Financial Sophistication and the Distribution of the Welfare Cost of Inflation: Appendices

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Appendix A: Helpful Derivations

A.1 The constrained-efficient allocation

Consider the allocation selected by a planner who maximizes the agents' lifetime utilities and treats agents identically. The planner is subject to the same physical and informational constraints faced by the agents and therefore cannot observe identities. However, the planner observes types. Basically, the planner can propose a type-dependent consumption plan in each trade cycle, but does not have the ability to transfer resources across agents over time. Equivalently, the planner maximizes expected utility of the arbitrary agent on each date. The planning problem thus corresponds to a sequence of static maximization problems, i.e., to maximizing ex-ante welfare of the representative agent, subject to technological feasibility.

Recall that on each date agents have identical preferences ex-ante and there is an identical proportion of buyers and sellers. Moreover, on each odd date agents that are active can produce or consume with equal probability.

Letting $\rho_j = \rho$ for $j = H$ and $1 - \rho$ for $j = L$, the planner problem is to choose $\{c_j, y_j\}_{j=H,L}$, q, and x to solve:

$$\begin{array}{ll} \max & \sum_{j=H,L} \frac{\alpha_j}{2} \rho_j [u(c_j) - \phi_j(y_j)] + U(q) - x \\ \text{s.t.} & \sum_{j=H,L} \rho_j c_j \leq \sum_{j=H,L} \rho_j y_j \quad \text{and } q \leq x \end{array}$$

By non-satiation, the feasibility constraints should hold with equality. Letting λ denote the Lagrange multiplier on the first feasibility constraint, the FOCs are thus

$$\frac{\alpha_j}{2}\rho_j[u'(c_j)-\lambda]=0$$

$$\frac{\alpha_j}{2}\rho_j[-\phi_j'(y_j)+\lambda]=0$$

$$U'(q)-1=0$$

That is, agents produce up to the point where the marginal utility of their consumption or labor equal the marginal utility of income, λ.

Hence, the efficient allocation is stationary across trade cycles, and it can be characterized as follows. On odd dates $c_j = c^* = \rho y_H + (1-\rho)y_L$ and $y_L = y_L^* < y_H = y_H^*$ where the starred output values are the unique positive solutions to the two equalities $u'(y_L + y_H) = \phi_j'(y_j)$ for $j = H, L$. It should be clear that $c^* = y^*$ such that $u'(c^*) = \phi'(c^*)$ if there is no heterogeneity in productivity. On even dates $q_j = x_j = q^*$ for each type j in each trade cycle, where q^* is the unique positive solution to $U'(q) = 1$.

A.2 Optimal choices in market one

The optimal choice $y_j \geq 0$ of a type-j producer must satisfy $\phi_j'(y_j) \geq \frac{\partial W_j(m_{j,s})}{\partial m_{j,s}}\frac{\partial m_{j,s}}{\partial y_j}$. The optimal c_j of a type-j buyer must satisfy $u'(c_j) + \frac{\partial W_j(m_{j,b})}{\partial m_{j,b}}\frac{\partial m_{j,b}}{\partial c_j} \geq 0$, omitting the multiplier on his budget constraint. Clearly, $\frac{\partial W_j(m_{j,k})}{\partial m_{j,k}} = 1$ and $\frac{\partial m_{j,s}}{\partial y_j} = -\frac{\partial m_{j,b}}{\partial c_j} = p$ from

$$m_{j,b} = m_j - pc_j, \quad m_{j,s} = m_j + py_j, \quad \text{and} \quad m_{j,n} = m_j. \tag{1}$$

Hence, one gets $p \leq \phi_j'(y_j)$ and $u'(c_j) \geq p$ for $j = H, L$.

Elasticities and the money demand ratio L

Consider a representative agent economy and focus on odd dates.

Elasticity of disutility of labor. The disutility of labor is $\phi(y) = \frac{y^\delta}{\delta}$, where y is production as well as labor effort. So, the elasticity of disutility of labor is

$$\varepsilon_y = \frac{d\phi(y)/\phi(y)}{dy/y} = \frac{d\ln\phi(y)}{d\ln y} = \frac{y^{\delta-1}y}{y^\delta}\delta = \delta,$$

since the differential $d\ln\phi(y) = d\ln(y^\delta/\delta) = d(\delta\ln y - \ln\delta) = \frac{\delta}{y}dy$. Since $\phi'(y) = p$, the labor supply $y(p)$ satisfies

$$y^{\delta-1} = p \Rightarrow y(p) = p^{\frac{1}{\delta-1}}.$$

Elasticity of labor supply. In our model the wage of a worker on odd dates is p. The elasticity of the labor supply with respect to the relative wage is

$$\varepsilon_p = \frac{dy(p)/y(p)}{dp/p} = \frac{d\ln y(p)}{d\ln p} = \frac{1}{\delta-1},$$

because the differential

$$d\ln y(p) = d(\ln p^{\frac{1}{\delta-1}}) = d\left(\frac{1}{\delta-1}\ln p\right) = \frac{1}{\delta-1} \times \frac{dp}{p}.$$

