Supporting Materials for:

A tractable analysis of contagious equilibria

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1 Closed-form transition matrices

In this section we show how to construct the transition matrix Q for decentralized punishment schemes associated to the following grim strategy.

Definition 1. *On $t = 0$, agent i is in state $s = C$ and selects action C. On all $t > 0$, agent i is either in state $s = C$ or $s = D$, and selects action s.*

- *If agent i is in state C in period t, then i switches state on $t + 1$ only if some agent in $O_i(t)$ selected D. Otherwise, i remains in state C.*
- *State D is absorbing.*

Let j denote the number of *mixed* pairs in a generic period.

For generality, consider the case in which an agent can see the actions of $a = 0, \dots, N-2$ agents *in addition* to his current opponent, i.e., $|O_i(t, a)| = a + 2$ in each period t for each agent i. It is assumed that these additional agents are randomly selected with a uniform probability, iid across agents (for example, due to random proximity). A player in state $s = C$ switches to state $s = D$ as soon as he observes one agent defect, either his opponent or someone else. Consequently, decentralized punishment proceeds either by *direct contagion*, when cooperators end up in mixed matches, or by *indirect contagion*, when cooperators do not meet defectors but see someone defecting in another match.

1.1 Private monitoring: direct contagion ($a = 0$)

Let $a = 0$, i.e., $O_i(t) = \{i, o_i(t)\}$ in each period t for all agents i. Suppose that there is an outcome of the random matching in which j pairs are *mixed*, then j additional agents will be defectors from next period on because if a defector meets a cooperator, then the cooperator becomes a defector from next period on (Definition 1).

Here, the contagious process has a key property: the number of defectors can only increase by an even number when k is even, and by an odd number when k is odd. Hence, the possible number of mixed pairs is

$$j \in J_k := \begin{cases} \{0, 2, 4, \ldots, J\} & \text{if } k = \text{even} \\ \{1, 3, 5, \ldots, J\} & \text{if } k = \text{odd} \end{cases} \qquad \text{with} \qquad J := \min(k, N-k) \quad (1)$$

To prove it, let the number of defectors k be even (resp. odd) and show that the number of mixed pairs cannot be odd (resp. even). By means of contradiction, suppose there is an odd (resp. even) number of mixed pairs. Recall that each agent is matched in each period. Hence, we must pair among themselves all remaining odd cooperators, and we must do the same for all remaining odd defectors; but this is impossible. This gives us the desired contradiction.[12]

Lemma 1. *Consider $N-k$ cooperators and $k=0,\dots,N$ defectors. The number of all possible random pairings that create j mixed pairs is given by*

$$\Lambda_{kj} := \begin{cases} j!\binom{k}{j}\binom{N-k}{j}(k-j-1)!!(N-k-j-1)!! & \text{if } j \in J_k \\ 0 & \text{if } j \notin J_k. \end{cases} \tag{2}$$

where $\sum_{j=0}^{N} \Lambda_{kj} = \sum_{j=0}^{J} \Lambda_{kj} = \Lambda_{00} = (N-1)!!$

Proof of Lemma 1. Fix $k=0,1,\dots,N$ defectors, and $N-k$ cooperators. Consider $j \notin J_k$; As explained above it is not feasible to create j mixed pairs (e.g., k is even and j is odd). Hence $\Lambda_{kj}=0$ for $j \notin J_k$.

Now consider $j \in J_k$. Start by randomly choosing the j defectors that must be matched to j cooperators. There are $\binom{k}{j}$ possible ways to choose j agents from a set of k defectors. Similarly, there are $\binom{N-k}{j}$ possible ways to choose j agents from the set of $N-k$ cooperators. Hence, the number of ways in which we can choose j defectors and j cooperators is

$$\binom{k}{j}\binom{N-k}{j}$$

Having selected j defectors and j cooperators, consider all possible ways to form j mixed pairings. Fix an agent from the first set (say, a defector); we can match him to any of the j agents from the second set (cooperators). Now, fix another agent from the first set; we can match him to any of the $j-1$ remaining agents from the second set. Repeating the process until we run out

[12] If everyone follows the strategy in Definition 1, then the number of defectors can only increase over time and only in even numbers (after the first defection). So, either $k=1$, or $k=0,2,4,\dots$. However, for generality we also consider the cases of odd number of defectors when $k>1$.

