \circ 1 = a \circ b + a \circ b^-$. But we just showed that $b^-(a \circ b) = 0$, consequently $b(a \circ b^-) = 0$ and therefore $a \circ b = b$.
Claim: $e + a \leq b \circ b \leq u$ and $e + a \leq c \circ c \leq v$. As in the previous claim, $a \circ b \leq b$ and hence $a(b \circ b^-) = 0$. Since $b \neq 0$ we have $a \leq b \circ 1 = b \circ b + b \circ b^-$, therefore $a \leq b \circ b$, and of course we always have $e \leq b \circ b$. On the other hand, since u is an equivalence element, $b \circ b \leq u \circ u = u$.
Claim: $b \circ w = c \circ w = d \circ w = w$, if $a \neq 0$ then $a \circ w = w$ and if $w \neq 0$ then $w^- \leq w \circ w$. The essential observation is that $w^- = u \circ v$ is an equivalence element, whence $w(w^- \circ w^-) = 0$ and therefore $x \circ w \leq w$ for any $x \leq w^-$. If $x \neq 0$ then the opposite inequality follows from $w \leq x \circ 1 = x \circ w + x \circ w^-$ and $w(x \circ w^-) = 0$. Furthermore, if $w \neq 0$ then $w^- \leq w \circ 1 = w \circ w + w \circ w^-$, and hence $w^- \leq w \circ w$.
Claim: $b \circ c = d$. In the previous claim we noted that $w(w^- \circ w^-) = 0$, so in particular $w(b \circ c) = 0$. Moreover, $u(b \circ c) = 0$ since $c(b \circ u) \leq cu = 0$, and similarly $v(b \circ c) = 0$. Therefore $b \circ c \leq d$. On the other hand

$$d = (u \circ v)u^- v^- = (e + u + v + a \circ a + a \circ b + a \circ c + b \circ c)(u + v)^- \leq b \circ c.$$

Claim: If $a \neq 0$ then $a \circ d = d$. Since u, v are equivalence elements $d(a \circ u + v) = 0$, and $d(a \circ w) = 0$ follows from $w(w^- \circ w^-) = 0$. Hence $a \circ d \leq d$. The opposite inclusion holds because $d \leq a \circ 1 = a \circ d + a \circ d^-$ and $d(a \circ d^-) = 0$.
Claim: $c \leq b \circ d \leq c + d$ and $b \leq c \circ d \leq b + d$. We certainly have $c \leq b \circ 1 = b \circ d + b \circ d^-$, and since $d^-(b \circ c) = d^- d = 0$ we conclude that $c \leq b \circ d$. The second inclusion follows from the observation that $d(b \circ (u + w)) = d(u + w) = 0$.
Claim: $e + a \leq d \circ d \leq w^-$. We argued earlier that d had to be nonzero, hence $e + a \leq d \circ 1 = d \circ d + d \circ d^-$. Since $(e + a) \circ d \leq d$ it follows that $e + a \leq d \circ d$. Finally $w(d \circ d) = 0$ holds because $w(w^- \circ w^-) = 0$.

This concludes the first part of the proof. We now show that the partially determined products cannot generated any new elements.

Claim: $a \le a \circ a$ or $a(a \circ a) = 0$. Suppose $a = a_1 + a_2$, $a_1 \le a \circ a$ and $a_2(a \circ a) = 0$. Then a_1, a_2 are disjoint, $a \circ a \le e + a$ and $a(a \circ a_2) = 0$. Together these statements imply $a_1 \circ a_2 = 0$. Since **A** is integral, either $a_1 = 0$ or $a_2 = 0$.
Claim: $b \le b \circ b$ or $b(b \circ b) = 0$. Suppose $b = b_1 + b_2$, $b_1 \le b \circ b$, $b_2(b \circ b) = 0$ and $b_2 \ne 0$. Then $b_1 \le b_2 \circ a$ since $b_1 \le b_2 \circ 1 = b_2 \circ a + b_2 \circ a^-$ and $b_1(b_2 \circ a^-) = 0$. Also $b_1 \le b \circ b$ and $b_1(b \circ b_2) = 0$ imply $b_1 \le b \circ b_1$. Therefore

$$b_1 \le b \circ b_1 \le b \circ (b_2 \circ a) = (b \circ b_2) \circ a \le (e + a) \circ a = e + a$$

and consequently $b_1 = 0$. Similarly $c \le c \circ c$ or $c(c \circ c) = 0$.
Claim: $d \le d \circ d$ or $d(d \circ d) = 0$. Suppose $d = d_1 + d_2$, $d_1 \le d \circ d$ and $d_2(d \circ d) = 0$. Then $d_1 \le d \circ d_1$, hence

$$d_2(d_1 \circ a) \le d_2(d \circ d_1 \circ a) \le d_2(d \circ d) = 0$$

where the second inequality holds since $d \circ a \le d$ (see Table 7). Also $d \le d \circ d_1 \le (c \circ b) \circ d_1 = c \circ (b \circ d_1) \le c \circ (c + d)$, and therefore $d_1(c \circ c) = 0$ implies $d_1 \le c \circ d$. Now

