Supplementary Information for:

Uniqueness of Equilibrium in Directed Search Models

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Changes in market structure on the price path

It is possible that some sellers could be out of the market along portions of the price path joining two candidate equilibria $\mathbf{v} < \mathbf{v}^*$. Here, we present an example of a possible change in market structure along the price path, and prove that this does not alter the analysis in the paper.

Suppose there are three different sellers. For simplicity, suppose seller 3 always keeps the promised utility fixed at v_3 along the price path. Let seller 1 be the one who has the largest percentage change in promised utility (this is called seller L in the paper). That is

$$\mathbf{v} = (v_1, v_2, v_3) \qquad \text{and} \qquad \mathbf{v}^* = (v_1^*, v_2^*, v_3)$$

with $v_1^* > v_2^*$ and $\dfrac{v_1^*}{v_1} > \dfrac{v_2^*}{v_2}$. An illustration is in the Figure below.

If v_2 is sufficiently low, then as we move along the path to $\mathbf{v}^*$ then seller 2 may exit and then re-enter the market. To see how this happens, note

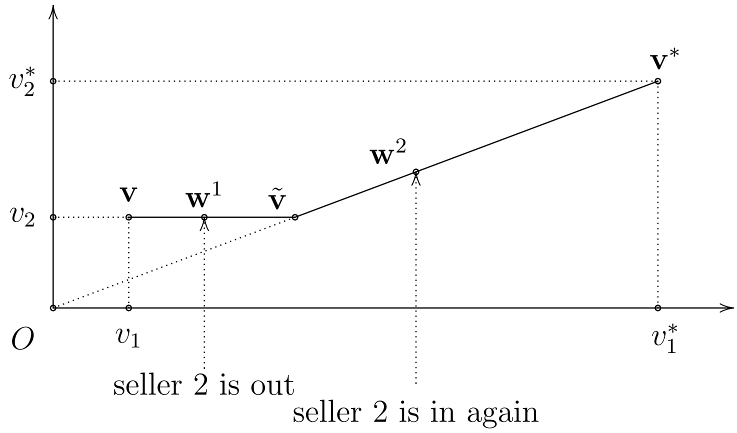

Figure 1: The price path between $\mathbf{v}^*$ and $\mathbf{v}$

that starting from $\mathbf{v}$, as seller 1 increases promised utility, buyers's payoff increases. At $\mathbf{w}^1$, if we have

$$\mathcal{H}(0)v_2 = \mathcal{H}(\pi_1(\mathbf{w}^1))w_1^1 = \mathcal{H}(\pi_3(\mathbf{w}^1))v_3, \qquad \text{with} \qquad \pi_1(\mathbf{w}^1)+\pi_3(\mathbf{w}^1) = 1,$$

i.e., seller 2 is out of the market as soon as we reach $\mathbf{w}^1$, because he does not increase promised utility.

From $\tilde{\mathbf{v}}$, seller 2 also starts to increase promised utility; as a consequence, at point $\mathbf{w}^2$, we can have

$$\mathcal{H}(0)w_2^2 = \mathcal{H}(\pi_1(\mathbf{w}^2))w_1^2 = \mathcal{H}(\pi_3(\mathbf{w}^2))v_3, \qquad \text{with} \qquad \pi_1(\mathbf{w}^2)+\pi_3(\mathbf{w}^2) = 1.$$

That is, as seller 2 increases promised utility above w_2^2, he again starts to attract buyers, since

$$v_1\frac{\partial \pi_2(\mathbf{v})}{\partial v_1} + v_1\frac{\partial \pi_2(\mathbf{v})}{\partial v_1} > 0,$$

on the segment of the price path going from $\mathbf{w}^2$ to $\mathbf{v}^*$, by Proposition 3.

The issue here is that the demand of seller 1 has kinks along the path, and therefore $\dfrac{\partial \pi_1}{\partial v_1}$ has jumps. This is illustrated in the figure below.

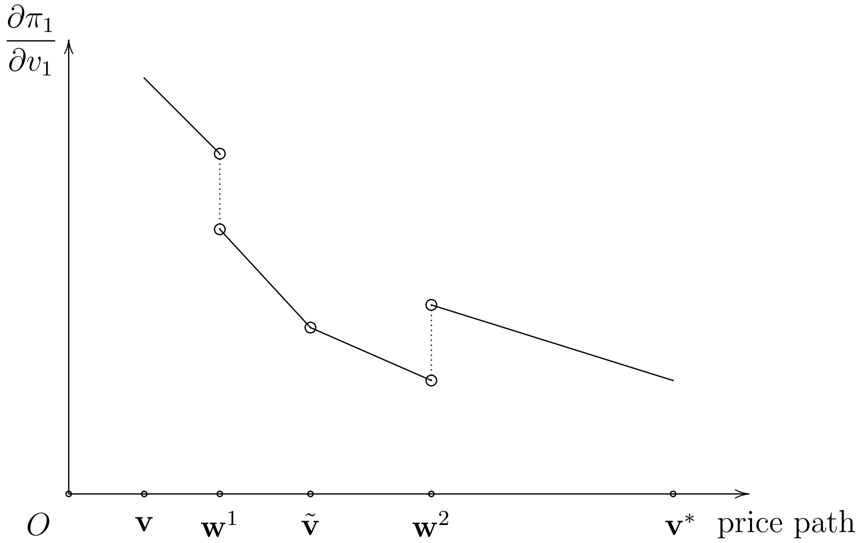

Figure 2: $\dfrac{\partial \pi_1}{\partial v_1}$ along the price path between $\mathbf{v}^*$ and $\mathbf{v}$

We will show that the analysis still goes through, despite these jumps in the derivatives of seller 1. We will do so in several steps.