of agents in each set, all possible ways to randomly pair each of the j defectors to one of the j cooperators is

$$j \cdot (j-1) \cdot (j-2) \cdots 3 \cdot 2 \cdot 1 = j!$$

Hence, given j the number of all possible mixed parings is

$$j! \binom{k}{j} \binom{N-k}{j}$$

Now consider all possible ways in which the remaining $k-j$ defectors can be matched among themselves and the remaining $N-k-j$ cooperators can be matched among themselves. Recall that $k-j$ and $N-k-j$ are necessarily even numbers. Considering the defectors, fix an agent in that set and match him to one of the remaining $k-j-1$ defectors. Now fix another defector and match him to one of the $k-j-3$ defectors who are left. Repeating this procedure until all $k-j$ defectors have been matched among themselves, gives

$$(k-j-1) \cdot (k-j-3) \cdots 3 \cdot 1 = (k-j-1)!!$$

possible pairings. Similarly, we have $(N-k-j-1)!!$ possible pairings among the cooperators.

Clearly, we have $\sum_{j=0}^{N} \Lambda_{kj} = \sum_{j \in J_k} \Lambda_{kj}$ because $\Lambda_{kj} = 0$ if $j \notin J_k$. To prove that $\sum_{j \in J_k} \Lambda_{kj} = (N-1)!!$, one can use direct calculation. Alternatively, notice that the summation $\sum_{j=0}^{N} \Lambda_{kj}$ is simply the number of *all possible* pairings in an economy of N agents, irrespective of whether they are cooperators or defectors. It should be clear that $\sum_{j=0}^{N} \Lambda_{kj}$ is independent of k because the matching process is random, and so it does not depend on the actions taken by agents. To obtain it, use the following recursive procedure.

Let S_N denote the number of all possible pairings among N individuals. Fix an agent; There are $N-1$ agents that can be matched to him. Once the agent is paired, there remain $N-2$ agents, who can be matches in S_{N-2} possible ways. Hence we have $S_N = (N-1)S_{N-2}$. Now recursively repeat the procedure above, fixing another agent among the remaining $N-2$. The number of pairings is recursively defined by $S_m = (m-1)S_{m-2}$ for $m = 2, \ldots, N$, with $S_0 = 1$. It follows that $S_N = (N-1)!!$. Clearly, $S_N = \Lambda_{kj}$ with $k = j = 0$. $\square$

Clearly, $\Lambda_{00} = (N-1)!!$ is the number of pairings we can have in a population of N cooperators or, equivalently, all possible pairings we can have for any configuration of mixed matches, i.e., $\sum_{j=0}^{J} \Lambda_{kj}$.

When the economy has k defectors, we let

$$\lambda_k(j) = \frac{\Lambda_{kj}}{(N-1)!!} \tag{3}$$

denote the probability associated to a pairing with j mixed pairs. This is also the probability that economy transitions to a state with $k+j$ defectors, i.e., letting $j = k' - k$ we have

$$Q_{kk'} = \begin{cases} \lambda_k(k'-k) & \text{if } 1 \leq k \leq k' \leq \min(2k, N) \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Note that (i) Q is upper-triangular; (ii) $Q_{kk'} = 0$ for $k' > \min(2k, N)$; (iii) for all k we have $Q_{kk'} = 0$ for k' odd because $\Lambda_{kk'} = 0$ when k' is odd. This last feature arises because the increment in defectors $k' - k$ depends on whether k is even or odd; the increment is even when k is even, and is odd when k is odd because each mixed match generates exactly one additional defector. Hence, in the case of $a = 0$ we have

$$Q := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & Q_{24} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{34} & 0 & Q_{36} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & Q_{N-2,N-2} & 0 & Q_{N-2,N} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

1.1.1 Conditional probabilities $\eta_{kk'}$ (private monitoring)

Consider $a = 0$, i.e. private monitoring. We wish to find exact expressions for the conditional probabilities $\eta_{kk'}$. Suppose agent i is one of k defectors. Fix a pairing in which $k' - k$ defectors meet $k' - k$ cooperators. Conditional on this pairing, the probability that defector i meets a cooperator is

$$\eta_{kk'} = \frac{k' - k}{k}, \qquad \text{for} \quad k \leq k' \leq \min(2k, N).$$

