Supplementary Material: Endogenizing the regularity properties

Key properties of matching and payoff functions are usually assumed in the literature to prove existence of equilibrium in the entire game.[1]
Here, we show that such properties emerge endogenously in a symmetric outcome. In doing so, we consider $\rho(i) = \frac{1-\theta^i}{i}$, for all $i = 1, \dots, I$, as it generalizes the typical case where $\theta = 0$.

Proposition. Let $q^i(I,\pi) := \frac{I!}{i!(I-i)!}\pi^i(1-\pi)^{I-i}$ for all $i = 0, \dots, I$ and $\rho(i) = \frac{1-\theta^i}{i}$, for all $i = 1, \dots, I$. For each $\pi \in [0,1]$, we have

- $\mathcal{M}(\pi)$ is twice continuously differentiable, strictly increasing, and concave.
- $\mathcal{H}(\pi)$ is twice continuously differentiable, strictly decreasing, and convex.
- $\mathcal{H}(\pi)^{-1}$ is convex, i.e. $2(\mathcal{H}'(\pi))^2 - \mathcal{H}''(\pi)\mathcal{H}(\pi) \geq 0$
- $\mathcal{H}(\pi)v$ is quasiconcave.

Proof. Consider a generic seller.

The function $\mathcal{M}(\pi)$. Notice that

$$\begin{aligned}\frac{I\pi}{i+1}q^i(I-1,\pi) =& \frac{I\pi}{i+1}\cdot\frac{(I-1)!}{i!(I-1-i)!}\pi^i(1-\pi)^{I-1-i}\\ =& \frac{I!}{(i+1)!(I-(i+1))!}\pi^{i+1}(1-\pi)^{I-(i+1)} = q^{i+1}(I,\pi)\ .\end{aligned}$$

From (2) in the paper, the functions $\mathcal{M}(\pi)$ and $\mathcal{H}(\pi)$ are twice continuously differentiable, because $q^i(I,\pi)$ is smooth in p for all i. Now notice

$$\begin{aligned}\mathcal{M}(\pi) &= \sum_{i=1}^{I} q^i(I,\pi) - \sum_{i=1}^{I} q^i(I,\pi)\theta^i \ = \ 1 - q^0(I,\pi) - \sum_{i=1}^{I} q^i(I,\pi)\theta^i\\ &= 1 - \sum_{i=0}^{I} q^i(I,\pi)\theta^i \ = \ 1 - \sum_{i=0}^{I}\binom{I}{i}(\pi\theta)^i(1-\pi)^{I-i}\\ &= 1-(1-(1-\theta)\pi)^I\end{aligned}$$

[1] For a recent example consider properties (i)-(iii) in [1, Assumption 2].

where the last line follows from the binomial formula. It is clear that $\mathcal{M}'(\pi) > 0$ and $\mathcal{M}''(\pi) < 0$ by direct differentiation.

The functions $\mathcal{H}(\pi)$ and $1/\mathcal{H}(\pi)$. Without loss of generality, set $\theta = 0$. Thus

$$\begin{aligned}
\mathcal{H}(\pi) &= \frac{\mathcal{M}(\pi)}{I\pi} = \frac{1-(1-\pi)^I}{I\pi} \\
\mathcal{H}'(\pi) &= \frac{1}{I\pi^2}\left[(1-\pi)^{I-1}(1-\pi+I\pi)-1\right].
\end{aligned}$$

- We show that $\mathcal{H}'(\pi) < 0$ for $I \geq 2$.
  The proof is by induction on I. Define the term

  $$A(\pi, I) := (1-\pi)^{I-1}(1-\pi+I\pi)-1.$$

  We have
  $$\mathcal{H}'(\pi) = \frac{1}{I\pi^2}A(\pi, I)$$
  Clearly $A(\pi,1) \leq 0$. Consider the following induction hypothesis: $A(\pi, I) < 0$ for some $I \geq 2$. We need to prove that $A(\pi, I+1) < 0$. We have

  $$\begin{aligned}
  A(\pi, I+1) &= (1-\pi)^I(1-\pi+I\pi+\pi)-1 \ = \ (1-\pi)^{I-1}(1-\pi)(1+I\pi)-1 \\
  &= (1-\pi)^{I-1}(1-\pi+I\pi-I\pi^2)-1 \ < \ (1-\pi)^{I-1}(1-\pi+I\pi)-1 \\
  &= A(\pi, I)
  \end{aligned}$$

  Hence $A(\pi, I) < 0$ for all $I \geq 1$. It follows that $\mathcal{H}'(\pi) < 0$.

- We show that $\mathcal{H}''(\pi) > 0$ for $I \geq 2$. By direct differentiation we have:

  $$\mathcal{H}''(\pi) = \frac{\pi}{I\pi^4}\left\{A'(\pi, I)\pi - 2A(\pi, I)\right\}$$

  where
  $$A'(\pi, I) := -I(I-1)(1-\pi)^{I-2}\pi.$$
  Hence, if $A'(\pi, I)\pi - 2A(\pi, I) > 0$, then $\mathcal{H}''(\pi) > 0$. The proof is by induction on I:

  $$A'(\pi, I)\pi - 2A(\pi, I) = -I(I-1)(1-\pi)^{I-2}\pi^2 - 2(1-\pi)^{I-1}(1-\pi+I\pi)+2$$

