Supporting Materials for: Understanding the Distributional Impact of Long-Run Inflation

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1 THE EFFICIENT ALLOCATION

Consider the allocation selected by a planner who treats agents identically. We call it the *efficient allocation.* The planner maximizes the expected lifetime utility of a representative agent subject to the physical and technological constraints. This means the optimal plan solves the following dynamic problem. Let the history of labor shocks of an arbitrary household be defined by

$$s^t = (h_1, h_2, \ldots, h_t),$$

where $s^t \in S^t$, with S^t the finite set of all possible histories for a household at date t. Define $p(s^t|s^{t-1})$ as the conditional probability of reaching the agent-specific state s^t, given the agent's previous state s^{t-1}; let $p(s^t)$ denote the unconditional probability. Notice that they are both independent of time as they are induced by the Markov process assumed in the model.

The planner chooses non-negative values $\{c_t(s^t), \ell_t(s^t)\}_{t=1}^{\infty}$ to solve

$$\begin{aligned}
\text{Maximize:} \quad & \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t \in S^t} p(s^t)[u(c_t(s^t)) - g(\ell_t(s^t))] \\
\text{Subject to:} \quad & \sum_{s^t \in S^t} p(s^t) c_t(s^t) \le Y(L_t) \quad \text{for each } \; t = 1, 2, \ldots \\
& \sum_{s^t \in S^t} p(s^t) \ell_t(s^t) h_t(s^t) = L_t \;\; \text{for each } \; t = 1, 2, \ldots
\end{aligned}$$

The problem above is solved before $t = 1$ shocks are realized and presumes that (i) the initial draw is from the long-run distribution and (ii) the population of households is identically given by the distribution of states, i.e., $\sum_{s^t \in S^t} p(s^t) = 1$ because a law of large numbers applies. In words, the planner maximizes average ex-ante utility of households, placing equal weight on each household "type," where a type is determined by the history of shocks s^t of the household. Here, slightly abusing notation, we are denoting the productivity shock to a household of "type" s^t by $h_t(s^t)$; it should be clear that h_t is independent of the agent's history of shocks (since we assume a first-order Markov process) and that s^t appears as an argument simply as an index denoting the "type" of agent. The planner takes as given the resource constraint reflecting aggregate and technological feasibility.

Necessary and sufficient condition for an interior optimum are

$$\begin{aligned} \beta^{t-1} g'(\ell_t(s^t)) &= \lambda_t Y'(L_t) h_t(s^t) \\ \beta^{t-1} u'(c_t(s^t)) &= \lambda_t \end{aligned}$$

for each agent in state $s^t \in S^t$ and $t = 1, 2, \ldots$, with the two constraints holding with equality. That is, on each date t the planner equates marginal consumption utility to the "shadow price" of aggregate output on date t, i.e., λ_t. In particular, in the efficient allocation $\lambda_t / \lambda_{t+j} = \beta^{-j}$. Hence, the efficient allocation involves perfect insurance for each household.

LEMMA 1. *The efficient allocation is stationary, unique, and is defined by constant individual consumption $c(s^t) = c^*$ for each $s^t \in S^t$, with $c^* := Y(L)$, and state-contingent individual labor supply satisfying*

$$g'(\ell_t(s^t)) = u'(c^*) Y'(L) h_t(s^t),$$

with $L = \sum_{s^t \in S^t} p(s^t) \ell_t(s^t) h_t(s^t)$ for each $t = 1, 2, \ldots$ The efficient allocation can be decentralizes using state-contingent prices

$$q_t(s^t) = \beta^{t-1} p(s^t) \quad \text{for each} \quad s^t \in S^t \text{ and } t = 1, 2, \ldots$$

and $w_t = Y'(L)$ for all $t = 1, 2, \ldots$.

