Technical Appendix to: Currency Competition in a Fundamental Model of Money.

(not intended for publication)

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Existence and uniqueness of a stationary distribution $\{m_j\}$ when $p^* = (1, 1, 1)$.

I. Sufficient conditions for existence. We use a Liapunov function approach as in Zhou (1997) to provide sufficient conditions such that there is a unique stationary distribution of money.

Specifically, consider the state space M and an equilibrium point $m^* \in M$. Define a real-valued function L on M, that satisfy the following requirements: (i) L is continuous and has continuous first-partial derivatives (ii) $L(m)$ has a unique minimum at m^* with respect to all other points in M. (iii) The function $\dot{L}(m)$ satisfies $\dot{L}(m) \leq 0$ for all $m \in M$. This function L is called a Liapunov function. We then rely on the Liapunov theorem stating that if there exists a Liapunov function the equilibrium point m^* is stable and if the function $\dot{L}(m) < 0$ at all $m \neq m^*$ then the stability is asymptotic. Equations

$$\begin{aligned} 1 &= m_0 + m_g + m_b + m_{2g} + m_{2b} + m_{gb} \\ M_g &= m_g + 2m_{2g} + m_{gb} \\ M_b &= m_b + 2m_{2b} + m_{gb} \end{aligned} \tag{1}$$

$$\tau(m_b + 2m_{2b} + m_{gb}) = \eta\left[m_0 + m_g + m_b\right]. \tag{2}$$

imply that $\{m_0, m_g, m_b, \eta\}$ are single-valued functions of $\{m_{2g}, m_{2b}, mg_b\}$. That is:

$$\begin{aligned} m_g &= M_g - m_{gb} - 2m_{2g} \\ m_b &= M_b - m_{gb} - 2m_{2b} \\ m_0 &= 1 - M_g - M_b + m_{2g} + m_{2b} + m_{gb} \end{aligned} \tag{3}$$

and the government budget constraint is

$$\eta = \frac{\tau M_b}{1-(m_{2g}+m_{2b}+m_{gb})} \tag{4}$$

Using (3) in

$$\dot{m}_{2g} = m_g\left(m_g + m_{gb}\right) - m_{2g}\left(m_0 + m_b\right) \tag{5}$$

$$\dot{m}_{2b} = x[m_b^2 - m_{2b}\left(m_0 + m_g\right)] + \eta m_b - \tau m_{2b} \tag{6}$$

$$\dot{m}_{gb} = x[m_g m_{2b} + m_b m_{2g} + 2m_b m_g - m_{gb}\left(m_0 + m_g\right)] + \eta m_g - \tau m_{gb} \tag{7}$$

we get:

$$\begin{aligned}
\dot{m}_{2g} = \quad & \left(M_g - m_{gb} - 2m_{2g}\right)\left(M_g - 2m_{2g}\right) - m_{2g}\left(1 - M_g + m_{2g} - m_{2b}\right) \\
\dot{m}_{2b} = \quad & x\left[\left(M_b - m_{gb} - 2m_{2b}\right)^2 - m_{2b}\left(1 - M_b - m_{2g} + m_{2b}\right)\right] \\
& +\eta\left(M_b - m_{gb} - 2m_{2b}\right) - \tau m_{2b} \\
\dot{m}_{gb} = \quad & x[\left(M_g - m_{gb} - 2m_{2g}\right)m_{2b} + \left(M_b - m_{gb} - 2m_{2b}\right)m_{2g} \\
& +2\left(M_b - m_{gb} - 2m_{2b}\right)\left(M_g - m_{gb} - 2m_{2g}\right) - m_{gb}\left(1 - M_b - m_{2g} + m_{2b}\right)] \\
& +\eta\left(M_g - m_{gb} - 2m_{2g}\right) - \tau m_{gb}
\end{aligned} \tag{8}$$

Define the 3x1 vector $m = [m_1, m_2, m_3]$ where $m_1 = m_{2g}$, $m_2 = m_{2b}$, $m_3 = m_{gb}$ and $m_i \in [0,1]$ with $m_1 + m_2 + m_3 \leq 1$. Then define the system in (8) as $\dot{m} = F(m)$ where $F(m)$ is a 3×1 vector. Denote by $F(m)[i]$ the i^{tb} row of $F(m)$. Then, letting $\frac{dF(m)[i]}{dm_j} = a(i,j)$, $j, i = 1, 2, 3$, the Jacobian of $F(m)$ is a 3x3 matrix

$$\frac{dF(m)}{dm} = \begin{bmatrix} a(1,1) & ... & a(1,3) \\ : & a(2,2) & : \\ a(3,1) & ... & a(3,3) \end{bmatrix}$$