$$d_2(d_1 \circ b) \le d_2(d \circ c \circ b) = d_2(d \circ d) = 0$$

and similarly $d_2(d_1 \circ c) = 0$. Since d_1, d_2 are disjoint and $d_2(d \circ d) = 0$ we have $d_2(d_1 \circ (e + d)) = 0$. We already showed that $d(d \circ w) = 0$, so we conclude that $d_2(d_1 \circ 1) = 0$ and hence $d_1 \circ d_2 = 0$. By integrality it follows that either $d_1 = 0$ or $d_2 = 0$.
Claim: $b \le d \circ d$ or $b(d \circ d) = 0$. Suppose $b = b_1 + b_2$, $b_1 \le b \circ b$, $b_2(b \circ b) = 0$ and $b_2 \ne 0$. Then $d \le b_2 \circ c$, since $d \le b_2 \circ 1 = b_2 \circ c + b_2 \circ c^-$ and $d(b_2 \circ c^-) = 0$ (see Table 7). Furthermore

$$b_1 \le d \circ d \le d \circ (b_2 \circ c) = (d \circ b_2) \circ c \le c \circ c.$$

This however implies that $b_1 = 0$ because $b(c \circ c) = 0$.
Claim: $w \le w \circ w$ or $w(w \circ w) = 0$. Suppose $w = w_1 + w_2$, $w_1 \le w \circ w$, $w_2(w \circ w) = 0$ and $w_1 \ne 0$. Then $w_2 \le w_1 \circ 1 = w_1 \circ w^- e^- + w \circ (e + w)$ and, since $w_2(w_1 \circ (e + w)) = 0$, it follows that $w_2 \le w_1 \circ w^- e^-$. From Table 7 we see that $w \circ w^- e^- = w$, hence

$$w_2 \le w_1 \circ w^- e^- \le (w \circ w) \circ w^- e^- = w \circ (w \circ w^- e^-) = w \circ w.$$

Since we assumed $w_2(w \circ w) = 0$ we now have to conclude that $w_2 = 0$.
Claim: $d \le b \circ d$ or $d(b \circ d) = 0$. Suppose $d = d_1 + d_2$, $d_1 \le b \circ d$ and $d_2(b \circ d) = 0$. Then

$$\begin{aligned}
&d_2(d_1 \circ a) \le d_2(b \circ d \circ a) = d_2(b \circ d) = 0 \\
&d_2(d_1 \circ b) = 0 \quad \text{by assumption} \\
&d_2(d_1 \circ c) \le d_2(b \circ d \circ c) \le d_2(b \circ (b + d)) = 0 \\
&d_2(d_1 \circ d) = d_1(d_2 \circ b \circ c) \le d_1(c \circ c) = 0 \quad \text{and} \\
&d_2(d_1 \circ w) = 0 \quad \text{since } d(d \circ w) = 0.
\end{aligned}$$

Therefore $d_2(d_1 \circ 1) = 0$, so $d_1 \circ d_2 = 0$ and by integrality either $d_1 = 0$ or $d_2 = 0$. Similarly $d \le c \circ d$ or $d(c \circ d) = 0$. □

Table 7 can be completed in several ways, depending on whether the terms

$$a, \quad w, \quad a(a \circ a), \quad b(b \circ b), \quad c(c \circ c), \quad \text{and} \quad w(w \circ w)$$

are zero or not (the values of $b(d \circ d)$, $c(d \circ d)$ and $d(d \circ d)$ are determined by these choices). Altogether there are up to isomorphism 27 symmetric relation algebras generated by two equivalence elements.

Problem 4.5 *Do three equivalence elements in a symmetric relation algebra always generate a finite subalgebra?*

A nonrepresentable absolute retract in SRA

An algebra $\mathbf{A}$ is an *absolute retract* in a variety $\mathcal{V}$ if for every embedding $f : \mathbf{A} \hookrightarrow \mathbf{B} \in \mathcal{V}$ there exists a homomorphism $g : \mathbf{B} \to \mathbf{A}$, called a *retraction*, such that $gf = id_A$. In *RRA* the absolute retracts are exactly the finite full relation algebras $\mathbf{Re}(n)$. This is mentioned in Andréka, Jónsson and Németi [**91**], and it is shown there that these algebras are also absolute retracts in the larger varieties *RA* and *SA*. However, it is not known whether there are any others. Below we give an example of a nonrepresentable absolute retract in *SRA*.

A variety $\mathcal{V}$ is *semisimple* if all members of $\mathrm{Si}(\mathcal{V})$ are simple. The following well-known result shows that for semisimple varieties the simple absolute retracts are exactly the maximal members of $\mathrm{Si}(\mathcal{V})$ (maximal in the sense that no member of $\mathrm{Si}(\mathcal{V})$ is a proper extension).

Lemma 4.6 *Let $\mathcal{V}$ be a variety. Then*

(i) *any maximal member of* $\mathrm{Si}(\mathcal{V})$ *is an absolute retract in $\mathcal{V}$;*

(ii) *if $\mathcal{V}$ is semisimple then any simple absolute retract in $\mathcal{V}$ is a maximal member of* $\mathrm{Si}(\mathcal{V})$.

