Section 4.1 - Vectors in $\mathbb{R}^n$

A vector in the plane is represented geometrically by a directed line segment whose initial point is the origin, and whose terminal point is the point $(x, y)$.

Properties of Vectors: Let $\vec{u}_1$, $\vec{v}_1$, and $\vec{w}$ be vectors in the plane, and let $c$ and $d$ be scalars.
1. $\vec{u} + \vec{v}$ is a vector in the plane.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. $\vec{u} + \vec{0} = \vec{u}$
5. $\vec{u} + (-\vec{u}) = \vec{0}$
6. $c\vec{u}$ is a vector in the plane
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$ and let $c$ be a real number.
Then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$
and $c\vec{u} = (cu_1, cu_2, \ldots, cu_n)$

The same 10 properties apply to $\mathbb{R}^n$.

Example:
$\vec{u} = (0, 5, -2, 1)$ and $\vec{v} = (3, 4, 1, -1)$ and $c = -2$
$c\vec{u} + \vec{v} = (-3, -6, 5, -3)$

To write a vector $\vec{x}$ as a linear combination of the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_n$, we need to find scalars $c_1$, $c_2$, and $c_n$ such that
$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \sum_{i=1}^{n} c_i\vec{v}_i$

Example: Let $\vec{x} = (-1, -2, -2)$ and $\vec{u} = (0, 1, 4)$ and $\vec{v} = (-1, 1, 2)$ and $\vec{w} = (3, 1, 2)$.
Find scalars $a$, $b$, and $c$ such that $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$
$(-1, -2, -2) = (0, a, 4a) + (-b, b, 2b) + (3c, c, 2c)$
So $(-1, -2, -2) = (-b + 3c, a + b + c, 4a + 2b + 2c)$
Section 4.2 - Vector Spaces

Definition of a vector space: Let \( V \) be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every element \( u, v \) and \( w \) and every scalar (real number) \( c \) and \( d \), then \( V \) is called a vector space and the elements are called vectors.

Addition:
1. \( \vec{u} + \vec{v} \) is in \( V \)
2. \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \)
3. \( \vec{u} + \left( \vec{v} + \vec{w} \right) = \left( \vec{u} + \vec{v} \right) + \vec{w} \)
4. \( V \) has a zero vector \( 0 \) such that for every \( \vec{u} \) in \( V \), \( \vec{u} + 0 = \vec{u} \)
5. For every \( \vec{u} \) in \( V \), there is a vector in \( V \) denoted by \( -\vec{u} \) such that \( \vec{u} + (-\vec{u}) = 0 \)

Scalar Multiplication
6. \( c\vec{u} \) is in \( V \).
7. \( c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \)
8. \( (c + d)\vec{u} = c\vec{u} + d\vec{u} \)
9. \( c(d\vec{u}) = (cd)\vec{u} \)
10. \( 1(\vec{u}) = \vec{u} \)

Some important vector spaces:
- \( \mathbb{R} \) = the set of all real numbers
- \( \mathbb{R}^2 \) = the set of all ordered pairs
- \( \mathbb{R}^3 \) = the set of all ordered triples
- \( \mathbb{R}^n \) = the set of all ordered n-tuples
- \( C(-\infty, \infty) \) = the set of all continuous functions defined on the real line.
- \( C[a, b] \) = the set of all continuous functions defined on the closed interval \([a, b]\)
- \( P \) = the set of all polynomials
- \( P_n \) = the set of all polynomials of degree \( \leq n \)
- \( M_{m,n} \) = the set of all \( m \times n \) matrices
- \( M_{n,n} \) = the set of all \( n \times n \) square matrices

Sets that are not vector spaces
- The set of integers
The set of \( n \)th degree polynomials

Example 1:
\[
p(x) = x^3 + x^2 \\
q(x) = -x^3 + x \\
p(x) + q(x) = x^2 + x <- \text{Failure of property 1}
\]

Example 2:
Let \( V = \mathbb{R}^2 \), the set of all ordered pairs of real numbers, with the standard operation of addition and the following nonstandard definition of scalar multiplication:
\[
c(x_1, x_2) = (cx_1, 0) \\
10. \ l\vec{u} = \vec{u} \\
1(x_1, y_1) = (1x_1, 0)
\]

Example 3:
The set of all \( n \times n \) singular matrices with the standard operations is not a vector space.
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} 
= 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

There are cases where two singular matrices, \( s \) and \( t \), when added will produce a nonsingular matrix \( n \).