Elasticity of money demand. From

$$c_j = \min\{\tfrac{m_j}{p}, c(p)\}, \tag{2}$$

one gets $pc = m$, so the Euler equation

$$i = \frac{\alpha_j}{2}\left[\frac{u'(c_j)}{\phi_j'(y_j)} - 1\right] \text{ for } j = H, L, \tag{3}$$

for the representative agent gives

$$F(m/p, i) = \frac{\alpha}{2}\left[\frac{u'(m/p)}{\phi'(y)} - 1\right] - i = 0.$$

Using the implicit function theorem we have

$$\frac{\partial m/p}{\partial i} = -\frac{\partial F/\partial i}{\partial F/\partial(m/p)} = -\frac{-1}{\frac{\alpha}{2\phi'(y)}u''(m/p)} = \frac{2\phi'(y)}{\alpha u''(m/p)}.$$

Given $c = m/p$ and market clearing $c = y$, the elasticity of money demand is

$$\varepsilon_m = \frac{\partial m/p}{\partial i} \times \frac{i}{m/p} = \frac{2\phi'(y)}{\alpha u''(c)} \times \frac{i}{c} = \frac{2i\phi'(y)}{\alpha c u''(c)} \tag{4}$$

We have $\phi'(y) = y^{\delta-1}$ and $y = c$. So (4) is $\frac{2ic^{\delta-1}}{\alpha c u''(c)}$. Substituting c from

$$c = \left(\frac{\alpha}{2i+\alpha}\right)^{\frac{1}{\delta+a-1}} \tag{5}$$

one gets

$$\varepsilon_m = -\frac{2i}{a(2i+\alpha)}.$$

The money demand ratio L. $L = \frac{m}{\frac{\alpha}{2}pc+A}$ and from (2) we have $pc = m$. Also, $p = \phi'(y)$. Since $\phi'(y) = y^{\delta-1}$ and $y = c$ from market clearing, then $L = \frac{1}{\alpha/2+Ac^{-\delta}}$, with c defined in (5) as a function of parameters and interest rate.

A.3 Explicit solutions for consumption and output

Heterogeneity in trade risk. Here $y_H = y_L = y$. Given the assumed functional forms we have $\phi'(y) = y^{\delta-1}$ and $u'(c_j) = c_j^{-a}$ so rewrite the Euler equation (3) as $1 + \frac{2i}{\alpha_j} = \frac{c_j^{-a}}{y^{\delta-1}}$ for $j = H, L$, which implies $c_L = \left[\frac{(2i+\alpha_L)\alpha_H}{\alpha_L(2i+\alpha_H)}\right]^{-\frac{1}{a}} c_H$. From market clearing

$$\alpha_H\rho y_H + \alpha_L(1-\rho)y_L = \alpha_H\rho c_H + \alpha_L(1-\rho)c_L \tag{6}$$

one gets

$$y = \frac{\rho\alpha_H c_H + (1-\rho)\alpha_L c_L}{\rho\alpha_H + (1-\rho)\alpha_L}.$$

Substituting for y and c_L in the Euler equation above

$$c_H = \left\{ \frac{\alpha_H}{2i+\alpha_H} \left[\frac{\alpha_H\rho+\alpha_L(1-\rho)\left(\frac{(2i+\alpha_L)\alpha_H}{\alpha_L(2i+\alpha_H)}\right)^{-\frac{1}{a}}}{\alpha_H\rho+\alpha_L(1-\rho)} \right]^{1-\delta} \right\}^{\frac{1}{a+\delta-1}}.$$

Heterogeneity in productivity. From Lemma 2 in the paper we have $c_H = c_L = c$. Given the assumed functional forms $\phi_j'(y_j) = \theta_j^{\delta} y_j^{\delta-1}$ and $u'(c) = c^{-a}$ so rewrite the Euler equation as $1 + \frac{2i}{\alpha} = \frac{c^{-a}}{\theta_j^{\delta} y_j^{\delta-1}}$ for $j = H, L$. From market clearing (6) we have $c = \rho y_H + (1-\rho) y_L$; from $p = \phi_j'(y_j)$ for $j = H, L$, we have $p = \phi_H'(y_H) = \phi_L'(y_L)$, which is

$$y_H = y_L \left(\frac{\theta_L}{\theta_H}\right)^{\frac{\delta}{\delta-1}} = y_L \theta^{\frac{\delta}{\delta-1}}$$

since we have normalized $\theta_L = \theta > \theta_H = 1$. So, market clearing implies $c = y_L \left(\rho\theta^{\frac{\delta}{\delta-1}} + 1 - \rho\right)$. Substituting for c in the Euler equation above

$$y_L = \left[\left(1+\frac{2i}{\alpha}\right)\left(\rho\theta^{\frac{\delta}{\delta-1}} + 1 - \rho\right)^{a} \theta^{\delta}\right]^{\frac{1}{1-a-\delta}}.$$