where, recalling that τ is a constant,

$$a(1,1) = m_{2b} + 2m_{gb} + 6m_{2g} - 3M_g - 1 < 0$$
$$a(1,2) = m_{2g}$$
$$a(1,3) = 2m_{2g} - M_g < 0 \ \text{(since } M_g > 2m_{2g}\text{)}$$
$$a(2,1) = xm_{2b} + \tfrac{d\eta}{dm_{2g}} (M_b - m_{gb} - 2m_{2b}) > 0$$
$$a(2,2) = x\left[4m_{gb} + m_{2g} + 6m_{2b} - 3M_b - 1\right] + \tfrac{d\eta}{dm_{2b}} (M_b - m_{gb} - 2m_{2b}) - 2\eta - \tau$$
$$a(2,3) = -2x\left[M_b - m_{gb} - 2m_{2b}\right] + \tfrac{d\eta}{dm_{gb}} (M_b - m_{gb} - 2m_{2b}) - \eta$$
$$a(3,1) = x\left[4m_{gb} + 4m_{2b} - 3M_b\right] + \tfrac{d\eta}{dm_{2g}} (M_b - m_{gb} - 2m_{2g}) - 2\eta$$
$$a(3,2) = x\left[2m_{gb} + 4m_{2g} - 3M_g\right] + \tfrac{d\eta}{dm_{2b}} (M_g - m_{gb} - 2m_{2g})$$
$$a(3,3) = x\left[4m_{gb} + 4m_{2g} + 2m_{2b} - M_b - 2M_g - 1\right] + \tfrac{d\eta}{dm_{gb}} (M_g - m_{gb} - 2m_{2g}) - \eta - \tau$$

where we note that $\frac{d\eta}{dm_i} > 0$ for all m_i. We note that $M_g > m_{gb} + 2m_{2g}$ and $M_b > m_{gb} + 2m_{2b}$ if $m_g, m_b > 0$, using (3)-(**??**). Substituting the infimum $M_g = m_{gb} + 2m_{2g}$ and $M_b = m_{gb} + 2m_{2b}$ in $a(2,2), a(3,2)$ and $a(3,3)$, it is easy to show that all of these terms are strictly negative as $\eta \to 0$ while $a(2,3) \to 0^-$. Thus there are small values of $\eta > 0$ such that $a(2,2), a(2,3), a(3,2)$ and $a(3,3)$ are all negative. Note that $\eta \to 0$ when either $\tau \to 0$ or $M_b \to 0$, and so does $\frac{d\eta}{dm_i} > 0$ for all m_i.

We want to show that $\frac{dF(m)}{dm}$ is negative definite. To do so we can consider the sign of its three principal minors:

$$D_1 = a(1,1), \quad D_2 = \begin{vmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{vmatrix}, \text{ and } D_3 = \begin{vmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \\ a(3,1) & a(3,2) & a(3,3) \end{vmatrix}$$

We note that $D_1 = a(1,1) < 0$. This is so because $M_g \geq m_{gb} + 2m_{2g}$ (with strict inequality if $m_g > 0$) using (3). Substituting $M_g = m_{gb} + 2m_{2g}$ in $a(1,1)$ provides a maximum for $a(1,1)$. This maximum is seen to be negative since $-m_{gb} + m_{2b} - 1 < 0$.

The minor $D_2 = a(1,1)a(2,2) - a(1,2)a(2,1)$. Note that $a(1,2)$ and $a(2,1)$ are both positive, and that their product tends to zero as x and η shrink to 0. Furthermore, $a(2,2) < 0$ as η tends to zero because $-3M_b - 1 + 4m_{gb} + m_{2g} + 6m_{2b} < 0$ (since $M_b \geq m_{gb} + 2m_{2b}$). Thus $D_2 > 0$ for x and η small (i.e. either τ or M_b small). The third minor is

$$\begin{aligned} D_3 = {} & a(1,1)\left[a(2,2)a(3,3) - a(2,3)a(3,2)\right] \\ & -a(1,2)\left[a(2,1)a(3,3) - a(2,3)a(3,1)\right] \\ & +a(1,3)\left[a(2,1)a(3,2) - a(2,2)a(3,1)\right] \end{aligned}$$

Note that as $\eta, x \to 0$ then the second and third line in D_3 vanish, and that the first line, is strictly negative and given by

$$\tau^2 \left(-3M_g - 1 + m_{2b} + 2m_{gb} + 6m_{2g}\right)$$

We conclude that there exist an M_b and x positive but sufficiently small such that $D_1 < 0$, $D_2 > 0$ and $D_3 < 0$. Thus, for M_b and x small the matrix $\frac{dF(m)}{dm}$ is negative definite (see Chiang).

Since $F(m)$ is a 3x1 vector ('′' transposes it), define the function

$$L(m) = [F(m)]' \, F(m) = (\dot{m}_{2g})^2 + (\dot{m}_{2b})^2 + (\dot{m}_{gb})^2 \geq 0$$

We show it is a Liapunov function. It is continuous (by construction) and it has continuous first partial derivatives. Recalling that the vector $F(m) = \dot{m}$, that $d[F(m)]'/dt = \dot{m}'\frac{dF(m)}{dm}$ (a 1x3 vector) and that $dF(m)/dt = \left[\frac{dF(m)}{dm}\right]' \dot{m}$ (a 3x1 vector) then the time derivative of $L(m)$ is the quadratic form (a scalar)

$$\dot{L}(m) = \dot{m}'\frac{dF(m)}{dm}\dot{m} + \dot{m}'\left[\frac{dF(m)}{dm}\right]'\dot{m}$$

so that $\dot{L}(m) = 0$ if $\dot{m} = 0$, and < 0 if $\dot{m} \neq 0$ for x and M_b small, since $\frac{dF(m)}{dm}$ is negative definite.