Proof. (i) Suppose $\mathbf{A}$ is maximal in $\mathrm{Si}(\mathcal{V})$ and suppose f is an embedding from $\mathbf{A}$ into $\mathbf{B} \in \mathcal{V}$. By Zorn's Lemma one can choose a maximal congruence θ on $\mathbf{B}$ that does not identify any elements of $f(A)$. Then $\mathbf{B}/\theta \in \mathrm{Si}(\mathcal{V})$ and $\mathbf{A}$ is embedded in $\mathbf{B}/\theta$. Since $\mathbf{A}$ is maximal in $\mathrm{Si}(\mathcal{V})$, it is isomorphic to $\mathbf{B}/\theta$, hence the map $x \mapsto x/\theta$ is (up to isomorphism) the required retraction.

(ii) Suppose $\mathbf{A} \in \mathrm{Si}(\mathcal{V})$ is an absolute retract in $\mathcal{V}$. From the assumption that $\mathcal{V}$ is semisimple, it follows that any extension of $\mathbf{A}$ in $\mathrm{Si}(\mathcal{V})$ is simple, hence the retraction onto $\mathbf{A}$ must be an isomorphism. Therefore $\mathbf{A}$ is maximal in $\mathrm{Si}(\mathcal{V})$. $\square$

Note that in a discriminator variety all simple absolute retracts must be finite since an infinite discriminator algebra $\mathbf{A}$ is a proper subalgebra of an ultrapower of $\mathbf{A}$ of suitably high cardinality, and any ultrapower of $\mathbf{A}$ is also simple. The next result illustrates how the above lemma is frequently applied. It is proved by essentially the same argument that Andréka, Jónsson and Németi use for the corresponding result about $\mathbf{Re}(n)$. An element a in a *ur*-algebra is *functional* if it satisfies $a \triangleright a \leq e$.

Theorem 4.7 *For every finite group $\mathbf{G}$, the group relation algebra $\mathbf{G}^+$ is an absolute retract in INA. In GRA every absolute retract is isomorphic to $\mathbf{G}^+$ for some finite group $\mathbf{G}$.*

Proof. In a nonassociative relation algebra every functional element is an atom since $e \geq (a_1 + a_2)^\smile \circ (a_1 + a_2) = a_1^\smile \circ a_1 + a_1^\smile \circ a_2 + a_2^\smile \circ a_1 + a_2^\smile \circ a_2$ implies $a_1^\smile \circ a_2 = 0$ and hence

$a_1 = 0$ or $a_2 = 0$. Suppose $\mathbf{A}$ is the complex algebra of a finite group. Then every atom is a functional element and therefore it is also an atom in any integral extension $\mathbf{B} \in \mathit{INA}$ of $\mathbf{A}$. Since $\mathbf{A}$ is finite, $\mathbf{B}$ must be isomorphic to $\mathbf{A}$, hence $\mathbf{A}$ is maximal in Si(INA) and, by Lemma 4.6, an absolute retract in INA.

Now suppose $\mathbf{A}$ is a simple absolute retract in GRA. Since GRA is generated by all complex algebras of groups, $\mathbf{A}$ is a subalgebra of $\mathbf{G}^+$ for some group $\mathbf{G}$. By Lemma 4.6 $\mathbf{A}$ is maximal in Si(GRA), hence $\mathbf{A}$ is isomorphic to $\mathbf{G}^+$. □

From the preceding theorem we can conclude that the symmetric relation algebras $(\mathbb{Z}_2^n)^+$ are absolute retracts in SRA. The lemma below provides us with stronger assumptions from which we can conclude that an element in a member of Si(SRA) is an atom whenever it satisfies these assumptions.

Lemma 4.8 *Let* $\mathbf{A}$ *be a simple symmetric relation algebra, and suppose* $a, b, c \leq e^-$ *are pairwise disjoint, nonzero elements that satisfy* $a \circ a = e + b$ *and* $b(b \circ b) = 0$.

(i) *If* $e < b \circ b$ *then* a *is an atom.*

(ii) b *is an atom.*

(iii) *If* a *is an atom,* $a \circ b = a + c$ *and* $b(c \circ c) = 0$ *then* c *is an atom.*

Proof. (i) Suppose a is not an atom, in which case we can write $a = a_1 + a_2$ for disjoint nonzero a_1, a_2. We show that this forces $b \circ b = e$. Since $\mathbf{A}$ is simple and symmetric, it is also integral, hence $e \leq a_1 \circ a_1 \leq e + b$. Moreover, $a_1 a_2 = 0$ implies $b^-(a_1 \circ a_2) \leq b^- b = 0$ and consequently $a_1 \circ a_2 \leq b$. By associativity

$$a_2 \leq e \circ a_2 \leq (a_1 \circ a_1) \circ a_2 = a_1 \circ (a_1 \circ a_2) \leq a_1 \circ b \quad \text{and hence}$$

$$a_2(a_2 \circ b) \leq a_2(a_1 \circ b \circ b) \leq a_2(a_1 \circ b^-) = 0.$$

Therefore $b(a_2 \circ a_2) = 0$ from which it follows that $a_2 \circ a_2 = e$, and similarly $a_1 \circ a_1 = e$. Now

$$a \circ a = a_1 \circ a_1 + a_1 \circ a_2 + a_2 \circ a_2 = e + a_1 \circ a_2 = e + b$$

implies that $a_1 \circ a_2 = b$. Finally $b \circ b = a_1 \circ a_2 \circ a_1 \circ a_2 = a_1 \circ a_1 \circ a_2 \circ a_2 = e$.