Money is not the only asset. Here $y_H = y_L = y$. The expression for c_L is obtained from

$$\frac{\pi}{\beta} = 1 + \frac{\alpha_j}{2}\left[\frac{u'(c_j)}{p} - 1\right] \text{ for } j = H, L, \tag{7}$$

and c_H is obtained from

$$\alpha_H\left(\frac{1}{\beta} - 1\right) = \frac{\alpha_H}{2}\left[\frac{u'(c_H)}{p} - 1\right]. \tag{8}$$

Given the assumed functional forms $\phi'(y) = y^{\delta-1}$ and $u'(c_j) = c_j^{-a}$ so the Euler equation (7) is $1 + \frac{2i}{\alpha_L} = \frac{c_L^{-a}}{y^{\delta-1}}$. From (8) one gets $\frac{2-\beta}{\beta} = \frac{c_H^{-a}}{y^{\delta-1}}$ which implies

$$c_L = \left(\frac{\alpha_L+2i}{\alpha_L} \times \frac{\beta}{2-\beta}\right)^{-\frac{1}{a}} c_H.$$

We have $y = \frac{\rho\alpha_H c_H + (1-\rho)\alpha_L c_L}{\rho\alpha_H + (1-\rho)\alpha_L}$ from (6). Substituting for y and c_L in (8) one gets

$$c_H = \left\{ \frac{\beta}{2-\beta} \left[\frac{\alpha_H\rho + \alpha_L(1-\rho)\left(\frac{\alpha_L+2i}{\alpha_L}\frac{\beta}{2-\beta}\right)^{-\frac{1}{a}}}{\alpha_H\rho + \alpha_L(1-\rho)} \right]^{1-\delta} \right\}^{\frac{1}{a+\delta-1}}.$$

Given $m_H = b_L = 0$ we have $(1-\rho)m_L = \bar{m}$ and $b = pc_H$. For a type L one has

$$(1-\beta)V_L(0,\bar{m}) = \frac{\alpha_L}{2}[u(c_L) - \phi(y)] + U(q^*) - q^* + \frac{\alpha_L}{2}p(y - c_L) - (\pi - 1)\frac{\rho}{1-\rho}\bar{m}.$$

Since $\bar{m} = (1-\rho)m_L = (1-\rho)pc_L$ then $(\pi-1)\bar{m}\frac{\rho}{1-\rho} = (\pi-1)\rho pc_L$. For a type H,

$$(1-\beta)V_H(b,0) = \frac{\alpha_H}{2}[u(c_H) - \phi(y)] + U(q^*) - q^* + \frac{\alpha_H}{2}p(y - c_H) + (\pi - 1)\bar{m}$$

because $\pi\theta = \alpha_H$, $m_H = 0$ and $b_H = b = pc_H$.

Appendix B: Model Fit

In this Appendix we present a rudimentary discussion about the fit of the theoretical money demand to the data for various model specifications.

1 Variations in preference parameters

In this section we calibrate market one preferences parameters in a different manner.

1.1 Linear disutility of labor in market one

If we set $\delta = 1$ as in Lagos and Wright (2005) and related papers, then the fit is virtually identical to the one we obtain with the specification found in the paper; the calibrated parameters α, A, and R^2 do not vary. Clearly, we need some convexity in disutility for coexistence of efficient/inefficient producers in the heterogeneous version of the model, hence we use $\delta = 1.1$.

1.2 Variations in a

Suppose now that we move away from unit elastic preferences in market one. For example suppose the parameter a is set to 0.71 to match a recent empirical study on risk aversion in Raj (2006).[1] In this case we get $\alpha = 0.248$, $A = 2.618$ and the fit falls

[1] Raj, C. (2006). A new method of estimating risk aversion. *American Economic Review*, 96 (5), 1821-1834.

slightly relative to our current model, $R^2 = 0.520$. So, there is not much difference in the fit. The welfare cost calculations do not change very much, either.