First, notice that the change in demand $\dfrac{\partial \pi_1}{\partial v_1}$ is decreasing along each interval of the price path.

If no seller leaves the market along an interval (of price path), then the directional derivative is negative (Proposition 3)

$$D\left(\frac{\partial \pi_L(\mathbf{v})}{\partial v_L}\right) := \left(v_L \frac{\partial}{\partial v_L} + \sum_{i \neq L} \lambda_i v_i \frac{\partial}{\partial v_i}\right)\left(\frac{\partial \pi_L(\mathbf{v})}{\partial v_L}\right) < 0.$$

We want to show that $\dfrac{\partial \pi_1(\mathbf{w})}{\partial v_1}$ achieves lower values on the last segment of the price path, as opposed to the first segment. That is

$$\lim_{\mathbf{w} \to \mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > \lim_{\mathbf{w} \to \mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1},$$

Going back to our example, using equation (2) we have

$$\lim_{\mathbf{w}\to\mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} = -\frac{\mathcal{H}(\pi_1(\mathbf{w}^1))}{\mathcal{H}'(\pi_1(\mathbf{w}^1))w_1^1 + \cfrac{1}{\cfrac{1}{\mathcal{H}'(\pi_3(\mathbf{w}^1))v_3} + \cfrac{1}{\mathcal{H}'(0)v_2}}},$$

$$\lim_{\mathbf{w}\to\mathbf{w}^{1+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} = -\frac{\mathcal{H}(\pi_1(\mathbf{w}^1))}{\mathcal{H}'(\pi_1(\mathbf{w}^1))w_1^1 + \cfrac{1}{\cfrac{1}{\mathcal{H}'(\pi_3(\mathbf{w}^1))v_3} + 0}},$$

$$\lim_{\mathbf{w}\to\mathbf{w}^{2-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} = -\frac{\mathcal{H}(\pi_1(\mathbf{w}^2))}{\mathcal{H}'(\pi_1(\mathbf{w}^2))w_1^2 + \cfrac{1}{\cfrac{1}{\mathcal{H}'(\pi_3(\mathbf{w}^2))v_3} + 0}},$$

$$\lim_{\mathbf{w}\to\mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} = -\frac{\mathcal{H}(\pi_1(\mathbf{w}^2))}{\mathcal{H}'(\pi_1(\mathbf{w}^2))w_1^2 + \cfrac{1}{\cfrac{1}{\mathcal{H}'(\pi_3(\mathbf{w}^2))v_3} + \cfrac{1}{\mathcal{H}'(0)w_2^2}}}.$$

The first and last equations above reflect the fact that as we move towards $\mathbf{w}^1$ from the left and $\mathbf{w}^2$ form the right, there are 3 sellers who are active in the market (1,2 and 3). Only sellers 1 and 3 are active along the direction $\mathbf{w} \to \mathbf{w}^{1+}$ and $\mathbf{w} \to \mathbf{w}^{2-}$. Hence

$$\begin{aligned}
\lim_{\mathbf{w}\to\mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} - \lim_{\mathbf{w}\to\mathbf{w}^{1+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > 0, \\
\lim_{\mathbf{w}\to\mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} - \lim_{\mathbf{w}\to\mathbf{w}^{2-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > 0.
\end{aligned}$$

For the general case, we can define a directional limit of $\dfrac{\partial \pi_j(\mathbf{v})}{\partial v_j}$ along the price path. Consider a change in market structure such that—at some point $\mathbf{w}$ along price path—we go from s sellers to t sellers. In this case, a directional limit of $\dfrac{\partial \pi_j(\mathbf{v})}{\partial v_j}$ to $\mathbf{w}$, reached from the direction that bring us from s sellers to t sellers, is denoted as

$$\frac{\partial \pi_j(\mathbf{w}; s \to t)}{\partial v_j}.$$

For example, in the simple case we have consider above we would have

$$\frac{\partial \pi_1(\mathbf{w}^1; 3 \to 2)}{\partial v_1} = \lim_{\mathbf{w} \to \mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1}, \qquad \frac{\partial \pi_1(\mathbf{w}^1; 2 \to 3)}{\partial v_1} = \lim_{\mathbf{w} \to \mathbf{w}^{1+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1},$$
$$\frac{\partial \pi_1(\mathbf{w}^2; 3 \to 2)}{\partial v_1} = \lim_{\mathbf{w} \to \mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1}, \qquad \frac{\partial \pi_1(\mathbf{w}^2; 2 \to 3)}{\partial v_1} = \lim_{\mathbf{w} \to \mathbf{w}^{2-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1}.$$

