Multi-player Bargaining with Endogenous Capacity
Appendix B - Not for publication

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Theorem 1 *The bargaining game between $n \geq 1$ buyers and a seller with $c = 1, ..., n$ goods has a unique subgame perfect equilibrium that is characterized as follows. The seller offers good i at price $q_i^s = q_i(c,n)$ with*

$$q_i(c,n) = 1 - \frac{\beta-\alpha}{n-i+1} \sum_{j=i}^{c} \beta^{j-i} \prod_{m=i}^{j} \frac{n-m+1}{n-m+1-\alpha} \tag{1}$$

and accepts any offer $q \geq \alpha q_i(c,n)$, where

$$\alpha = \frac{\beta\gamma}{\beta\gamma+1-\beta}. \tag{2}$$

Each buyer offers to buy good i at price $q_i^b = \alpha q_i(c,n)$ and accepts any offer $q \leq q_i(c,n)$.

Lemma 2 *The subgame perfect equilibrium described in Theorem 1 is the unique subgame perfect equilibrium of this game.*

Proof of Lemma 2

To prove uniqueness, we will demonstrate all SPE of this game must satisfy stationarity and no-delay. The proof involves three steps following the method by Shaked and Sutton (1984),[1] i.e., showing that the supremum and infimum of the set of SPE payoffs coincide.

[1] See Shaked, A. and J. Sutton (1984). Involuntary Unemployment as a Perfect Equilibrium in a Bargaining Model. *Econometrica* 52(6), 1351-1364.

As described in Muthoo (1999),[2] we exploit the stationary structure of the game. Any two subgames that start with the same player's offer (for the i^{th} good) are strategically equivalent. This means that the sets of subgame perfect equilibria in such subgames are identical. Hence the sets of SPE payoffs to the player making the offer are the same.

Let $\mathcal{B}_{k,i}$ denote the set of SPE payoffs to buyer k in any subgame in which buyer k makes an offer for good $i = 1, ..., c$. Similarly let $\mathcal{S}_{k,i}$ denote the set of SPE payoffs to the seller in any subgame starting with the seller making an offer to some buyer $k \in A_i$. Denote $\underline{b}_{k,i} = \inf \mathcal{B}_{k,i}$, $\overline{b}_{k,i} = \sup \mathcal{B}_{k,i}$, $\underline{s}_{k,i} = \inf \mathcal{S}_{k,i}$, $\overline{s}_{k,i} = \sup \mathcal{S}_{k,i}$.

Notice that the sets of payoffs depend on $k \in A_i$ because in principle different buyers may behave differently. Therefore, since a buyer is selected with uniform probability, we can define the expected infimum and supremum of the set of payoffs for each player as follows. First define the expectations

$$\begin{array}{l} \underline{\mu}_i(s) = \sum_{j \in A_i} \frac{\underline{s}_{j,i}}{n-i+1} \\ \overline{\mu}_i(s) = \sum_{j \in A_i} \frac{\overline{s}_{j,i}}{n-i+1} \\ \underline{\mu}_i(b) = \sum_{j \in A_i} \frac{\underline{b}_{j,i}}{n-i+1} \\ \overline{\mu}_i(b) = \sum_{j \in A_i} \frac{\overline{b}_{j,i}}{n-i+1}, \end{array}$$

which are conditional on the selection of, respectively, the seller and the buyer to make the offer. Therefore, the unconditional expected infima and suprema of the set of payoffs in the subgame where good i is sold are:

$$\begin{array}{rcl} \underline{w}_i & = & \gamma \underline{\mu}_i(s) + (1-\gamma)\left[1 - \overline{\mu}_i(b)\right] \\ \overline{w}_i & = & \gamma \overline{\mu}_i(s) + (1-\gamma)[1 - \underline{\mu}_i(b)] \end{array} \tag{3}$$

$$\begin{array}{rcl} \underline{u}_{k,i} & = & \frac{\gamma}{n-i+1}(1 - \overline{s}_{k,i}) + \frac{1-\gamma}{n-i+1}\underline{b}_{k,i} + \frac{\beta(n-i)}{n-i+1}\underline{u}_{k,i+1} \\ \overline{u}_{k,i} & = & \frac{\gamma}{n-i+1}(1 - \underline{s}_{k,i}) + \frac{1-\gamma}{n-i+1}\overline{b}_{k,i} + \frac{\beta(n-i)}{n-i+1}\overline{u}_{k,i+1} \end{array} \tag{4}$$