1.3 Variations in δ

Now suppose that, in addition to fixing $a = 0.71$, we also vary the disutility of labor in market one to match data on labor elasticities for the U.S.. Notice that δ corresponds to the elasticity of disutility of labor with respect to labor effort in market one. The elasticity of the labor supply with respect to the relative wage is $\frac{1}{\delta-1}$. So, set δ to match average elasticity of labor supply with respect to own wage in the U.S.. Estimates of the elasticity of labor supply vary according to the group considered (e.g., male versus female). From Filer, Hamermesh, and Rees (1996)[2] estimates of labor supply elasticities are 0.00 for men and 0.80 for women. Consequently, we set δ to match the average of the two values with weights given by the proportion of men (0.55) and women (0.45) in the labor force for the period 1960-2006 as reported by the Bureau of Labor Statistics. We get $\delta = 3.78$. Now fix $(a, \delta) = (0.71, 3.78)$. We obtain $\alpha = 0.248, A = 2.801$, and the fit falls to $R^2 = 0.460$. Figure B1 illustrates how the model fits the data.

[2] Filer, R. K., D. S. Hamermesh, and A. E. Rees (1996). *The Economics of Work and Pay*, Sixth Edition. New York: Harper-Collins.

1.4 Using off-the-shelf estimates of elasticity of money demand

Still, fix $(a, \delta) = (0.71, 3.78)$. We can calibrate the model to match an off-the-shelf elasticity of money demand, instead of using our own estimate to see how the model performs. For example, suppose we consider theelasticity in Aruoba, Waller and Wright (2007), who consider a different sample period for the U.S. and obtain an estimated elasticity of money demand of -0.226. In this way we obtain $\alpha = 0.427$ and $A = 3.052$. The fit is poorer because R^2 is 0.328. Figure B2 illustrates the fit of the model to the data.

2 Quarterly specification

Suppose instead we use a quarterly specification of our model. Hence, fix $(a, \delta) = (1, 1.1)$. We pin α down to match our estimate of the yearly elasticity of money demand (i.e., -0.3376). This implies $\alpha = 0.041$, $A = 0.781$, and $R^2 = 0.400$. So, the fit is worse than for a yearly model. Figure B3 illustrates the fit of the model to the quarterly data. With trade shock heterogeneity $\alpha_L = 0.001$ and $\alpha_H = 0.101$. The percentage of output produced in market one for this calibration is slightly lower than in the calibration presented in the paper. The upper bound on the share of market one output is not much different than the specification reported in the paper. The welfare costs are reported in Table 1 in the paper.

3 The trading friction α and the fit of the implied money demand

We have run the following sensitivity analysis for the representative agent model. Suppose we fix α, i.e., we do not calibrate it specifically to match the elasticity of money demand. Given α, choose A to match the empirical money demand, as usual. What is the α value that generates the best possible fit? How does the fit change with α? How does the share of market one output vary with α? The result is in Figure B4. It shows (horizontal axis) α versus R^2, and the ratio of output produced in market one to overall output.

The best possible fit is obtained for $\alpha = 0.075$, a value even smaller than the one that matches the estimated yearly elasticity of money demand. This measure of fit is hump-shaped in α. This means that if the model assigns too much or too little importance to monetary trade (market one trade is exclusively monetary, unlike market two transactions), then the implied money demand fits the data very poorly. See Figures B5-B6.

The calibrated value $\alpha = 0.145$ generates a fit close to the best possible fit (R^2 is 0.55 vs. 0.61). Greater values of α result in an even worse fit. To best match the empirical money demand the model should exhibit a sufficiently small share of monetary trade (out of total trade). In this sense the calibrated value 0.145 of the trade parameter α is not too small.

Now consider the share of market one output. It rises in α and we know from previous work that this share should not be too large (e.g., see Aruoba, Waller and Wright, 2007). Even in this sense, our trade parameter is not too small.