Proof. The claim in the Lemma regarding the optimal allocation immediately follows from the conditions for an interior Pareto optimum. To find the price vector that decentralizes this allocation suppose the household can trade a full set of contingent claims, before $t = 1$ shocks are realized. The household chooses $\{c_t(s^t), \ell_t(s^t)\}_{t=1}^{\infty}$ given the prices $\{q_t(s^t), w_t\}_{t=1}^{\infty}$ to solve the problem

$$\begin{aligned} \text{Maximize:} \quad & \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t \in S^t} p(s^t)[u(c_t(s^t)) - g(\ell_t(s^t))] \\ \text{Subject to:} \quad & \sum_{t=1}^{\infty} \sum_{s^t \in S^t} q_t(s^t) c_t(s^t) = y. \end{aligned}$$

Here $y := \sum_{t=1}^{\infty} \sum_{s^t \in S^t} q_t(s^t)[w_t \ell_t(s^t) h_t(s^t) + \xi_t]$ is the present discounted value of income; $\{\xi_t\}_{t=1}^{\infty}$ denotes the dividend income stream (the household owns a share of the firm) and $\{w_t\}_{t=1}^{\infty}$ denotes the stream of competitive wage rates. Dividends and income do not depend on the individual state and there is no aggregate risk. Denoting λ the marginal value of the household's permanent income y, the FOC is

$$\beta^{t-1} p(s^t) u'(c_t(s^t)) = q_t(s^t)\lambda \text{ for each } s^t \in S^t \text{ and } t = 1, 2, \ldots$$

In the efficient allocation $c(s^t) = c^*$ for each $s^t \in S^t$. Hence, fixing the initial period 1, the Euler equation for the household becomes

$$\frac{p(s^1)}{\beta^{t-1} p(s^t)} = \frac{q_1(s^1)}{q_t(s^t)} \text{ for any period } t \geq 2.$$

Now normalize the state-contingent price on the initial date in state $s^1 \in S^1$ by fixing $q_1(s^1) := p(s^1)$. We obtain the state-contingent relative price vector $(q_t(s^t))_{t,s^t}$ defined by

$$q_t(s^t) := \beta^{t-1} p(s^t) \text{ for any date } t = 1, 2, \ldots \text{ and state } s^t \in S^t.$$

Given that there is a constant aggregate labor supply, then $w_t = Y'(L)$ for all t. □

Lemma 1 makes it clear that the efficient allocation can be decentralized by introducing a full set of contingent claims to be traded before period 1 shocks are realized. The notation $q_t(s^t)$ denotes the price of consumption traded in the initial period and delivered on date t if the household's state is s^t. As usual, we have normalized $q_1(s^1) := p(s^1)$.

2 COMPUTATION METHOD

2.1 COMPUTATIONAL ALGORITHM

It could be a challenging task to compute a heterogeneous agent economy with occasionally binding constraints. In our economy, there are three state variables in the household problem with two inequality constraints. These sometimes binding constraints make the optimal policy function non-differentiable and hence complicate the computational task. On top of that, we need to reply on a large panel of simulation data to clear four markets in our economy. In order to compute our economy more efficiently and accurately, we follow the algorithm described by

(?), who use a flexible simplicial interpolation method (called Delaunay Interpolation) to endogenously pick the grid points at state space for the kink points induced by non-differentiabilities. Our computational algorithm follows the following steps

1. Given an inflation rate $\pi \geq 1$, guess a risk-free interest rate $i = i_0 > 1$, a wage rate w and a real money balance $\overline{M}$.

2. Given the guess, compute household's policy functions. Note that the policy functions are computed by solving a system of non-linear equation consisting of the first order conditions and constraints of household problem. Please refer to the details discussion on the computational strategy of policy function below. This step generates the policy functions $c(\omega), l(\omega), m'(\omega)$ and $b'(\omega)$, where $\omega := (m, b, h)$.

3. Given the policy functions, the economy is simulated over $T = 2,000$ periods with population $N = 20,000$. This step allows to obtain an approximately invariant distribution of states, with respect to the mean. This gives an associated distribution function $\phi^{sim}(\omega)$.

4. Given the approximately invariant distribution of states, the excess supply of bond, labor and real money holdings are calculated, which correspond to

$$\varepsilon_b = \sum_{h\in H} \int_{m\in M} \int_{b\in B} b'(\omega)\phi^{sim}(\omega) dm\ db,$$

$$\varepsilon_L = \sum_{h\in H} \int_{m\in M} \int_{b\in B} l(\omega) h \phi^{sim}(\omega) dm\ db - L,$$

$$\varepsilon_m = \overline{M} - \sum_{h\in H} \int_{m\in M} \int_{b\in B} \pi m'(\omega)\phi^{sim}(\omega) dm\ db$$

5. Repeat the steps above and update the guess of $(i, w, \overline{M})$ until $(\varepsilon_b, \varepsilon_L, \varepsilon_m)$ are all sufficient close to 0.