Section 4.3 - Subspaces of Vector Spaces

Definition: A nonempty subset \( W \) of a vector space \( V \) is called a subspace of \( V \) if \( W \) is itself a vector space under the operations of addition and scalar multiplication defined in \( V \). \((W \in V)\)

Test for a subspace: If \( W \) is a nonempty subset of a vector space \( V \), then \( W \) is a subspace of \( V \) if and only if the following closure conditions hold:
1. If \( \vec{u} \) and \( \vec{v} \) are in \( W \), then \( \vec{u} + \vec{v} \in W \)
2. If \( \vec{u} \in W \) and \( c \) is a scalar, then \( c\vec{u} \in W \)

Example: Let \( W \) be the set of all \( 2 \times 2 \) symmetric matrices.
\( W \subset M_{2,2} \), which is a vector space
1. Let \( A, B \in W \). \( (A + B)^T = A^T + B^T = A + B \). Therefore, \( A + B \) is symmetric, and \( A + B \in W \).
2. Let \( A \in W \) and \( c \in \mathbb{R} \). \( (cA)^T = cA^T = cA \). Therefore, \( cA \in W \)

Theorem: If \( V \) and \( W \) are both subspaces of a vector space \( U \), then the intersection
of $V$ and $W$, denoted by $V \cap W$, is also a subspace of $U$.

$V \cap W \subset U$

1. Let $\overrightarrow{u}, \overrightarrow{v} \in V \cap W$. Then $\overrightarrow{u}, \overrightarrow{v} \in V$ and $\overrightarrow{u}, \overrightarrow{v} \in W \Rightarrow \overrightarrow{u} + \overrightarrow{v} = V$ and $\overrightarrow{u} + \overrightarrow{v} = W$. Therefore, $\overrightarrow{u} + \overrightarrow{v} \in V \cap W$.

2. Let $\overrightarrow{u} \in V \cap W$ and $c \in \mathbb{R}$. Then $\overrightarrow{u} \in V$ and $\overrightarrow{u} \in W$. $c\overrightarrow{u} \in V$ and $c\overrightarrow{u} \in W \Rightarrow c\overrightarrow{u} \in V \cap W$.

What about the union of two subspaces?

$V = \{(x, 0) \text{ where } x \in \mathbb{R}\}$

$W = \{(0,y) \text{ where } y \in \mathbb{R}\}$

$(1,0) \in V \cup W$

$(0,1) \in V \cup W$

But $(1,0) + (0,1) = (1,1)$ and $V \cup W$. So it is not a subspace of $\mathbb{R}^2$.

Section 4.4 - Spanning Sets and Linear Independence

A vector $\overrightarrow{v}$ in a vector space $V$ is called a linear combination of the vectors $\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_r}$ if $\overrightarrow{v}$ can be written in the form $\overrightarrow{v} = c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + \ldots + c_k\overrightarrow{u_k}$ where $c_1, c_2, \ldots, c_k$ are scalars.

Example: $V = M_{2,2}$

$$\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\overrightarrow{v} = c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3}$$

$$\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -c_2 - 2c_3 \\ 2 & c_1 + c_2 + c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

So $\overrightarrow{v}$ is a linear combination of $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}$.

Spanning sets:

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_r}\}$ be a subspace of a vector space $V$. The set $S$ is called a
spanning set of $V$ if every vector in the vector space $V$ can be written as a linear combination of vectors in $S$. In such cases, we say that $S$ spans $V$.

Example: The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ spans $\mathbb{R}^3$ since every vector $\overrightarrow{u} = (u_1,u_2,u_3) = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$.

\[
S = \{(1,2,3),(0,1,2),(-1,0,1)\}
\]

\[
(u_1 - u_3, 2u_1 + u_2, 3u_1 + u_2 + u_3)
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
2 & 1 & 0 \\
3 & 2 & 1
\end{bmatrix}
\]

\[\text{det} = 0\]

Therefore, $S$ is not a spanning set.

A set of vectors $S = \{v_1,v_2,\ldots,v_k\}$ in a vector space $V$ is called linearly independent if the vector equation $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_k\overrightarrow{v_k} = \overrightarrow{0}$ has only the trivial solution $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If not, then $S$ is linearly dependent.

Example: Determine whether the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ is dependent or not.

\[
1\overrightarrow{v_1} + 2\overrightarrow{v_2} + \ldots + c_k\overrightarrow{v_k} = \overrightarrow{0}
\]

\[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}
\]

row echelon form: \[\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}\]

Theorem: A set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\}, k \geq 2$ is linearly dependent if and only if at least one of the vectors $v_j$ can be written as a linear combination of the other vectors.

\[
c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + \ldots + c_k\overrightarrow{v_k} = \overrightarrow{0}
\]

Without loss of generality (WLG), suppose $c_1 \neq 0$.

\[
c_1\overrightarrow{v_1} = -c_2\overrightarrow{v_2} - \ldots - c_k\overrightarrow{v_k}, \text{ so } \overrightarrow{v_1} = -\frac{c_1}{c_1}\overrightarrow{v_2} - \ldots - \frac{c_1}{c_1}\overrightarrow{v_k}.
\]

Conversely, if $\overrightarrow{v_1} = c_2\overrightarrow{v_2} + \ldots + c_k\overrightarrow{v_k}$, $\overrightarrow{v_1} = c_2\overrightarrow{v_2} + \ldots + c_k\overrightarrow{v_k}$

Therefore, if a $c_n \neq 0$, then the equation is dependent.