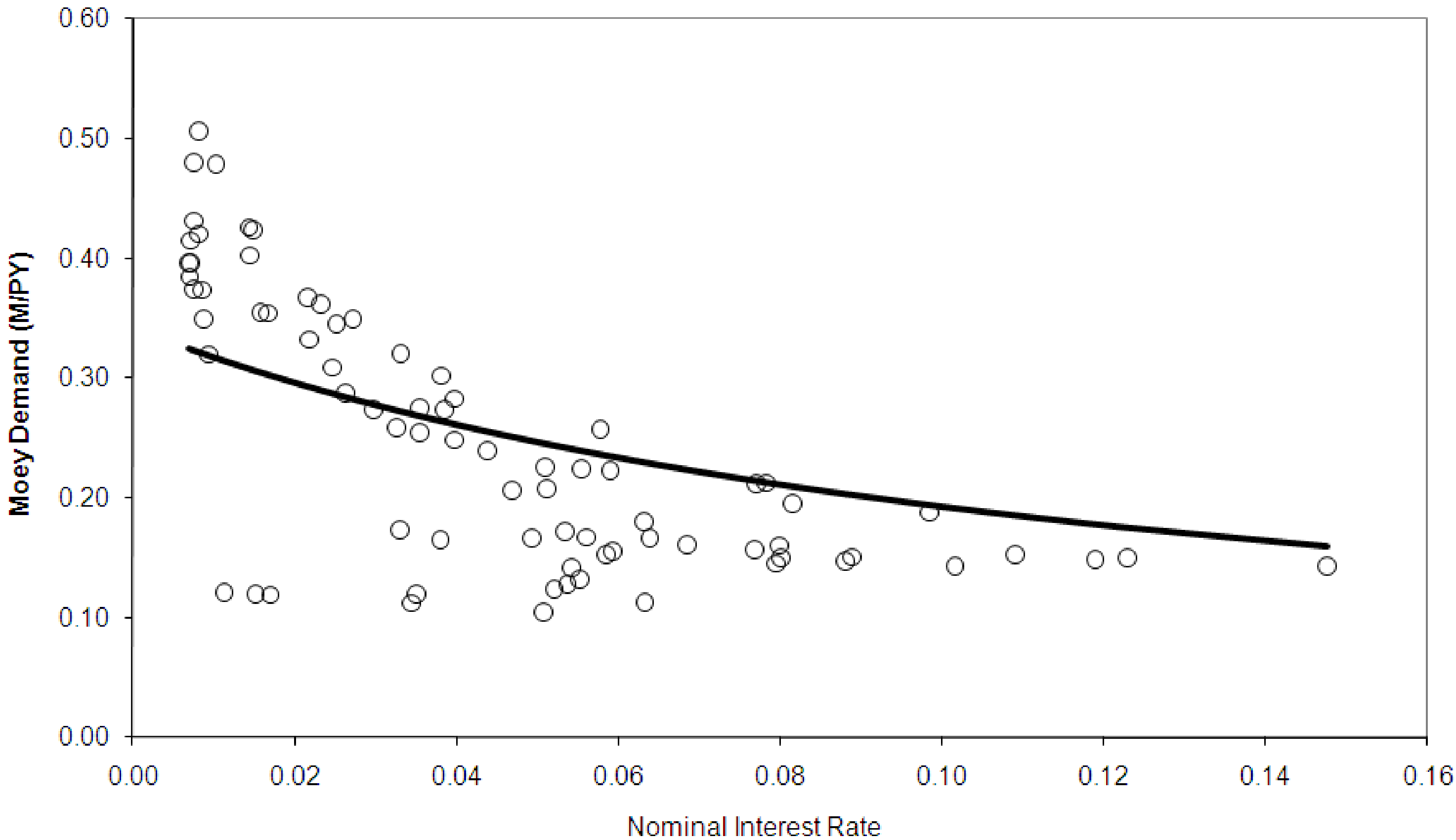

FIGURE B1: fit for α=0.71 and δ=3.78

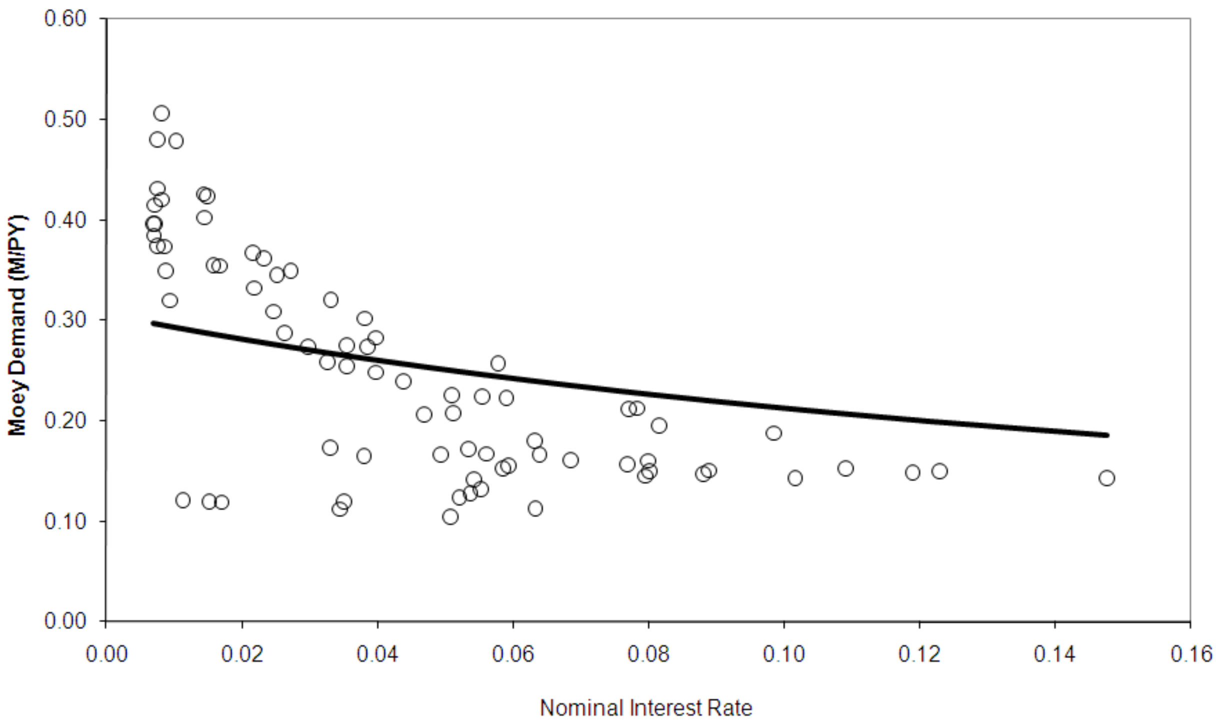

FIGURE B2: fit for α=0.71, δ=3.78 and A