New Directions in Function Theory:

From Complex to Hypercomplex to Non-Commutative

Chapman University, November 21-26, 2019

Abstracts

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SUPEROSCILLATIONS AND ANALYTIC EXTENSION

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Question: we are given the values of a polynomial in $[-1, 1]$, but we do not know the degree of the polynomial. Is there a constructive way to get the values of the polynomial outside $[-1, 1]$? Using the theory of superoscillations we consider the above question, and give applications to the trigonometric moment problem and to the theory of stationary second order stochastic processes and stationary-increment stochastic processes. We obtain in particular a new constructive solution of the prediction problem.

This is joint work with Fabrizio Colombo and Irene Sabadini.

NEVANLINNA FUNCTIONS, OPERATOR REPRESENTATIONS, AND SPECTRAL THEORY

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In this talk we recall the concept of operator models for scalar, matrix and operator-valued Nevanlinna functions in reproducing kernel Hilbert spaces and L^2-spaces, and we discuss the connection between the spectrum of the representing operator (or relation) and the limit behaviour of the Nevanlinna function near the real axis. As a classical application of the abstract theory we consider singular Sturm-Liouville differential operators and their Fourier transforms.

ATOMS OF DISTRIBUTIONS OF FREE CONVOLUTIONS OF OPERATOR-VALUED SELF-ADJOINT RANDOM VARIABLES

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It has been known for a long time that the operation $\boxplus$ of free additive convolution of probability measures on $\mathbb{R}$ is highly nonlinear. As a consequence, the relation between the atomic part of measures μ, ν and $\mu \boxplus \nu$ is more involved than in the case of classical convolution. This relation has been established by Bercovici and Voiculescu in [3]: it essentially states that $(\mu \boxplus \nu)(\{a\}) = \max\{0, \mu(\{s\}) + \nu(\{t\}) - 1 \colon s + t = a\}$. In this talk, we show that a similar formula holds for the much more general free additive convolution of operator-valued distributions. We shall start by introducing the concept of operator-valued noncommutative probability space and operator-valued distributions, and showing some motivating examples. We present next the main result and the tools for proving it, namely the free noncommutative analytic transforms associated to operator-valued distributions, and their Julia-Carathéodory derivatives [1]. As an application, we shall provide a characterization of the atomic part of the distribution of an arbitrary selfadjoint polynomial in free variables in terms of its realization matrices, thus providing what is effectively a converse to the main results of Shlyakhtenko and Skoufranis [4]. The talk is based on the joint work [2] with Hari Berkovici and Weihua Liu.

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TOWARD CLIFFORS INFINITE DIMENSIONAL ANALYSIS

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We discuss Gaussian measures and white noise in the context of Clifford analysis. We will construct an abstract Wiener space for Clifford vectors following the construction in [1, 2] for the case of real-valued functions. To construct the abstract Wiener spaces we use Clifford Hermite polynomials [3] and their properties.
As a result we obtain a Fock space decomposition of the space $L^2(\mathbb{R}^m, \mathbb{C}_m)$ of Clifford-valued functions of m variables. Because the Clifford Hermite polynomials $\varphi_{s,k,j}(\underline{x})$ are eigenfunctions of the operator
$A = \frac{1}{2}(D_{\underline{x}}^2 - \underline{x}^2 - m)$ we can define the norm $|f|_p = ||A^p f||$ for any $p \in \mathbb{R}$ and obtain that
$(L^2(\mathbb{R}^m, \mathbb{C}_m), \mathcal{S}_{-p}(\mathbb{R}^m, \mathbb{C}_m))$ is an abstract Wiener space if and only if $p > 1$. Here, $\mathcal{S}_{-p}(\mathbb{R}^m, \mathbb{C}_m)$ is the completion of $(L^2(\mathbb{R}^m), \mathbb{C}_m)$ with respect to the norm $| \cdot |_p$.