Two vectors are linearly dependent if one is a scalar multiple of the other.

$S = \{(1,1,1),(2,2,2)\}$ is a linearly dependant set.

Section 4.5 - Basis and Dimensions
A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space $V$ is called a basis for $V$ if the following conditions are true:

1. $S$ spans $V$
2. $S$ is linearly independent

- A standard basis for $\mathbb{R}^2$ is $\{e_1, e_2, \ldots, e_n\}$ where $e_i = (0, 0, \ldots, 1, \ldots, 0)$
- A monostandard basis for $\mathbb{R}^2$ is $S = \{(1, 2), (2, 1)\}$

The standard basis is $\vec{i}, \vec{j}, \vec{k}$

For $P_n$ (polynomials degree $\leq n$), a standard basis is $\{1, x, x^2, \ldots, x^{n-1}, x^n\}$

For $M_{2,2}$,

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
$$

*Theorem: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a space $V$. Then every vector in $V$ can be written in one and only one way as linear combinations of vectors in $S$.

Proof: Let $\vec{u} \in V$. Then, there exist $c_1, c_2, \ldots, c_n : u = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

(Spanning set)

Suppose $u = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$. Then

$c_1 - b_1) v_1 + (c_2 - b_2) v_2 + \ldots + (c_n - b_n) v_n = 0$.

But $S$ is a basis, therefore it is linearly independent. So

$c_1 - b_1 = c_2 - b_2 = \ldots = c_n - b_n = 0$. Therefore, $c_i = b_i$ for every $i \in \{1, \ldots, n\}$.

Consequently, the representation is unique.

*Theorem: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for vector space $V$, then every set containing more than $n$ vectors in $V$ is linearly dependent.

Corollary: If a vector space $V$ has one basis with $n$ vectors, then every basis for the vector space has the same number of elements.

If a vector space $V$ has a basis consisting of $n$ vectors, then the number $n$ is called the dimension of $V$, denoted by $\dim(V) = n$.

Examples: $\dim(\mathbb{R}^n) = n$; $\dim(M_{n,m}) = m \times n$

$V = \text{subspace of symmetric matrices in } M_{2,2}$

$\dim(V) = 3$

basis: $\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}$
Theorem: Let \( V \) be a vector space of dimension \( n \).
1. If \( S = \{v_1, \ldots, v_n\} \) is a linearly independent set of vectors in \( V \), then \( S \) is a basis for \( V \).
2. If \( S = \{v_1, v_2, \ldots, v_n\} \) spans \( V \), then \( S \) is a basis for \( V \).

Section 4.6 - Rank of a Matrix and Systems of Linear Equations

Let \( A \) be a \( m \times n \) matrix.
1. The row space of \( A \) is the subspace of \( \mathbb{R}^n \) spanned by the row vectors of \( A \).
2. The column space of \( A \) is the subspace of \( \mathbb{R}^m \) spanned by the column vectors of \( A \).

If \( A \) is an \( m \times n \) matrix, then the row space and column space of \( A \) have the same dimensions.

The dimension of the row space or the column space is called the rank of matrix \( A \). Rank is denoted by \( \text{rank}(A) \).

Example: Find the rank of the matrix \( A \) given by
\[
A = \begin{bmatrix}
1 & -2 & 0 & 1 \\
2 & 1 & 5 & -3 \\
0 & 1 & 3 & 5
\end{bmatrix}
\]
row echelon form:
\[
\begin{bmatrix}
1 & 0 & 0 & -7 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]
The dimension is 3, so the rank is 3.

If \( A \) is am \( m \times n \) matrix, then the set of all solutions of the homogenous system of linear equations \( Ax = 0 \) is a subspace of \( \mathbb{R}^n \), called the null space of \( A \), denoted by \( \text{N}(A) \). \( \text{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\} \). The dimension of the null space of \( A \) is called the nullity of \( A \).

Example 1: \[
\begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
is a null space.
\( \text{N}(A) = \{(0,0)\} \)
nullity(A) = 0
Example 2: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(A) = \{(−2t, t), t \in \mathbb{R}\} \quad \text{nullity}(A) = 1$

If $A$ is a $m \times n$ matrix of rank $r$, then $n = rank(A) + nullity(A)$.

For square matrices:
If $A$ is an $n \times n$ matrix, then the following conditions are equivalent:
1. $A$ is invertable
2. $\det(A) \neq 0$
3. $Ax = b$ has a unique solution for any $n \times 1$ matrix $b$ which is $x = A^{-1}b$
4. Rank$(A) = n$
5. nullity$(A) = 0$
6. The $n$ row vectors of $A$ are linearly independent.
7. The $n$ column vectors of $A$ are linearly independent.