calibrated to elasticity from AWW 2007

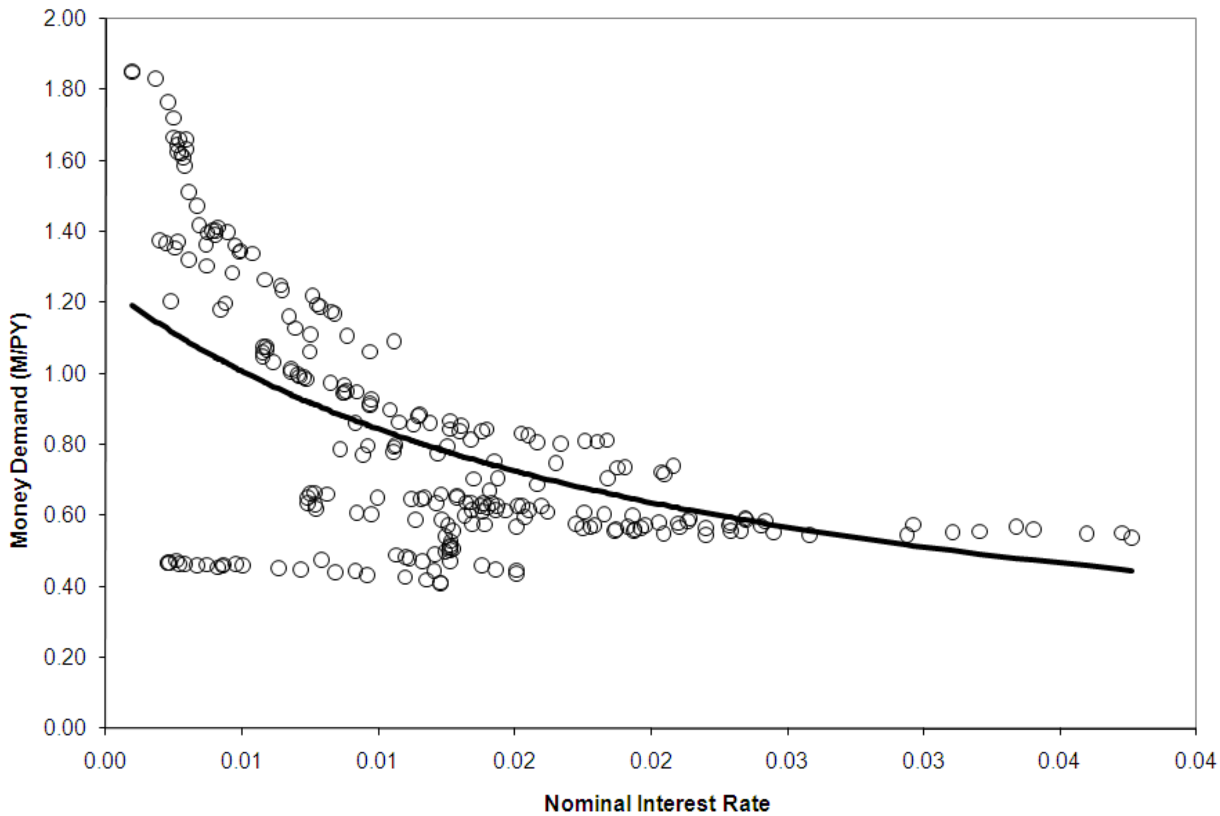

FIGURE B3: fit for quarterly specification

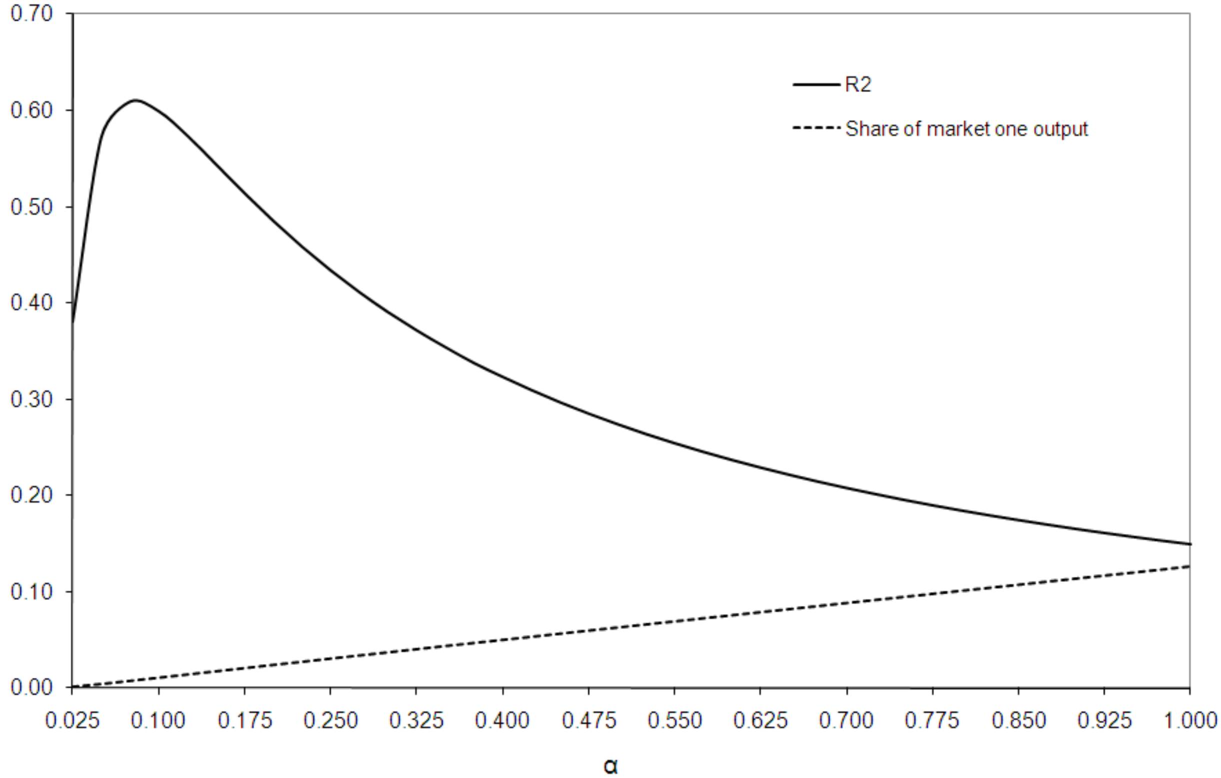