2.2 POLICY FUNCTIONS

The computational strategy is described next. It follows ?. The strategy revolves around solving a system of equations.

1. We start by selecting a set of grid points G_{exo} on state variable m and b, and by guessing initial policy functions. Then, we solve a system of equations by collocation methods. For each grid point $g \in G_{exo}$, we solve a system of equations given next period policy functions. The policy functions are updated until it converges.

2. Given the policy functions obtained above, compute the endogenous grid points which corresponding to the exactly bringing points of inequality constraints. The set of these endogenous grid points is denoted by G_{end}.

3. Finally, combine the two set of grind points, $G = G_{exo} \cup G_{end}$ and compute the implied policy functions.

2.3 INEQUALITY CONSTRAINTS

In order to deal with the Kühn-Tucker conditions induced by the inequality constraints, we first rewrite Kühn-Tucker condition into nonlinear-complementarity-problem (NCF) function by observing the following relationships

$$\begin{aligned}\varphi(a,b) &= (a^2+b^2)^{1/2} - a - b = 0 \\ &\Leftrightarrow a \geq 0, b \geq 0, ab = 0\end{aligned}$$

where a and b variables that need to satisfy Kühn-Tucker conditions. $\varphi(a,b)$ denotes for NCP function.

Once we formulate the Kühn-Tucker conditions into NCP function, we can solve the inequality constraints by using a standard non-linear equation solver since NCP function is continuous and differentiable. The set of non-linear equations in our household problem is therefore given by

$$\begin{aligned}
u'(c)\pi - \lambda_1\pi &= \beta E u'(c') \\
g'(l) &= (u'(c) - \lambda_1)wh \\
\varphi_1(m-c, \lambda_1) &= ((m-c)^2 + \lambda_1^2)^{1/2} - (m-c) - \lambda_1 = 0 \\
wl + m + \tau + \xi + ib &= c + m'\pi + b'\pi \\
u'(c)\pi - \lambda_1\pi &= \beta i E(u'(c') - \lambda_1') + \lambda_2 \\
\varphi_2(b' - \underline{b}, \lambda_2) &= ((b' - \underline{b})^2 + \lambda_2^2)^{1/2} - (b' - \underline{b}) - \lambda_2 = 0
\end{aligned}$$

where λ_1 and λ_2 are the shadow price of liquidity constraints and borrowing constraints. There are 6 functions to be solved, $c(\omega), l(\omega), m'(\omega)$, $b'(\omega), \lambda_1(\omega)$ and $\lambda_2(\omega)$.

3 TRADE BONDS BEFORE TRADING GOODS

Suppose households can trade bonds—hence can borrow—*after* productivity shocks are realized and before the good market opens. In this case, constraints no longer bind for anyone, unless money and bonds are equivalent assets or households can borrow insufficient amounts. In this case we have just one constraint to consider

$$c + \pi b' + \pi m' = m + \tau + bi + w\ell h + \xi,$$

where $\pi m' = w\ell h + \xi$, i.e., money savings are entirely determined by labor and capital income.

The main differences in the optimality conditions previously derived are (i) $\lambda = 0$ so that $V_m = \mu = u'(c)$ and (ii) the optimal choice of labor supply requires

$$g'(\ell) = \frac{w\ell}{\pi}\beta E[V_{m'}].$$

These considerations imply that the analysis is conducted with two Euler conditions

$$\begin{aligned}
&-g'(\ell) + \beta E[u'(c')]\tfrac{w\ell}{\pi} = 0 \\
&-\pi u'(c) + \beta E[u'(c')]i \leq 0 \quad (\text{with} = \text{if } b' > \underline{b}).
\end{aligned}$$

4 THE MODEL WITH CAPITAL

We report some additional data for the model economy with capital.

[Table 1 about here.]

[Table 2 about here.]

[Figure 1 about here.]