To show that there exists an m^* such that $L(m^*) = 0$ we use a proof by contradiction. If $L(m) = \dot{m} \neq 0$ for all m defined above then $\dot{L}(m) \neq 0$. Since m is defined on a compact set it follows that $\dot{L}(m)$ has a maximum, say $l < 0$ (because of negative definiteness). But this cannot be since, defining $m(t)$ to be the state of the system at date t,

$$\int_0^t \dot{L}(m(s))ds = L(m(t)) - L(m(0)) \leq lt \Rightarrow L(m(t)) \leq lt + L(m(0))$$

which in turn implies $L(m(t)) \to -\infty$ as $t \to \infty$. This can't be since at every date, by construction, $L(m) \geq 0$. Thus $L(m)$ must be reaching a minimum 0 at some m^*. To show that m^* is unique, see below.

Thus $L(m)$ is a Liapunov function, and applying the Liapunov Theorem (see Azariadis, 1993, for a discrete time version) the unique equilibrium m^* is asymptotically stable if x and M_b are positive but sufficiently small. The money distribution m^* is unique and stationary.

II. Uniqueness. Using (3)-(4) and $M_g + M_b < 2$, then $m_i > 0$ and $\eta < 1$ require

$$m_{gb} + 2m_{2g} < M_g < 2 - M_b < 2 - m_{gb} - 2m_{2b}$$

$$m_{gb} + 2m_{2b} < M_b < \frac{1-(m_{2g}+m_{2b}+m_{gb})}{\tau} \quad .$$

We now show that for a feasible pair $\{m_{2b}, m_{2g}\}$, if m^*_{gb} solves (7), then it must be unique. Using (7) and (a1)-(a4) we obtain

$$m_{gb} = \frac{\left(M_g-m_{gb}-2m_{2g}\right)\left(\frac{\tau M_b}{1-m_{2g}-m_{2b}-m_{gb}}+xm_{2b}\right)}{\tau+x(1-M_b+m_{2b}-m_{2g})} + \frac{x\left(M_b-m_{gb}-2m_{2b}\right)[m_{2g}+2\left(M_g-m_{gb}-2m_{2g}\right)]}{\tau+x(1-M_b+m_{2b}-m_{2g})}$$

The right hand side can be shown to be strictly decreasing in m_{gb} for all feasible values of m_{gb}, m_{2b}, and m_{2g}. It then follows that if there is a feasible m^*_{gb} that solves this expression, then it is unique.

We now show that for a feasible value of m_{gb}, a unique pair $\left\{m^*_{2b}, m^*_{2g}\right\}$ solves (5) and (6). Using (5) and (a1)-(a4) we obtain $m_{2b} = g(m_{gb}, m_{2g})$ where

$$g(m_{gb}, m_{2g}) = 1 - M_g + m_{2g} - \frac{(M_g - m_{gb} - 2m_{2g})(M_g - 2m_{2g})}{m_{2g}}$$

seen to be increasing in m_{2g} for feasible values $m_{2g} \leq (M_g - m_{gb})/2$, and it is concave in m_{2g}.

Using (6) and (a1)-(a4) we obtain $m_{2g} = b(m_{gb}, m_{2b})$ where

$$b(m_{gb}, m_{2b}) = \frac{\tau}{x} + 1 - M_b + m_{2b} - \frac{[\eta + x(M_b - m_{gb} - 2m_{2b})](M_b - m_{gb} - 2m_{2b})}{xm_{2b}}$$

which is easily seen to be increasing and concave in m_{2b} for feasible values $m_{2b} \leq (M_b - m_{gb})/2$,since η is increasing in m_{2b}. Note also that $g(m_{gb}, m_{2g}) \to -\infty$ as $m_{2g} \to 0$ and $b(m_{gb}, m_{2b}) \to -\infty$ as $m_{2b} \to 0$. Note that $m_{2b} \leq (M_b - m_{gb})/2 < g(m_{gb}, (M_g - m_{gb})/2)$ and $m_{2g} \leq (M_g - m_{gb})/2 < b(m_{gb}, (M_b - m_{gb})/2)$. The properties of the two functions imply there is a single crossing point for the two functions in the feasible part of the (m_{2b}, m_{2g}) plane. Thus, for any feasible value of m_{gb} and η, there is a unique pair $\left\{m^*_{2b}, m^*_{2g}\right\}$ that solves the system

$$\begin{cases} m_{2b} = g(m_{gb}, m_{2g}) \\ m_{2g} = b(m_{gb}, m_{2b}) \end{cases}$$

Given the uniqueness of the values in (3)-(4) and m_{gb},then if a feasible distribution exists, it is unique.