FIGURE B4: R2 and share of market one output as a function of α

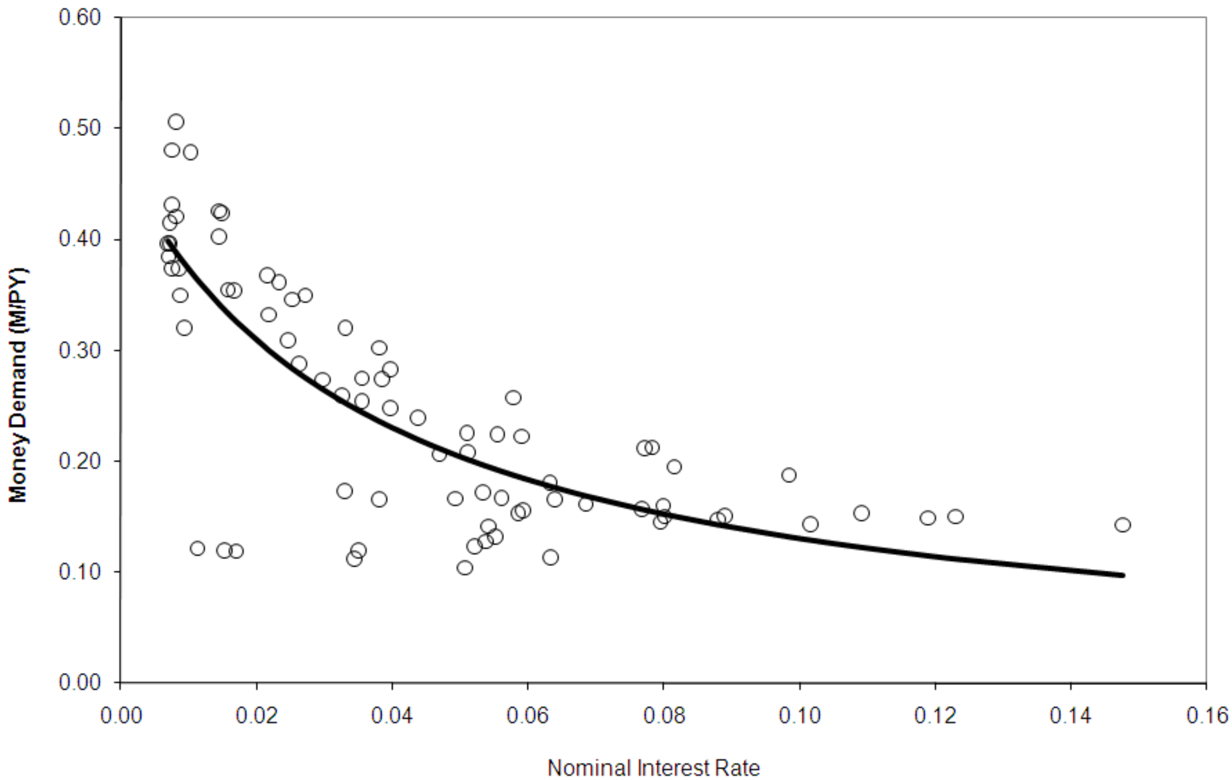

FIGURE B5: fit for parameter specification with highest