(ii) If $b \circ b = e$ then b is functional, hence cannot be split in an integral relation algebra. On the other hand, if $b \circ b > e$ then a is an atom by (i). Suppose $0 \neq b' \leq b$. Then $b' \leq a \circ a$ implies $a \leq b' \circ a$, so

$$b \leq a \circ a \leq b' \circ a \circ a \leq b' \circ (e + b) = b' + b' \circ b$$

and since $b(b \circ b) = 0$ we must have $b \leq b'$. Hence b is also an atom.

(iii) Suppose $0 \neq c' \leq c$. Since a is assumed to be an atom, $c' \leq a \circ b$ implies $a \leq b \circ c'$. Now

$$b \leq a \circ a \leq a \circ b \circ c' \leq (a + c) \circ c' = a \circ c' + c \circ c'$$

and since $b(c \circ c) = 0$ we have $b \leq a \circ c'$. Consequently

$$c \leq a \circ b \leq a \circ a \circ c' \leq (e + b) \circ c' = c' + b \circ c'$$

and now $b(c \circ c) = 0$ implies $c \leq c'$. Therefore c is an atom. $\square$

Theorem 4.9 *For every prime p, the symmetric relation algebra $S(\mathbb{Z}_p)$ is an absolute retract in SRA.*

Proof. We show that $S(\mathbb{Z}_p)$ is maximal in $\mathrm{Si}(SRA)$, then the result follows by Lemma 4.6. Since $S(\mathbb{Z}_2) = \mathbb{Z}_2^+$, this case is covered by Theorem 4.7, so we may assume that $p > 2$. Let a be any atom below e^-. Then $a = \{-n, n\}$ for some nonzero $n \in \mathbb{Z}_p$, and $b = e^-(a \circ a) = \{-2n(\mathrm{mod} p), 2n(\mathrm{mod} p)\}$. Therefore $ab = 0$, $a \circ a = e + b$, $b(b \circ b) = 0$ and $e < b \circ b$. If $S(\mathbb{Z}_p)$ is embedded in any member of $\mathrm{Si}(SRA)$ then the images of a and b also satisfy these requirements, hence the preceding lemma implies that the image of a is an atom. Since $S(\mathbb{Z}_p)$ is finite and atoms are mapped to atoms, the embedding is an isomorphism. $\square$

The previous result can be used to argue that the varieties *NGRA* and *NBGRA* are distinct. For example the algebra $\mathbf{B}_5 = S(\mathbb{Z}_5) \in NGRA$ is maximal in $\mathrm{Si}(SRA)$, hence it cannot be a subdirect product of simple members of *NBGRA*.

Theorem 4.10 *There exist non-representable absolute retracts in the variety of symmetric relation algebras.*

Proof. The atoms of $\mathbf{B}_{10}$ (see Table 4) satisfy the conditions of the Lemma 4.8. As in the proof above, it follows that $\mathbf{B}_{10}$ is maximal in $\mathrm{Si}(SRA)$. Using techniques from Maddux [**78**] this algebra is easily seen to be nonrepresentable. $\square$

APPENDIX

Discussion and description of the program

The computational aspect of this dissertation began with a program that counted finite relation algebras. Several people assisted in this venture, most notably E. Lukács and S. Tschantz. As the program became more sophisticated, so did the methods by which it reduced the search space, eliminating those parts of the search tree that could never be completed to a relation algebra. This development lead to the algorithm described in the last section of Chapter III. In its present form the program is better suited to proving that sets of axioms are inconsistent rather than finding models of such axioms.

Design and implementation. The algorithm is applicable to any type of Boolean algebras with finitely many additional operations, but in practice it is only useful for a small number of unary and binary operations. Some effort has been made to implement the algorithm in a flexible manner. The current version allows for the analysis of modal algebras, tense algebras, r-algebras, nonassociative relation algebras and symmetric relation algebras (the last two types could be treated as r-algebras, but for reasons of efficiency they are separated out).

The set $\mathcal{E}_{\mathbf{A}_{\underline{n}}}$ is not implemented explicitly. Instead the structure of binary numbers is used to implicitly represent the Boolean algebras $\mathbf{A}_{\underline{n}}$. Initially $\mathbf{A}$ is the two-element Boolean algebra. With statements like $1 = e \dotplus d$, $d = a \dotplus b$, etc., the atom 1 of $\mathbf{A}$ is split into new constants of the language $\mathcal{L}_{A_{\underline{n}}}$ that represent the atoms of $\mathbf{A}_{\underline{n}}$. The set $\mathcal{E}$ of universal sentences is specified in the input, although some computationally intensive BAO axioms (additivity, associativity) are coded into the program. The initial set $\mathcal{C}$ of inclusion and exclusion formulas is also specified in the input file. The sets $\mathcal{C}_{\underline{n}}$ are represented internally by collections of linked lists of irredundant inclusion and exclusion formulas (2 list for each operator). From these lists the upper and lower approximations (f^{μ} and f^{λ}) are calculated.