In any subgame in which good i is up for sale, the seller's smallest expected payoff is $\underline{w}_i$. With probability γ he gets to make an offer. The offer is made to buyer $j \in A_i$ with equal probability $\frac{1}{n-i+1}$. The seller's smallest payoff in this case is $\underline{s}_{j,i}$ and the expected

[2] See Muthoo, A. (1999). *Bargaining Theory with Applications.* Cambridge University Press07-637.

smallest payoff is $\underline{\mu}_i(s)$. With probability $1-\gamma$ some buyer makes the offer, and the seller's smallest expected payoff in this case is $1-\overline{\mu}_i(b)$.

In any subgame in which good i is up for sale, buyer k's smallest expected payoff is $\underline{u}_{k,i}$. With probability $\frac{\gamma}{n-i+1}$ buyer k is in a subgame in which the seller makes him an offer. This gives the buyer at least $1-\overline{s}_{k,i}$ payoff. With the probability $\frac{1-\gamma}{n-i+1}$ the buyer is in a subgame in which he makes an offer. In this case his smallest payoff is $\underline{b}_{k,i}$. With the complementary probability $\frac{n-i}{n-i+1}$ the buyer is not involved in negotiations. Since good i is sold to some other buyer, and good $i+1$ is put up for sale with probability β, then buyer k's smallest expected payoff is $\beta\underline{u}_{k,i+1}$.

Step 1. For all i and $k \in A_i$ we have

$$\overline{b}_{k,i} \leq 1-\beta\underline{w}_i \quad \text{and} \quad \underline{b}_{k,i} \geq 1-\beta\overline{w}_i \tag{5}$$

$$\overline{s}_{k,i} \leq 1-\beta\underline{u}_{k,i} \quad \text{and} \quad \underline{s}_{k,i} \geq 1-\beta\overline{u}_{k,i} \tag{6}$$

To prove it start with (5). In any SPE the seller's smallest expected payoff from negotiating over good i is $\underline{w}_i$. Therefore, if buyer k makes an offer, it cannot be less than $\beta\underline{w}_i$ (or the seller would not accept it). Thus, the buyer gets no more than $1-\beta\underline{w}_i$. The second inequality in (5) can be explained similarly. Now consider (6). In any subgame in which good i is put up for sale, buyer k's minimum expected payoff is $\underline{u}_{k,i}$. Therefore the seller cannot offer less than $\beta\underline{u}_{k,i}$ and so will gets no more than $1-\beta\underline{u}_{k,i}$.

Step 2. We prove that, for each player, the smallest and highest payoffs coincide. That is, for all i and $k \in A_i$ we have

$$\overline{s}_{k,i} = \underline{s}_{k,i} = q_i(c,n) \quad \text{and} \quad \overline{b}_{k,i} = \underline{b}_{k,i} = 1-\alpha q_i(c,n),$$

where $q_i(c,n)$ denotes the seller's equilibrium offer.

Start by noticing that, from (5) and (3) we have

$$\overline{b}_{k,i} \leq 1-\beta\left[\gamma\underline{\mu}_i(s)+(1-\gamma)\left[1-\overline{\mu}_i(b)\right]\right] \quad \text{for all } k \in A_i. \tag{7}$$

Take the average of both sides of (7) over all buyers in A_i. The left side becomes $\overline{\mu}_i(b)$ since $\sum_{k\in A_i}\frac{\overline{b}_{k,i}}{n-i+1} = \overline{\mu}_i(b)$. The right side is unchanged since it is independent of k, i.e.,