fit (α=0.075, A=2.084, R^2 =0.61)

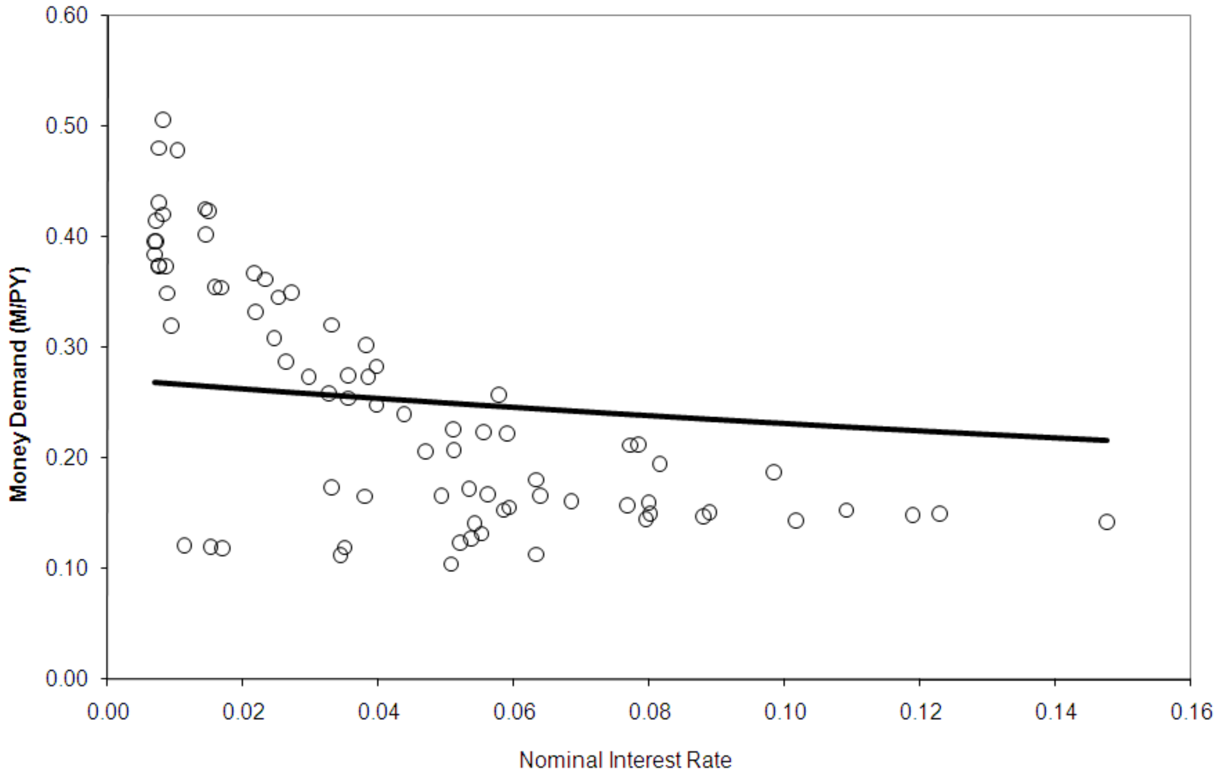

FIGURE B6: fit for parameter specification with lowest fit (α=1, A=3.176, R^2 =0.15)