The program has three main components. The first component interprets the input and sets up the initial parameters and formula lists for the root node. The second part computes some (or all possible) proof trees originating from this root node. This is done level by level in a breath first search, with various built-in options to heuristically limit the search space. When (if) every branch in the tree terminates with an inconsistent node, the third component of the program searches for, and prints out, the sequence of steps that lead to the various inconsistencies. The proof in the next section shows how the output is presented.

Program correctness. This is an important issue for any program, especially one that claims to prove mathematical truths. At this point all verification of proofs produces by the program is done by hand. We use the program as a tool rather than a complete solution. Most proofs are unnecessarily long and tedious because the algorithm does not look for subresults or symmetries that might simplify the argument.

Sample output from the program

The proof below was found and typeset by the program based on the algorithm from the last section of Chapter II. The first 11 lines are the input for the program. (The lines numbered (1)-(6) are formulas universally quantified over x.) From this input the program searches a tree of all possible proofs in the direction of likely contradictions and produces the output lines 1-76.

A few explanatory remarks are needed to read the proof. The symbol '$\dot{+}$' below denotes *disjoint* boolean join of *nonzero* elements (i.e. $a \dot{+} b$ means $a+b$ where $ab = 0$ and $a, b \neq 0$). An expression like $a \mathbin{?} b \circ b$ signifies the start of three new subbranches corresponding to the following (mutually exclusive) assumptions.

(i) $a \leq b \circ b$

(e) $a \cdot (b \circ b) = 0$ and

(s) $a = a_1 \dot{+} a_2, \quad a_1 \leq b \circ b, \quad a_2 \cdot (b \circ b) = 0.$

The result proved is Theorem 3.47. For convenience, the statement of this theorem is repeated here. An outline of the proof tree is given in Figure 6.

Theorem *Let* $\mathbf{B}$ *be a finite simple symmetric relation algebra. If* $\mathbf{B}$ *has more than 4 elements and satisfies the equation* $xe^-((xe^- \circ xe^-)^- \circ (xe^- \circ xe^-)^-) = 0$ *then it has a subalgebra isomorphic to* $\mathbf{B}_5$, $\mathbf{B}_6$, $\mathbf{B}_7$ *or* $\mathbf{B}_8$.

As mentioned before, several formulas of $\mathcal{E}$ are coded into the program, hence these formulas are not specified in the input. For symmetric relation algebras the built-in formulas are

(i) distributivity: $x \leq y \circ (z + w)$ and $x(y \circ w) = 0$ imply $x \leq y \circ z$,

(ii) conjugation: $x(y \circ z) = 0$ implies $y(x \circ z) = 0$,

(iii) commutativity (i): $x \leq y \circ z$ implies $x \leq z \circ y$,

(iv) commutativity (e): $x(y \circ z) = 0$ implies $x(z \circ y) = 0$,

(v) associativity (i): $u \leq v \circ z$ and $v \leq x \circ y$ and $y \circ z \leq w$ imply $u \leq x \circ w$,

(vi) associativity (e): $u(x \circ w) = 0$ and $y \circ z \leq w$ and $v \leq x \circ y$ imply $u(v \circ z) = 0$,

(vii) atom rule: $x \leq a \circ y$ and a an atom implies $a \leq x \circ y$.

From the form of these rules it is apparent that they can be imposed directly on the lists of inclusion and exclusion formulas for $\circ$. This results in a more efficient implementation as compared with including equivalent equations in $\mathcal{E}$ and deriving inclusion and exclusion formulas from them using Lemma 2.15 (iii) and (iv).

The diagrams of the symmetric relation algebras $\mathbf{B}_1, \ldots, \mathbf{B}_{12}$ are also coded into the program. This allows the program to recognize them when they appear as subalgebras during the search. Lemma 3.45 is used to recognize $\mathbf{B}_8$ early on, so that the formulas that follow from the assumptions of this lemma do not have to be derived again.

We briefly indicate how the input is interpreted by the program. The first line designates the letter x as a variable that ranges over the elements of $\mathbf{A}_{\underline{n}}$. The next line splits the top