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ZERO STRUCTURE AND SOME CHARACTERIZATIONS OF FINITE BLASCHKE PRODUCTS OVER QUATERNIONS

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Given a complex polynomial $p(z) = p_0 + p_1 z + \ldots + p_n z^n$ with all roots $a_1, \ldots, a_n$ inside the open unit disk $\mathbb{D}$, the rational function

$$f(z) = \frac{p(z)}{z^n p(1/\overline{z})} = \frac{p_0 + p_1 z + \ldots + p_n z^n}{\overline{p}_n + \overline{p}_{n-1} z + \ldots + \overline{p}_0 z^n} \tag{1}$$

can be written in the form

$$f(z) = c \cdot \prod_{i=1}^{k} \frac{z - a_i}{1 - z\overline{a}_i}, \qquad |c| = 1, \ |a_i| < 1, \tag{2}$$

and is called a finite Blaschke product of degree n. To construct a Blaschke product with prescribed zeros or with the same zero structure (zeros counted with multiplicities) as a given polynomial p, one can directly use formulas (2) and (1). In the talk, we will discuss the analogs of these formulas in the context of Blaschke products over quaternions; the latter were introduced in [2] within the theory of quaternionic slice-regular Schur-class functions. We will also survey several intrinsic characterizations of quaternionic finite Blaschke products, which specify some results from [1, 3] on general slice-regular Schur-class functions.

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THE TOPOLOGY OF HARMONIC MAPPINGS WITH FINITE BLASCHKE DILATIONS

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Let $f(z)$ be a complex function defined on the unit disk $\mathbb{D}$. f is a harmonic mapping if $\delta f = 0$ and in such a case f admits the unique representation $f(z) = h(z) + \overline{g(z)}$ up to a complex constant. f is sense-preserving if, and only if, $J(f) = |h'(z)|^2 - |g'(z)|^2 > 0$ where h and g are analytic functions in $\mathbb{D}$, equivalently, $|g'(z)| < |h'(z)|$. Written otherwise, f is a sense-preserving harmonic mapping if, and only if, it is a solution to the linear elliptic equation

$$\overline{f_{\overline{z}}(z)} = a(z) f_z(z);\ a(z) = g'(z)/h'(z) \text{ is analytic and } |a(z)| < 1 \text{ in } \mathbb{D}.$$

Such functions are quasiregular if $|a(z)| < k < 1$ but generally not when $|a(z)| = 1;\ z \in \partial\mathbb{D}$. The special case where $a(z)$ is a finite Blaschke product, is a typical example of the abnormality of $f(z)$. We shall discuss the topology of such mappings and their connection to analytic functions via topological considerations.

This is joint work with Melike Aydogan, Abdallah Lyzzaik and Muge Sakar.

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A SPINOR REPRESENTATION OF ACOUSTIC FIELD THEORY

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We present a new representation of acoustic field theory and show that varying the usual acoustic Lagrangian with respect to a spinor potential rather than a scalar potential yields a new set of conserved quantities, consistent with the original theory, that provides a corrected account of the local forces and torques experienced by subwavelength probe particles suspended in acoustic fields.

Joint work with Konstantin Bliokh and Justin Dressel.

BANACH SPACE OPERATORS ACTING ON SEMICIRCULAR ELEMENTS INDUCED BY p-ADIC NUMBER FIELDS ON p-ADIC SEMICIRCULAR ELEMENTS

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In this talk, we consider (i) how p-adic number fields induce semicircular elements, (ii) how to construct Banach $*$-probability space generated by the semicircular elements of (i), (iii) how certain shifting processes on the set of all primes, and the set of all integers generate $*$-homomorphisms preserving free probability on the Banach $*$-probability space of (ii), (iv) how the $*$-homomorphisms of (iii) induce adjointable Banach-space operators acting on our semicircular elements, and (v) how such operators deform the semicircular law.

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DIRECT AND INVERSE FUETER-SCE-QIAN MAPPING THEOREM AND SPECTRAL THEORIES

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In classical complex operator theory, the Cauchy formula of holomorphic functions is a fundamental tool for defining functions of operators. Moreover, the Cauchy-Riemann operator factorizes the Laplace operator, so holomorphic functions play also a crucial role in harmonic analysis and in boundary value problems. In higher dimensions, for quaternion-valued functions or more in general for Clifford-algebra-valued functions, there appear two different notions of hyper-holomorphicity. The first one is called slice hyperholomorphicity and the second one is known under different names, depending on the dimension of the algebra and the range of the functions: Cauchy-Fueter regularity for quaternion-valued functions and monogenicity for Clifford-algebra-valued functions. The Fueter-Sce-Qian mapping theorem reveals a fundamental relation between the different notions of hyperholomorphicity and it can be illustrated by the following two maps

$$F_1 : Hol(\Omega) \mapsto \mathcal{N}(U) \quad \text{and} \quad F_2 : \mathcal{N}(U) \mapsto \mathcal{AM}(U).$$

The map F_1 transforms holomorphic functions in $Hol(\Omega)$, where Ω is a suitable open set Ω in $\mathbb{C}$, into intrinsic slice hyperholomorphic functions in $\mathcal{N}(U)$ defined on the open set U in $\mathbb{H}$. Applying the second transformation F_2 to intrinsic slice hyperholomorphic functions, we get axially Fueter-regular resp. axially monogenic functions. Roughly speaking the map F_1 is defined as follows:

1. We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex halfplane. (In order to distinguish the imaginary unit of $\mathbb{C}$ from the quaternionic imaginary units, we denote it by ι). We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

   where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy-Riemann system.