Clearly if $k' = k$ (which occurs with probability Q_{kk}), then there are no mixed matches so none of the k defectors meets a cooperator, hence $\eta_{kk} = 0$. If there is only one mixed match, then the probability that i is in the mixed match is $\frac{1}{k}$. If there are $k' - k \geq 2$ mixed matches, then the random matching process may pair defector i to any one of $k' - k \geq 2$ cooperators. For illustrative purposes, suppose that pairings are formed sequentially. With probability $\frac{1}{k}$ agent i is selected in the first round with equal probability among all k defectors. With probability $\left(1 - \frac{1}{k}\right)\frac{1}{k-1}$ agent i is not selected in the first round and is selected in the second, and so on. Agent i is selected in a generic round $j = 2, \ldots, (k' - k)$ with round-invariant probability

$$\begin{aligned} & \Big(1 - \frac{1}{k}\Big) \times \Big(1 - \frac{1}{k-1}\Big) \times \cdots \times \Big(1 - \frac{1}{k-j+2}\Big) \times \frac{1}{k-j+1} \\ = \; & \frac{k-1}{k} \times \frac{k-2}{k-1} \times \cdots \times \frac{k-j+1}{k-j+2} \times \frac{1}{k-j+1} \\ = \; & \frac{1}{k}. \end{aligned}$$

Hence, since there are summing all probabilities to be selected in each round $j = 1, \ldots, k' - k$ we have that the defector is matched to a cooperator with probability

$$\eta_{kk'} = \frac{1}{k} + \sum_{j=2}^{k'-k} \frac{1}{k} = \frac{k' - k}{k}.$$

It is easy to verify that

$$\sigma_k = \sum_{k'=k}^{N} Q_{kk'} \eta_{kk'}.$$

where $\sigma_k = \dfrac{N-k}{N-1}$. In particular, $\sigma_1 = Q_{12} = 1$, because from $k = 1$ the economy can only reach the state $k' = 2$.

1.1.2 Noisy transitions (private monitoring)

An interesting extension is to take in account the possibility of noise in implementing punishment; for instance, cooperators may hesitate to switch state and start punishing when they become aware of a defection. In order to capture this feature, we amend the strategy in Definition 1 by assuming that a player in state $s = C$ who observes a defection, switches to state $s = D$ with probability ϵ. We will say that in this economy transitions are "noisy" and we wish to find a closed form for the transition matrix Q in this noisy scenario.

The central implication is that, given k initial defectors, if on date t a realization of the random matching generates $j \leq J := \min(k, N-k)$ mixed matches, then the number of defectors *may* increase by $h \leq j$. Conditional on j mixed matches, the probability to have h additional defectors is

$$\varepsilon_j(h) = \binom{j}{h} \epsilon^h (1-\epsilon)^{j-h}.$$

It is immediate that for $\epsilon = 1$ we have $\varepsilon_j(h) = 0$ for all $j > h$, while $\varepsilon_j(j) = 1$ (which is the case previously discussed).

Now, note that we can have h additional defectors as a consequence of having at least h mixed matches (= every cooperator in a mixed match switches state) and at most J mixed matches (= only the cooperators in h mixed matches switch state). The matching process randomly selects a pairing with equal probability among all possible pairings. Hence, the probability of having mixed matches that may generate h new defectors is

$$\sum_{j=h}^{J} \lambda_k(j)\,,$$

where $\lambda_k(j)$ is the probability in (3). When the economy has k defectors, the probability that the number of defectors in the economy increases from k to

$k' = k + h$ given in (4) must be is modified as

$$Q_{kk'} = \begin{cases} \sum_{j=k'-k}^{J} \lambda_k(j)\varepsilon_j(k'-k) & \text{if } 1 \leq k \leq k' \leq \min(2k, N) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Clearly, because of the noise in transitions, $Q_{kk'}$ is no longer equivalent to the probability associated to obtaining a random pairing with $k' - k$ mixed pairs.

1.2 Public monitoring: direct & indirect contagion ($a \geq 0$)

Here we study the general case in which each agent i in period t observes the actions of a set of agents denoted $O_i(t, a)$, which includes agent i, i's opponent $o_i(t)$, and $a = 0, \ldots, N-2$ other randomly selected agents, i.e., $|O_i(t, a)| = a+2$ in each period t for each agent i. These additional a agents are randomly selected with a uniform probability, iid across agents (for example, due to random proximity). Again, we will presume that a player in state $s = C$ switches to state $s = D$ as long as he observes one defection. This implies that the probability that a cooperator in a mixed match switches to state $s = D$ does not depend on whether defections were observed outside the match. For a cooperator not in a mixed match, the probability to switch to state $s = D$ depends on the probability he observes at least one defection in some other match.