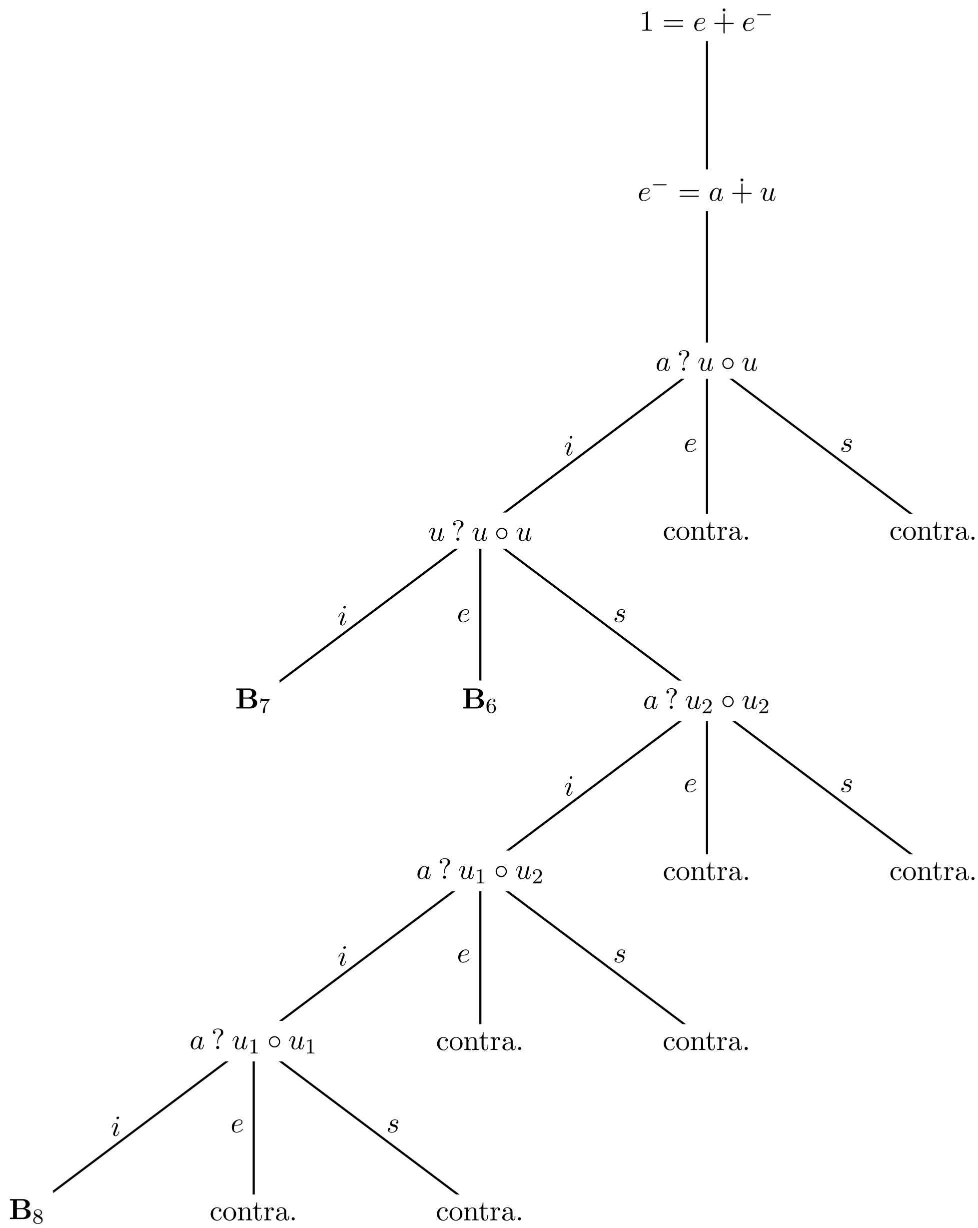

Figure 6: An outline of the prooftree of Theorem 3.47

element of $\mathbf{A}$ into two disjoint nonzero parts e and e^-, that are atoms of $\mathbf{A}_s = \mathbf{A}_{\underline{n}}$. The third line stipulates that the constant e is an atom in any extension of $\mathbf{A}_{\underline{n}}$. The universal sentence labelled (1) makes e the unit element of $\mathbf{A}_{\underline{n}}$. The restriction $0 \prec x$ results in a more efficient implementation of this sentence since by additivity it suffices if e is a unit with respect to all atoms of $\mathbf{A}_{\underline{n}}$. The term 'implement' refers to the fact that this condition can be implemented to give an inclusion and an exclusion formula ($a \leq a \circ e$ and $(a \circ e)a^-$ for each atom a). 'No reference' indicates that these formulae will not be referred to explicitly in the output, thereby reducing the length of the printout.

The universal sentence (2) is equivalent to the equation assumed in the statement of the theorem. This sentence fails in the symmetric algebras $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ and $\mathbf{B}_4$. Therefore Theorem 3.36 implies that we can add the third universal sentence. In the proof we also assume that $\mathbf{B}_5$, $\mathbf{B}_6$, $\mathbf{B}_7$ and $\mathbf{B}_8$ are not subalgebras of $\mathbf{B}$, whence (4) holds. The following line splits e^- into two disjoint nonzero parts a and u that are atoms of $\mathbf{A}_{ss}$. Since $\mathbf{B}$ is finite and has no subalgebra isomorphic to $\mathbf{B}_1, \ldots, \mathbf{B}_5$, we can assume by Corollary 3.40 that a is minimal with respect to the condition $1 \leq a \circ a$. This is expressed by the last three lines of the input. The restriction $x \prec a$ again improves the efficiency of the algorithm since it limits the number of values that have to be substituted for x to check that (5) and (6) are still compatible.

The output is indented to show the tree structure of the proof (cf. Figure 6). A standard output line begins with a number, followed by an inclusion or exclusion formula, how it was derived, and references of the form $[n]$ to earlier lines that justify this derivation. Lines that correspond to the start of three new subbranches have forward references to each of the three subbranches. The computer generated output follows.