We can also define what happens to the change in demand at the point $\mathbf{w}$, when the number of active sellers increases from t to $s > t$:

$$\Delta_j(\mathbf{w}; t \to s) := \overbrace{\frac{\partial \pi_j(\mathbf{w}; s \to t)}{\partial v_j}}^{\text{sellers leave}} - \overbrace{\frac{\partial \pi_j(\mathbf{w}; t \to s)}{\partial v_j}}^{\text{sellers come in}} > 0.$$

In our simple example,

$$\Delta_1(\mathbf{w}^1; 2 \to 3) = \lim_{\mathbf{w} \to \mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} - \lim_{\mathbf{w} \to \mathbf{w}^{1+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > 0,$$
$$\Delta_1(\mathbf{w}^2; 2 \to 3) = \lim_{\mathbf{w} \to \mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} - \lim_{\mathbf{w} \to \mathbf{w}^{2-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > 0.$$

Recall that we want to show that $\dfrac{\partial \pi_1(\mathbf{w})}{\partial v_1}$ achieves lower values on the last segment of the price path, as opposed to the first segment. That is

$$\lim_{\mathbf{w} \to \mathbf{w}^{1-}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1} > \lim_{\mathbf{w} \to \mathbf{w}^{2+}} \frac{\partial \pi_1(\mathbf{w})}{\partial v_1},$$

To do so, now we use Δ.

We know that $\Delta_j(\mathbf{w}; t \to s)$ is differentiable at any $\mathbf{w}$ along the price path. We need to show that

$$D\left(\Delta_L(\mathbf{v};t\to s)+\frac{\partial\pi_L(\mathbf{v};t\to s)}{\partial v_L}\right)<0,\qquad \mathbf{v}\in(\mathbf{w}^1,\tilde{\mathbf{v}})\cup(\tilde{\mathbf{v}},\mathbf{w}^2)$$

where D is a directional derivative. An illustration is given in the figure below.

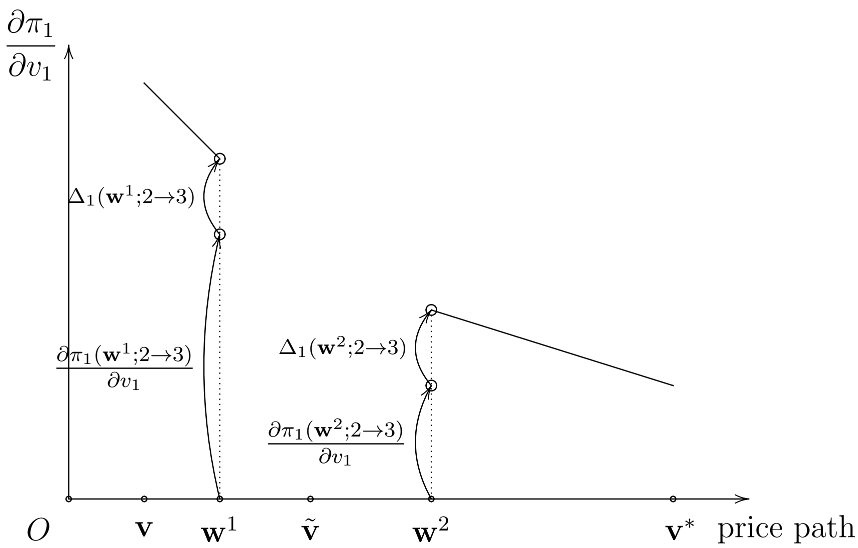

Figure 3: Breakdown of $\dfrac{\partial\pi_1}{\partial v_1}$ as the number of sellers grows from 2 to 3

Because

$$\begin{aligned}D\left(\Delta_L(\mathbf{v};t\to s)+\frac{\partial\pi_L(\mathbf{v};t\to s)}{\partial v_L}\right)&=D\left(\frac{\partial\pi_L(\mathbf{v};s\to t)}{\partial v_L}\right)\\&=D\left(-\frac{\mathcal{H}(\pi_L(\mathbf{v}))}{h_L+\dfrac{1}{\sum_{i\neq L}\dfrac{1}{h_i}+\dfrac{s-t}{h_0}}}\right)<0\end{aligned}$$

where $h_0:=\mathcal{H}'(0)\mathcal{H}(\pi_L(\mathbf{v}))v_L$. The inequality follows from Proposition 3;

in the last equation of proof of Proposition 3 there is one more term:

$$-\mathcal{H}(\pi_L)v_j \frac{\dfrac{(s-t)\mathcal{H}'(\pi_L)\dfrac{\partial \pi_L}{\partial v_j}v_L}{\mathcal{H}'(0)(\mathcal{H}(\pi_L)v_L)^2}}{\left(\sum_{i\neq L}\dfrac{1}{h_i}+\dfrac{s-t}{h_0}\right)^2} > 0, \qquad j \neq L$$

Everything else is same if we consider $1/h_0$ as any other $1/h_i$. Therefore we have the desired result.