$\sum_{k\in A_i}\frac{X}{n-i+1}=X$ for X constant, since $|A_i|=n-i+1$. Then we have

$$\overline{\mu}_i(b)\leq 1-\beta\left[\gamma\underline{\mu}_i(s)+(1-\gamma)\left[1-\overline{\mu}_i(b)\right]\right]\quad\Rightarrow\quad\overline{\mu}_i(b)\leq 1-\alpha\underline{\mu}_i(s)$$

given our definition of α. Using the latter inequality jointly with (7) we obtain

$$\overline{b}_{k,i}\leq 1-\alpha\underline{\mu}_i(s). \tag{8}$$

We can similarly establish

$$\underline{b}_{k,i}\geq 1-\alpha\overline{\mu}_i(s). \tag{9}$$

Now use backward induction on i. Let $i=c$. Using (6), (4) and $\underline{u}_{k,c+1}=0$ we have

$$\overline{s}_{k,c}\leq 1-\frac{\beta\gamma(1-\overline{s}_{k,c})}{n-c+1}-\frac{\beta(1-\gamma)\,\underline{b}_{k,c}}{n-c+1}.$$

Then considering $\underline{b}_{k,c}$ from inequality (9) we have

$$\overline{s}_{k,c}\leq\frac{n-c+1-\beta}{n-c+1-\beta\gamma}+\frac{\alpha\beta(1-\gamma)}{n-c+1-\beta\gamma}\overline{\mu}_c(s). \tag{10}$$

Since this is true for all $k\in A_c$, we take the average of both sides of (10) over all buyers in A_c. The left side becomes $\sum_{k\in A_c}\frac{\overline{s}_{k,i}}{n-i+1}=\overline{\mu}_c(s)$ while the right side is unaffected. Rearranging (10) we get

$$\overline{\mu}_c(s)\leq\frac{n-c+1-\beta}{n-c+1-\alpha}=q_c(c,n).$$

This finding and (10) imply $\overline{s}_{k,c}\leq q_c(c,n)$. We can similarly establish $\underline{s}_{k,c}\geq q_c(c,n)$. Since $\overline{s}_{k,c}\geq\underline{s}_{k,c}$ we have

$$\overline{s}_{k,c}=\underline{s}_{k,c}=q_c(c,n).$$

Then (8) and (9) imply $\overline{b}_{k,c}=\underline{b}_{k,c}=1-\alpha q_c(c,n)$ because $\overline{\mu}_c(s)=\underline{\mu}_c(s)=q_c(c,n)$.

For the induction step suppose it is true that $\overline{s}_{k,j}=\underline{s}_{k,j}=q_j(c,n)$ for all $i+1\leq j\leq c-1$, and $k\in A_j$. Then it is also true that $\overline{b}_{k,j}=\underline{b}_{k,j}=1-\alpha q_j(c,n)$, and therefore $\overline{u}_{k,j}=\underline{u}_{k,j}=u_j$. When $j=i+1$, use

$$q_i(c,n)=1-\tfrac{\beta\Phi_i(c,n)}{n-i+1} \tag{11}$$

(from (9) in the paper)and (6) to get

$$\beta u_{i+1} = 1 - q_{i+1}(c,n) = \frac{\beta \Phi_{i+1}(c,n)}{n-i}. \tag{12}$$

Now we prove that $\overline{s}_{k,j} = \underline{s}_{k,j} = q_j(c,n)$ and $\overline{b}_{k,j} = \underline{b}_{k,j} = 1 - \alpha q_j(c,n)$ for $j = i$. Using (6), (4) we have

$$\begin{aligned}
\overline{s}_{k,i} &\leq 1 - \beta \underline{u}_{k,i} \\
&\leq 1 - \tfrac{\beta\gamma\left(1-\overline{s}_{k,i}\right)}{n-i+1} - \tfrac{\beta(1-\gamma)\underline{b}_{k,i}}{n-i+1} - \tfrac{\beta^2(n-i)}{n-i+1} u_{i+1} \\
&\leq 1 - \tfrac{\beta\gamma\left(1-\overline{s}_{k,i}\right)}{n-i+1} - \tfrac{\beta(1-\gamma)(1-\alpha\overline{\mu}_i(s))}{n-i+1} - \tfrac{\beta^2(n-i)}{n-i+1} u_{i+1}.
\end{aligned}$$