  For $I = 2$ we have $A'\pi - 2A = 0$. For the induction hypothesis suppose $A'\pi - 2A > 0$ for some $I \geq 3$, that is

  $$2[(1-\pi)^{I-1}(1-\pi+I\pi)-1] < -I(I-1)(1-\pi)^{I-2}\pi^2$$

We need to show that this is true for $I+1$, i.e.,

$$\begin{aligned}
2(1-\pi)^I(1-\pi+I\pi+\pi)-2 &\leq -I(I+1)(1-\pi)^{I-1}\pi^2 \\
2[(1-\pi)^{I-1}(1-\pi)(1+I\pi)-1] &\leq -I(I-1)(1-\pi)^{I-1}\pi^2-2I(1-\pi)^{I-1}\pi^2 \\
2[(1-\pi)^{I-1}(1-\pi+I\pi)-1] &\leq -I(I-1)(1-\pi)^{I-1}\pi^2
\end{aligned}$$

By virtue of the induction hypothesis we have

$$2[(1-\pi)^{I-1}(1-\pi+I\pi)-1] < -I(I-1)(1-\pi)^{I-2}\pi^2 < -I(I-1)(1-\pi)^{I-1}\pi^2$$

so the statement is true for $I+1$.

- We prove $\mathcal{H}''(\pi)\mathcal{H}(\pi)-2\left(\mathcal{H}'\pi)\right)^2<0$.
Notice that

$$\begin{aligned}
&\mathcal{H}''(\pi)\mathcal{H}(\pi)-2\left(\mathcal{H}'\pi)\right)^2 \\
&\quad = \frac{\pi}{I\pi^4}\left\{A'(\pi,I)p-2A(\pi,I)\right\}\frac{1-(1-\pi)^I}{I\pi}-2\left(\frac{1}{I\pi^2}\right)^2\left[(1-\pi)^{I-1}(1-\pi+I\pi)-1\right]^2 \\
&\quad \propto \left[A'(\pi,I)\pi-2A(\pi,I)\right]\left[1-(1-\pi)^I\right]-2\left[A(\pi,I)\right]^2
\end{aligned}$$

We want to prove that

$$\left[A'(\pi,I)\pi-2A(\pi,I)\right]\left[1-(1-\pi)^I\right]-2\left[A(\pi,I)\right]^2<0.$$

Using the definition for A' and A this can be rearranged as

$$-(I-1)\pi\left[1-(1-\pi)^I\right]<2A(\pi,I)(1-\pi).$$

Again, we use the proof by induction. Notice that for $I=2$ the inequality above holds. For the induction hypothesis, suppose it also holds for some $I\geq 2$. Then we must show that it holds for $I+1$, i.e.

$$-I\pi\left[1-(1-\pi)^{I+1}\right]<2A(\pi,I+1)(1-\pi).$$

This inequality is rearranged as

$$\begin{aligned}
-I\pi\left[1-(1-\pi)^I\right]+I\pi^2(1-\pi)^I &< (1-\pi)2[(1-\pi)^{I-1}(1-\pi+I\pi)-1] \\
&= 2(1-\pi)A(\pi,I)
\end{aligned}$$

The left hand side can be rearranged as follows:

$$\begin{aligned}
&-(I-1)\pi\left[1-(1-\pi)^I\right]-\pi\left[1-(1-\pi)^I\right]+I\pi^2(1-\pi)^I\\
&\qquad=-(I-1)\pi\left[1-(1-\pi)^I\right]+\pi\left[-1+(1-\pi)^{I-1}(1-\pi+I\pi)-I\pi^2(1-\pi)^{I-1}\right]\\
&\qquad=-(I-1)\pi\left[1-(1-\pi)^I\right]+\pi\left[A(I,\pi)-I\pi^2(1-\pi)^{I-1}\right],
\end{aligned}$$

therefore it is smaller than $2(1-\pi)A(\pi,I)$ because (i) $A(\pi,I)-I\pi^2(1-\pi)^{I-1}<0$ due to $A(\pi,I)\leq 0$, and (ii) by induction hypothesis above. Hence $\mathcal{H}''(\pi)\mathcal{H}(\pi)-2\left(\mathcal{H}'(\pi)\right)^2<0$ for all $I\geq 2$.

Quasiconcavity of $\mathcal{H}(\pi)v$. If $\mathcal{H}(\pi)v=r\geq 0$, then the superior set is defined by $S_r=\{(\pi,v)\in[0,1]\times[\underline{v},\overline{v}]:\mathcal{H}(\pi)v\geq r\}$. The set S_r is convex because if $(\pi,v),(\pi',v')\in S_r$ then for $\lambda\in(0,1)$, we have

$$\frac{r}{\mathcal{H}(\pi^\lambda)}\quad\leq\quad\lambda\frac{r}{\mathcal{H}(\pi')}+(1-\lambda)\frac{r}{\mathcal{H}(\pi)}\quad\leq\quad\lambda v'+(1-\lambda)v$$

where $\pi^\lambda=\lambda\pi'+(1-\lambda)\pi$. The first line follows from convexity of $\frac{1}{\mathcal{H}(\pi)}$; the second line follows from $(\pi,v),(\pi',v')\in S_r$. $\square$

References

[1] Galenianos, M. and P. Kircher, (2012). On the Game-theoretic Foundations of Competitive Search Equilibrium. *International Economic Review*, 53 (1), 121.