Input

var x

$1 = e \dot{+} e^-$

atom e

(1) $0 \prec x \;\Rightarrow\; x \circ e = x$, (implement) (no reference)

(2) $x < e^- \;\Rightarrow\; x{\cdot}((x \circ x)^- \circ (x \circ x)^-) = 0$, (implement)

(3) $0 < x$ and $x < e^- \;\Rightarrow\; e^- \leq x \circ x^-{\cdot}e^-$, (implement)

(4) $0 < x$ and $x < e^-$ and $x^-{\cdot}e^- \circ x^-{\cdot}e^- < 1 \;\Rightarrow\; x \circ x = 1$, (implement)

$e^- = a \dot{+} u$

$1 \leq a \circ a$ by assumption

(5) $x \prec a \;\Rightarrow\; 1 \leq x^-{\cdot}e^- \circ x^-{\cdot}e^-$, (implement)

(6) $x \prec a \;\Rightarrow\; x \circ x < 1$, (check)

Output

$1 = e \dot{+} e^-$

$e^- = a \dot{+} u$

1 $1 \leq a \circ a$ by assumption

2 $e^- \leq u \circ a$ from $x = u$ and $e^- \leq x \circ x^-{\cdot}e^-$

3 $e \leq u \circ u$ since $u \leq e \circ u$ and e is an atom

4 $a \,?\, u \circ u$, (i5)(e60)(s62)

5 . $a \leq u \circ u$ by assumption

6 . $u \,?\, u \circ u$, (i7)(e9)(s11)

7 . $u \le u \circ u$ by assumption
8 . u generates $\mathbf{B}_7$ $[2,1,7,5,3]$
9 . $u \cdot (u \circ u) = 0$ by assumption
10 . u generates $\mathbf{B}_6$ $[2,1,9,5,3]$
11 . $u = u_1 \dot{+} u_2$ by assumption
12 . $u_1 \le u \circ u$ by assumption
13 . $u_2 \cdot (u \circ u) = 0$ by assumption
14 . $e^- \le (a+u_2) \circ u_1$ from $x = a + u_2$ and $e^- \le x \circ x^- \cdot e^-$
15 . $e^- \le u_2 \circ (a+u_1)$ from $x = u_2$ and $e^- \le x \circ x^- \cdot e^-$
16 . $e \le u_1 \circ u_1$ since $u_1 \le e \circ u_1$ and e is an atom
17 . $e \le u_2 \circ u_2$ since $u_2 \le e \circ u_2$ and e is an atom
18 . $u \le u_2 \circ a$ since $u \le u_2 \circ (a+u_1)$ and $u \cdot (u_2 \circ u_1) = 0$ $[13,15]$
19 . $u \le u_1 \circ a$ since $u \le u_1 \circ (a+u_2)$ and $u \cdot (u_1 \circ u_2) = 0$ $[13,14]$
20 . $u_1 \le u \circ u_1$ since $u_1 \le u \circ u$ and $u_1 \cdot (u \circ u_2) = 0$ $[13,12]$
21 . $u_1 \le u_1 \circ u_1$ since $u_1 \le u_1 \circ u$ and $u_1 \cdot (u_1 \circ u_2) = 0$ $[13,20]$
22 . $a \,?\, u_2 \circ u_2$, (i23)(e53)(s55)
23 . . $a \le u_2 \circ u_2$ by assumption
24 . . $a \le u_2 \circ (e+a)$ since $a \le a \circ u$, $a \le u_2 \circ u_2$ and $u_2 \circ u \le e + a$ $[13,23,2]$
25 . . $a \le u_2 \circ a$ since $a \le u_2 \circ (e+a)$ and $a \cdot (u_2 \circ e) = 0$ $[24]$
26 . . $a \,?\, u_1 \circ u_2$, (i27)(e38)(s40)
27 . . $a \le u_1 \circ u_2$ by assumption
28 . . $a \le u_1 \circ a$ since $a \le u_1 \circ u_2$, $u_1 \le u_1 \circ u_1$ and $u_1 \circ u_2 \le a$ $[13,21,27]$
29 . . $a \,?\, u_1 \circ u_1$, (i30)(e32)(s34)
30 . . . $a \le u_1 \circ u_1$ by assumption
31 . . . a generates $\mathbf{B}_8$ $[27,23,17,12,5,3,28,19,13,30,21,16,1]$
32 . . . $a \cdot (u_1 \circ u_1) = 0$ by assumption
33 . . . **contradicts** $u_1 \le a \circ u_1$ $[32,19]$
34 . . . $a = a_1 \dot{+} a_2$ by assumption
35 . . . $a_2 \cdot (u_1 \circ u_1) = 0$ by assumption
36 . . . $u_1 \cdot ((u_2 + a_2) \circ (u_2 + a_2)) = 0$ from $x = u_1$ and $x \cdot ((x \circ x)^- \circ (x \circ x)^-) = 0$ $[35,13]$
37 . . . **contradicts** $a_2 \le u_1 \circ (u_2 + a_2)$ $[36,13,27]$
38 . . $a \cdot (u_1 \circ u_2) = 0$ by assumption
39 . . **contradicts** $u_1 \le a \circ u_2$ $[38,18]$
40 . . $a = a_1 \dot{+} a_2$ by assumption
41 . . $a_1 \le u_1 \circ u_2$ by assumption
42 . . $a_2 \cdot (u_1 \circ u_2) = 0$ by assumption
43 . . $e \le a_1 \circ a_1$ since $a_1 \le e \circ a_1$ and e is an atom
44 . . $u_2 \le u_1 \circ a_1$ since $u_2 \le u_1 \circ a$ and $u_2 \cdot (u_1 \circ a_2) = 0$ $[42,19]$
45 . . $u_1 \le u_2 \circ a_1$ since $u_1 \le u_2 \circ a$ and $u_1 \cdot (u_2 \circ a_2) = 0$ $[42,18]$
46 . . $u_2 \le u_2 \circ a_1$ since $u_2 \le a_1 \circ u_1$, $a_1 \le u_2 \circ u_2$ and $u_2 \circ u_1 \le a_1$ $[42,13,23,44]$
47 . . $a_1 \le a_1 \circ a_1$ since $a_1 \le u_2 \circ u_1$, $u_2 \le a_1 \circ u_2$ and $u_2 \circ u_1 \le a_1$ $[42,13,46,41]$
48 . . $a_1 \le u_1 \circ a_1$ since $a_1 \le u_1 \circ u_2$, $u_1 \le u_1 \circ u_1$ and $u_1 \circ u_2 \le a_1$ $[42,13,21,41]$
49 . . $e + a \le a_1 \circ a_1$ since $e + a \le u_2 \circ u_2$, $u_2 \le a_1 \circ u_1$ and $u_1 \circ u_2 \le a_1$ $[42,13,44,23]$