Let there be $k \geq 1$ defectors in a generic period. Start by noting that if $a + 2 > N - k$, then each cooperator necessarily sees at least one defection, because the number of agents whose actions are observed, $a + 2$, is greater than the number of cooperators, $N - k$. Hence, in this case, every cooperator will become a defector form next period on, with probability 1.

Now, consider $a + 2 \leq N - k$. Here, each player observes the actions of a number of agents that is at most equal to the number of cooperators, $N-k$. Each realization of the random matching process partitions cooperators into two sets: those who are in mixed matches and those who meet other cooperators. Indirect contagion may only spread among the latter group of cooperators. Suppose that an outcome of the random matching generates j mixed pairs. Consequently, there are $N - k - j$ cooperators who are not in mixed matches. These cooperators may or may not observe a defection. Since for any agent i the a agents in $O_i(t) \setminus \{i, o_i(t)\}$ are randomly selected by an independent process, a cooperator who is not in a mixed match sees *at least* a defection in $O_i(t) \setminus \{i, o_i(t)\}$ with probability

$$\delta_{ka} := \begin{cases} 0 & \text{if } a = 0 \\ 1 - \prod_{m=0}^{a-1} \left(1 - \frac{k}{N-m-2}\right) & \text{if } a = 1, \ldots, N-k-2 \\ 1 & \text{if } a > N-k-2. \end{cases}$$

To obtain δ_{ka}, consider a cooperator, say, agent i, whose opponent $o_i(t)$ is also a cooperator. Agent i observes the actions of $a \geq 0$ individuals, in addition to the actions of agents $\{i, o_i(t)\}$ and he starts defecting only if he sees *at least* one defection. Switching state does not depend on how many defections are observed; one is all that is needed. For illustrative purposes, suppose that agent i draws a observations sequentially, one at a time. Partition the population into cooperator i, cooperator $o_i(t)$, and $N-2$ remaining agents.

Clearly if $a = 0$, then there is no additional observation so we are back to the case of private monitoring and $\delta_{ka} = 0$. Now consider the case $a > 0$. Given k defectors, if $1 \leq a \leq N-2-k$, then cooperator i sees a defection with probability $\frac{k}{N-2}$ on his first observation, with probability $\left(1 - \frac{k}{N-2}\right)\frac{k}{N-3}$ on his second observation, and so on. Therefore, agent i sees no defection after $a = 1, \ldots, N-2-k$, observations with probability

$$\left(1 - \frac{k}{N-2}\right) \times \cdots \times \left(1 - \frac{k}{N-2-(a-1)}\right) = \prod_{m=0}^{a-1} \left(1 - \frac{k}{N-2-m}\right)$$

With complementary probability $1 - \prod_{m=0}^{a-1} \left(1 - \frac{k}{N-2-m}\right)$ cooperator i sees at least one defection, when $a = 1, \ldots, N-2-k$. This gives us the second line, above.

Finally, if $a > N-k-2$ then cooperator i will certainly observe a defection because he observes the actions of a number a of agents outside his match, which is greater than the number of cooperators outside his match $N-k-2$;

hence, we have $\delta_{ka} = 1$ as in the last line of the expression above.[13]

Given k and j, we are interested in studying the probability $\Delta^a_{kj}(m)$ that m cooperators who *are not* in mixed matches, see at least one defection outside of their match. Clearly, the number of cooperators outside of mixed matches is $N - j - k$ and so $m \leq N - k - j$. Consequently,

$$\Delta^a_{kj}(m) = \binom{N-k-j}{m} (\delta_{ka})^m \; (1 - \delta_{ka})^{N-k-j-m}.$$

Now, consider the case in which we transition from k defectors today to $k + h \geq k$ defectors tomorrow. The h additional defectors can be due to a direct (private monitoring) or indirect (public monitoring) observation of defections. As before, $\lambda_k(j)$ is the probability that j cooperators are in j mixed pairs, which leads to a direct observation of defections. In addition, $\Delta^a_{kj}(m)$ is the probability that, conditional on having k defectors and j mixed matches, m cooperators who are *not* in mixed matches observe outside of their match that at least one of a agents has defected. Hence, the probability to go from k to $k + h$ defectors, where the number of additional defectors is

[13]Suppose the agent has observed the actions of $a - 1$ agents outside his match and has only seen cooperation. Then, in his a^{th} observation the probability to see a defection would be $\min\left(\frac{k}{N-2-(a-1)}, 1\right)$ and notice that $\frac{k}{N-2-(a-1)} \geq 1$ whenever $a + 2 > N - k$.