In the second line we have used the fact that $\underline{u}_{k,i+1} = u_{i+1}$ from the induction step. In the third line we have used (9). Inserting (12) into the last line and rearranging we obtain

$$\overline{s}_{k,i} \leq \tfrac{n-i+1-\beta}{n-i+1-\beta\gamma} + \tfrac{\alpha\beta(1-\gamma)\overline{\mu}_i(s)}{n-i+1-\beta\gamma} - \tfrac{\beta^2\Phi_{i+1}(c,n)}{n-i+1-\beta\gamma}. \tag{13}$$

Since this is true for all $k \in A_i$, we take the average of both sides of (13) over all buyers in A_i. The left side becomes $\sum_{k\in A_i} \frac{\overline{s}_{k,i}}{n-i+1} = \overline{\mu}_i(s)$ while the right side is unchanged. Rearranging (13) we get

$$\overline{\mu}_i(s) \leq \tfrac{n-i+1-\beta}{n-i+1-\alpha} - \tfrac{\beta^2\Phi_{i+1}(c,n)}{n-i+1-\alpha}.$$

Using this and (13) we obtain

$$\begin{aligned}
\overline{s}_{k,i} &\leq \tfrac{n-i+1-\beta}{n-i+1-\alpha} - \tfrac{\beta^2\Phi_{i+1}(c,n)}{n-i+1-\alpha} \\
&\leq 1 - \tfrac{\beta\Phi_i(c,n)}{n-i+1} = q_i(c,n).
\end{aligned}$$

In the second line we have used

$$\Phi_i(c,n) = \tfrac{n-i+1}{n-i+1-\alpha}\left[\tfrac{\beta-\alpha}{\beta} + \beta\Phi_{i+1}(c,n)\right], \tag{14}$$

i.e., the result in (28) from the proof of Lemma 3 in the Appendix to the paper. Similarly we can establish $\underline{s}_{k,i} \geq q_i(c,n)$. Since $\overline{s}_{k,i} \geq \underline{s}_{k,i}$, we have $\overline{s}_{k,i} = \underline{s}_{k,i} = q_i(c,n)$. Then (8) and (9) imply $\overline{b}_{k,i} = \underline{b}_{k,i} = b_i = 1 - \alpha q_i(c,n)$, because $\overline{\mu}_i(s) = \underline{\mu}_i(s) = q_i(c,n)$.

Using the result in Step 2, we can rearrange (3) and (4) to obtain

$$\begin{aligned}
&\underline{w}_i = \overline{w}_i = w_i = \tfrac{\alpha}{\beta} q_i(c,n) \\
&\underline{u}_{k,i} = \overline{u}_{k,i} = u_i = \tfrac{1-\frac{\alpha}{\beta}q_i(c,n)}{n-i+1} + \tfrac{\beta(n-i)}{n-i+1} u_{i+1}.
\end{aligned}$$

Compare these two expressions with

$$\pi_i = \tfrac{\alpha}{\beta} q_i^s \tag{15}$$

$$u_i = \tfrac{1-\frac{\alpha}{\beta}q_i^s}{n-i+1} + \tfrac{\beta(n-i)}{n-i+1} u_{i+1}. \tag{16}$$

respectively (from Lemma 2 in the paper).

We establish that in any SPE, when the seller and buyers in A_i negotiate over good i, the seller's and every buyer's expected payoffs are

$$w_i = \pi_i(c,n) = \frac{\alpha}{\beta} q_i(c,n)$$
$$u_i = u_i(c,n) = \frac{\Phi_i(c,n)}{n-i+1}.$$

Step 3. We want to prove that in any SPE offers are accepted without delay and are stationary. We first prove that in any SPE offers are immediately accepted. Suppose we are in a subgame in which the seller is making an offer to some buyer k. The argument above shows that he must offer exactly $q_i(c,n) = 1 - \frac{\beta\Phi_i(c,n)}{n-i+1}$. If the buyer's strategy is to accept any offer $q < q_i(c,n)$ and randomize when $q = q_i(c,n)$, then no best response for the seller exists. Randomization by the buyer is inconsistent with equilibrium. A similar argument applies in any subgame that starts with some buyer's offer. We now prove that offers are stationary. From Step 2 it is obvious that whenever the seller gets to make an offer, he proposes $q_i(c,n)$ and whenever a buyer in A_i makes an offer he proposes $\alpha q_i(c,n)$. This completes the proof of uniqueness.■