50 . . $e + u_1 \leq a_1 \circ a_1$ since $e + u_1 \leq u_1 \circ u_1$, $u_1 \leq a_1 \circ u_2$ and $u_2 \circ u_1 \leq a_1$ [42, 13, 45, 21]
51 . . $u + a_2 \leq a_1 \circ a_1$ since $u + a_2 \leq a_1 \circ u_2$, $a_1 \leq a_1 \circ u_1$ and $u_1 \circ u_2 \leq a_1$ [42, 13, 48, 46]
52 . . **contradicts** $x = a_1$ and $x \circ x < 1$ [51, 50, 49, 47, 43]
53 . . $a{\cdot}(u_2 \circ u_2) = 0$ by assumption
54 . . **contradicts** $u_2 \leq a \circ u_2$ [53, 18]
55 . . $a = a_1 \dotplus a_2$ by assumption
56 . . $a_2{\cdot}(u_2 \circ u_2) = 0$ by assumption
57 . . $u_2{\cdot}((u + a_2) \circ (u + a_2)) = 0$ from $x = u_2$ and $x{\cdot}((x \circ x)^- \circ (x \circ x)^-) = 0$ [56, 13]
58 . . $u_2 \leq (u + a_2) \circ (u + a_2)$ from $x = a_1$ and $1 \leq x^-{\cdot}e^- \circ x^-{\cdot}e^-$
59 . . **contradicts** $u_2{\cdot}((u + a_2) \circ (u + a_2)) = 0$ [58, 57, 56, 13]
60 . $a{\cdot}(u \circ u) = 0$ by assumption
61 . **contradicts** $u \leq a \circ u$ [60, 2]
62 . $a = a_1 \dotplus a_2$ by assumption
63 . $a_1 \leq u \circ u$ by assumption
64 . $a_2{\cdot}(u \circ u) = 0$ by assumption
65 . $e^- \leq a_2 \circ (u + a_1)$ from $x = a_2$ and $e^- \leq x \circ x^-{\cdot}e^-$
66 . $u{\cdot}(a_2 \circ a_2) = 0$ from $x = u$ and $x{\cdot}((x \circ x)^- \circ (x \circ x)^-) = 0$ [64]
67 . $1 \leq (u + a_2) \circ (u + a_2)$ from $x = a_1$ and $1 \leq x^-{\cdot}e^- \circ x^-{\cdot}e^-$
68 . $e \leq a_1 \circ a_1$ since $a_1 \leq e \circ a_1$ and e is an atom
69 . $u \leq (u + a_2) \circ u$ since $u \leq (u + a_2) \circ (u + a_2)$ and $u{\cdot}((u + a_2) \circ a_2) = 0$ [66, 64, 67]
70 . $a_2 \leq (u + a_2) \circ a_2$ since $a_2 \leq (u + a_2) \circ (u + a_2)$ and $a_2{\cdot}((u + a_2) \circ u) = 0$ [66, 64, 67]
71 . $u + a_2 \leq a_2 \circ a_1$ since $u + a_2 \leq a_2 \circ (u + a_1)$ and $(u + a_2){\cdot}(a_2 \circ u) = 0$ [66, 64, 65]
72 . $u + a_2 \leq u \circ a_1$ since $u + a_2 \leq u \circ a$ and $(u + a_2){\cdot}(u \circ a_2) = 0$ [66, 64, 2]
73 . $u \leq a_1 \circ a_1$ since $u \leq (u + a_2) \circ u$, $u + a_2 \leq a_1 \circ a_2$ and $a_2 \circ u \leq a_1$ [66, 64, 71, 69]
74 . $a_2 \leq a_1 \circ a_1$ since $a_2 \leq (u + a_2) \circ a_2$, $u + a_2 \leq a_1 \circ u$ and $u \circ a_2 \leq a_1$ [66, 64, 72, 70]
75 . $e + a_1 \leq a_1 \circ a_1$ since $e + a_1 \leq u \circ u$, $u \leq a_1 \circ a_2$ and $a_2 \circ u \leq a_1$ [66, 64, 71, 63]
76 . **contradicts** $x = a_1$ and $x \circ x < 1$ [75, 74, 73, 68]

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