$h = j + m \leq N - k$, is given by

$$Q_{k,k+h} := \begin{cases} 1 & \text{if } a+2 > N-k \text{ and } h = N-k \\ 0 & \text{if } a+2 > N-k \text{ and } h < N-k \\ \sum_{j=0}^{\min(h,J)} \lambda_k(j)\Delta_{kj}^a(h-j) & \text{if } a+2 \leq N-k \text{ and } h \leq N-k \end{cases}$$

It should be clear that $Q_{k,k'} = 0$ for $k' < k$ since there is no reversion to cooperation. To understand the expression above recall that every agent i observes the actions of $a+2$ agents in each period, where the number 2 refers to the actions of i and her current opponent, and a refers to the actions of a agents in some other pair(s).

The first two lines correspond to the case in which the number of agents whose actions are observed exceeds the number of remaining cooperators in the economy; clearly, in this case *every* cooperator must observe at least one defection. Consequently, there is certainty to transition to a state of full defection ($h = N - k$), while it is impossible to transition to any other state with less than full defection ($h < N - k$). The last line corresponds to the case when cooperators do not necessarily see a defection, i.e., the number of agents whose actions are observed, $a+2$, is less than the number of cooperators in the economy $N - k$. The summation is over j, which is the number of additional defectors due to *direct* observation (in mixed matches). This number can go from 0 (defections are only observed in matches other than the player's own match) to $\min(h, J)$ (the smallest of the number of additional defectors h, and the number of mixed matches, J); therefore, we need to sum across $j = 0, \ldots, \min(h, J)$.

1.2.1 Noisy transitions ($a \geq 0$)

Here we extend the analysis of the transition matrix to the case where there is noise in implementing punishment. Again, assume that whenever a cooperator observes a defection he switches to defection state with probability ϵ. In the general case, a player in state $s = C$ switches to state $s = D$ with probability ϵ if he observes one or more defections. This implies that the probability ϵ that a cooperator in a mixed match switches to state $s = D$ does not depend on whether defections were observed outside the match. Similarly, a cooperator who is not in a mixed match switches to state $s = D$ with a probability ϵ that is independent on how many defections he has observed, outside of his match. That is to say, ϵ is not a function of the number of defections observed.

Supposed we have k defectors. Suppose that $j + m \leq N - k$ cooperators observe a defection, where j denotes the number of cooperators who directly observes a defection (they are in a mixed match), and m denotes the number of cooperators who indirectly observes a defection (they randomly observes a actions outside their match). Given the noise in the transition from a cooperative state to a defection state, the probability to have $h \leq j + m$ additional defectors is

$$\varepsilon_{j+m}(h) = \binom{j+m}{h} \epsilon^h (1-\epsilon)^{j+m-h}.$$

Hence, the probability that the number of defectors increases from k to $k+h$, due either to direct or indirect contagion, is

$$Q_{k,k+h} = \begin{cases} \sum_{j=0}^{J} \lambda_k(j) \sum_{m=\max(0,h-j)}^{N-k-j} \Delta^a_{kj}(m) \varepsilon_{j+m}(h) & \text{if } a \leq N-2-k \\ \\ \varepsilon_{N-k}(h) & \text{if } a > N-2-k \end{cases}$$

The second lines corresponds to the case in which the number of agents whose actions are observed exceeds the number of remaining cooperators in the economy, i.e., $a+2 > N-k$. Clearly, in this case *every* remaining cooperator observes at least one defection. The transition to h additional defectors therefore only depends on the random noise ϵ.

The first line, instead, corresponds to the case when the remaining cooperators do not necessarily see a defection, i.e., the number of agents whose actions are observed, $a+2$, is not greater than the number of remaining cooperators in the economy, $N-k$. The summation $\sum_{m=\max(0,h-j)}^{N-j-k}$ considers the number $m \leq N-k-j$ of cooperators who are not in mixed matches and indirectly observes at least one defection. Conditional on having $j = 0, \ldots, J$ mixed matches, the probability that $m \leq N-k-j$ cooperators indirectly observe a defection is $\Delta^a_{kj}(m)$. Clearly, we need to consider all possible pairs (j, m) leading to h new defectors; for example, if $j \geq h$ then we might have $m = 0$, i.e., we might not need any indirect observation to reach h new additional defectors. Conversely, if $j < h$ then we need at least $m = h - j$ cooperators outside mixed matches that observe a defection somewhere in the economy (and maybe more, due to randomness in transitions).