Table 1: The Economy with Capital

	Persistent Shocks				IID Shocks			
$\pi - 1$	$\overline{m}$	$\overline{c}$	$\overline{l}$	$\overline{k}$	$\overline{m}$	$\overline{c}$	$\overline{l}$	$\overline{k}$
0%	1.185	1.100	0.986	3.942	1.184	1.175	1.065	3.587
1%	1.137	1.095	0.981	3.925	1.177	1.169	1.061	3.570
2%	1.110	1.090	0.977	3.907	1.170	1.164	1.056	3.555
3%	1.096	1.085	0.972	3.884	1.164	1.160	1.052	3.540
4%	1.090	1.084	0.972	3.871	1.158	1.155	1.047	3.524
5%	1.085	1.076	0.964	3.837	1.153	1.150	1.043	3.510
10%	1.062	1.058	0.949	3.739	1.129	1.128	1.023	3.437
15%	1.040	1.035	0.928	3.627	1.107	1.106	1.004	3.365
20%	1.020	1.018	0.914	3.540	1.086	1.086	0.985	3.298
25%	1.001	0.999	0.897	3.448	1.068	1.068	0.969	3.237
30%	0.984	0.983	0.883	3.368	1.049	1.049	0.952	3.177
35%	0.968	0.967	0.870	3.292	1.032	1.032	0.937	3.121
40%	0.954	0.954	0.859	3.226	1.016	1.015	0.922	3.067

Note: $\gamma = 1.3$ and $\delta = 2$; $\pi - 1$ is the net inflation rate; $\overline{x}$ is the mean value, and $Gini_x$ is the Gini coefficient associated to the equilibrium random variable x. We define $m =$ money balances (in real terms), $c =$ consumption, $I =$ income net of taxes/transfers, wealth is $w = m + \omega$

Table 2: Distribution of Welfare Costs in the Economy with Capital

	Panel A: Persistent Shocks						
	$\delta = 1.5$			$\delta = 2$			$\delta = 3$
$\pi - 1$	Δ_π	Δ_π	Q_1	Q_2	Q_3	Q_4	Δ_π
0%	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1%	−0.061	−0.120	−0.372	−0.318	0.196	0.140	−0.134
2%	−0.155	−0.226	−0.589	−0.482	0.209	0.183	−0.281
3%	−0.264	−0.381	−0.994	−0.609	0.189	0.246	−0.470
4%	−0.436	−0.607	−1.405	−0.860	0.164	0.175	−0.695
5%	−0.610	−0.742	−1.800	−1.014	0.183	0.293	−0.853
10%	−1.353	−1.531	−3.827	−1.451	0.155	0.265	−1.645
15%	−1.833	−2.003	−5.175	−1.821	0.322	0.411	−2.247
20%	−2.126	−2.424	−6.180	−2.320	0.311	0.672	−2.760
25%	−2.282	−2.591	−6.975	−2.679	0.597	1.225	−3.038
30%	−2.322	−2.754	−7.619	−2.890	0.571	1.764	−3.270
35%	−2.352	−2.782	−8.358	−2.727	0.740	2.348	−3.529
40%	−2.160	−2.851	−9.031	−2.718	0.976	2.780	−3.547

	Panel B: IID Shocks						
	$\delta = 1.5$			$\delta = 2$			$\delta = 3$
$\pi - 1$	Δ_π	Δ_π	Q_1	Q_2	Q_3	Q_4	Δ_π
0%	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1%	0.061	0.035	−0.017	−0.008	0.041	0.094	0.019
2%	0.136	0.086	−0.056	0.009	0.124	0.223	0.037
3%	0.208	0.135	−0.071	0.029	0.197	0.319	0.080
4%	0.293	0.197	−0.083	0.086	0.281	0.424	0.098
5%	0.373	0.246	−0.094	0.134	0.339	0.506	0.145
10%	0.801	0.581	−0.016	0.462	0.795	0.940	0.342
15%	1.302	0.945	0.158	0.849	1.236	1.371	0.587
20%	1.869	1.373	0.453	1.269	1.818	1.807	0.866
25%	2.455	1.730	0.721	1.630	2.219	2.200	1.156
30%	3.069	2.267	1.193	2.146	2.860	2.805	1.466
35%	3.721	2.759	1.594	2.696	3.424	3.332	1.881
40%	4.396	3.259	2.019	3.236	4.026	3.848	2.225

Note: $\gamma = 1.3$ and $\delta = 2$ unless otherwise noted; $\pi - 1$ is the net inflation rate. The welfare cost Δ_π is reported as the percent of current consumption the average household would give up to be at zero inflation (two-digit approximation; a negative number indicates a welfare gain). Q_i denotes the i-th quartile of the wealth distribution.

Figure 1: Real interest rates comparison