2 Reverting to cooperation

Here we extend the study of cooperative equilibrium with private monitoring to the case in which defection is not an absorbing state, following [4]. Suppose a public randomization device is available to players. At the start of each date, the device randomly selects and makes public a number $\tilde{q}_t \in [0, 1]$ with uniform probability. Defectors switch state if $\tilde{q}_t$ is sufficiently high, say, higher than $q \in (0, 1)$; everyone else remains in their state. Consequently, the strategy in Definition 1 is modified as follows:

Definition 2. *On $t = 0$, agent i is in state $s = C$ and selects action C. On all $t > 0$, agent i is either in state $s = C$ or $s = D$ and selects action s.*

- *If agent i is in state C in period t, then i selects action $s = C$ and switches state on $t + 1$ only if some agent in $O_i(t)$ selected D and if $\tilde{q}_{t+1} < q$. Otherwise, i remains in state C.*

- *If agent i is in state D in period t, then i switches state on $t + 1$ only if*

$\tilde{q}_{t+1} \geq q$. *Otherwise, i remains in state D.*

The key difference with the strategy in Definition 1 is that, out of equilibrium, the economy can revert back to full cooperation, with probability $1 - q$.

We will show that the statements in Propositions 1 and 2 remain true   the basic difference amounts to replacing the discount factor β with an adjusted discount factor $q\beta$. In addition, we will show that relying on a public randomization device reduces the incentives to cooperate in equilibrium but increases the incentives to punish off equilibrium.

Clearly, the continuation payoff in equilibrium is still v_0. Consider out-of-equilibrium situations in which there are $k \geq 1$ defectors at the start of some period, and fix one, say, agent i. Since decentralized punishment is characterized by matrix Q, then using standard recursive methods the payoff w_k to defector i is

$$w_k = \sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}\pi_{DC} + (1 - \eta_{kk'})\pi_{DD} + \beta q w_{k'} + \beta(1 - q)v_0], \text{ for } k \geq 1. \quad (6)$$

The continuation payoff $\beta q w_{k'} + \beta(1 - q)v_0$ accounts for the fact that the economy may revert to cooperation with probability $1 - q$, while punishment continues with probability q. Letting $w := (w_1, \ldots, w_N)^\intercal$, we can write

$$w_k \quad = \quad \sigma_k\pi_{DC} + (1 - \sigma_k)\pi_{DD} + \beta q e_k^\intercal Q w + \beta(1 - q)v_0, \text{ for } k = 1, \ldots, N.$$

We can now prove a result similar to that reported in Theorem 2. Letting $\bar{v}_0$ denote a $N-$dimensional vector with all elements equal to v_0, the off-equilibrium payoffs can now be defined by

$$
\begin{aligned}
& w = \sigma\pi_{DC} + (\mathbf{1} - \sigma)\pi_{DD} + \beta q Q w + \beta(1-q)\bar{v}_0 \\
\Rightarrow \quad & w = (I - \beta q Q)^{-1}[\sigma\pi_{DC} + (\mathbf{1} - \sigma)\pi_{DD} + \beta(1-q)\bar{v}_0],
\end{aligned}
$$

where $\mathbf{1}$ is the N-dimensional unit vector. The contact rate has now an adjusted discount factor, βq, so we denote it

$$\phi_k(\beta q) = (1 - \beta q)e_k^\mathsf{T}(I - \beta q Q)^{-1}\sigma, \qquad k = 1, \ldots, N,$$

to make explicit the difference relative to the contact rate $\phi_k \equiv \phi_k(\beta)$ defined in Theorem 2. Noting that $\phi_k(x) = (1-x)e_k^\mathsf{T}(I - xQ)^{-1}\sigma$ is a decreasing function of $x \in (0,1)$ for $k = 1, \ldots, N$ (see the Proof of Proposition 1 in the Appendix), we have that $\phi_k(\beta q) \leq \phi_k(\beta)$ for all $q \in (0,1]$.

Consequently, the off equilibrium continuation payoff given that k defectors are present is

$$w_k \;=\; \frac{1}{1-\beta q}[\phi_k(\beta q)\pi_{DC} + (1 - \phi_k(\beta q))\pi_{DD} + \beta(1-q)v_0], \qquad k = 1, \ldots, N$$

where we have used the fact that $e_k^\mathsf{T}(I - \beta q Q)^{-1}\mathbf{1} = \dfrac{1}{1-\beta q}$ for all k. Note that as $q \to 1$ we have $w_k \to v_k$ for all $k \geq 1$.

To prove existence of equilibrium we must consider one-shot deviations from the strategy in Definition 2 and simply substitute w_k for v_k in the relevant expressions in Section 5.

In equilibrium a generic agent must choose C and not D, which holds if $v_0 - w_1 \geq 0$. Substituting for $\pi_{CC}, \pi_{DC}, \pi_{DD}$ we have

$$v_0 - w_1 = \frac{1}{1-\beta q}[c - d - \phi_1(\beta q)(c + g - d)]. \tag{7}$$

From Theorem 2 we have $\lim_{\beta q \to 1^-} \frac{\phi_1(\beta q)}{1-\beta q} < \infty$. From Proposition 1 we know that there exists a value $q\beta = \beta^* \in (0,1)$ satisfying $c - d = \phi_1(\beta q)(c + g - d)$, so that $v_0 - w_1 \geq 0$ for all $q\beta \in [\beta^*, 1)$. Equivalently, it is suboptimal to deviate in equilibrium for all $\beta \in \left[\frac{\beta^*}{q}, 1\right)$. Therefore, the possibility to use a public randomization device makes it harder to sustain cooperation on the equilibrium path. Intuitively, the possibility to revert to cooperation after a defection increases the continuation payoff off-equilibrium, hence strengthens the incentive to defect.

We now show that, off-equilibrium, the possibility to use a public randomization device increases the incentive to follow the strategy proposed in Definition 1. That is, it strengthens the incentives to punish a defection.

Let agent i be one of $k \geq 2$ defectors, out of equilibrium. Agent i chooses

D, whenever

$$\sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}(c+\beta q w_{k'-1}+\beta(1-q)v_0)+(1-\eta_{kk'})(d-l+\beta q w_{k'}+\beta(1-q)v_0)]$$
$$\leq \sum_{k'=k}^{N} Q_{kk'}[\eta_{kk'}(c+g)+(1-\eta_{kk'})d+\beta q w_{k'}+\beta(1-q)v_0]$$

The expression above differs from the corresponding expression in Section 5.1 in the continuation payoffs and the discount factor, now adjusted for the use of the public randomization device. First, there can be reversion to cooperation, with probability $1-q$, in which case the payoff is v_0. But these terms cancel out from the two sides of the inequality. Second, if there is no reversion to cooperation, then the continuation payoff is either $w_{k'}$, if the agent punishes as he should or if he meets a defector, or $w_{k'-1}$, if the agent does not punish a cooperator. Rewrite the above inequality as

$$q\beta \sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}(w_{k'-1}-w_{k'}) \leq \sigma_k g+(1-\sigma_k)l.$$

Deviating out of equilibrium is suboptimal for any β and l, if q is sufficiently small. Alternatively, deviating out of equilibrium is suboptimal for any β and q, when l is sufficiently large.

The inequality above is satisfied when

$$l \geq l(q\beta,k) := \frac{1}{1-\sigma_k}\left\{(c+g-d)\sum_{k'=k}^{N} Q_{kk'}\eta_{kk'}\frac{q\beta\Big(\phi_{k'-1}(q\beta)-\phi_{k'}(q\beta)\Big)}{1-q\beta}-\sigma_k g\right\},$$

where we have used

$$w_{k'-1} - w_{k'} = \frac{c+g-d}{1-\beta q}\Big[\phi_{k'-1}(\beta q) - \phi_{k'}(\beta q)\Big]$$

Recalling that $l(x,k)$ is a non-decreasing function of $x \in (0,1)$ (Proposition 2), it follows that $l(q\beta,k) \leq l(\beta,k)$ for all $q \in (0,1]$, and for all $k \geq 1$. This means that using a public randomization device to revert to equilibrium makes it easier to follow the actions prescribed in Definition 2, off the equilibrium path. Put differently, off equilibrium punishment is incentive compatible even if the cost from cooperating is small, and even if the population is large, as long as agents can coordinate on reverting back to cooperation with sufficiently high probability.