2. For suitable quaternions q, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the quaternionic imaginary unit $\frac{\Im(q)}{|\Im(q)|}$ and we set $u = \Re(q) = q_0$ and $v = |\Im(q)|$. We then define

$$f(q) = f_0(q_0, |\Im(q)|) + \frac{\Im(q)}{|\Im(q)|} f_1(q_0, |\Im(q)|).$$

   The function $f(q)$ turns out to be slice hyperholomorphic by construction.

When considering quaternion-valued functions, the map F_2 is the the Laplace operator, i.e. $F_2 = \Delta$. When we work with Clifford-algebra-valued functions then $F_2 = \Delta_{n+1}^{(n-1)/2}$, where n is the number of generating units of the Clifford algebra and Δ_{n+1} is the Laplace operator in dimension $n + 1$. The Fueter-Sce-Qian mapping theorem can be adapted to the more general case in which $\mathcal{N}(U)$ is replaced by slice hyperholomorphic functions and the axially regular (or axially monogenic) functions $\mathcal{AM}(U)$ are replaced by monogenic functions. The generalization of holomorphicity to quaternion- or Clifford-algebra-valued functions produces two different notions of hyper-holomorphicity that are useful for different purposes. Precisely, we have that:

(I) The Cauchy formula of slice hyperholomorphic functions leads to the definition of the S-spectrum and the S-functional calculus for quaternionic linear operators. Moreover, the spectral theorem for quaternionic linear operators is based on the S-spectrum.
(II) The Cauchy formula associated with Cauchy-Fueter regularity resp. monogenicity leads to the notion of monogenic spectrum and produces the Cauchy-Fueter functional calculus for quaternion-valued functions and the monogenic functional calculus for Clifford-algebra-valued functions. This theory has applications in harmonic analysis in higher dimension and in boundary value problems.

A FUNCTIONAL ANALYTIC APPROACH TO SINGULAR PERTURBATION PROBLEMS.

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This talk is dedicated to the analysis of singularly perturbed problems on singularly perturbed domains by an approach which is alternative to those of asymptotic analysis, and has an expository character.
In particular, we will consider some specific problems depending on a positive parameter ϵ and we will consider a family of solutions depending on ϵ as ϵ approaches 0. Then we shall represent the dependence on ϵ of the family of solutions, or of corresponding functionals of the solutions in terms of possibly singular at 0 but known functions of ϵ such as ϵ^{-1} or $\log \epsilon$, and in terms of possibly unknown real analytic operators.

POLYNOMIAL APPROXIMATION IN SLICE REGULAR FOCK SPACES

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In this talk, we introduce the Fock spaces of slice regular functions both of the first and second kind. In particular, we prove several approximation results on these different spaces, some of them are based on constructive methods that make use of the Taylor expansion and the convolution polynomials. The techniques used in these two cases are different. For the second kind theory, we can discuss also the density of reproducing kernels. This is joint work with Prof. Gal, S.G. and Prof. Sabadini, I.

MATRIX CONVEXITY, CHOQUET BOUNDARIES AND TSIRELSON PROBLEMS

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Following work of Evert, Helton, Klep and McCullough on free LMI domains, we ask when a matrix convex set is the closed convex hull of its (finite dimensional) Choquet points. This is a finite-dimensional version of Arveson's non-commutative Krein-Milman theorem, and some matrix convex sets can fail to have any finite-dimensional Choquet points. The general problem of determining whether a given matrix convex set has this property turns out to be difficult because for certain correlation sets studied by Tsirelson we show that a positive answer is equivalent to Connes' embedding conjecture. Our approach provides new geometric variants of Tsirelson type problems for pairs of convex polytopes which may be easier to rule out than the original Trsirelson problems.

QUANTUM CONTEXTUALITY AS ERASURE

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We formulate multiparticle quantum mechanics as an algebraic quotient space constructed from redundant copies of spacetime. We demonstrate that the needed equivalence relation erases geometric distinctions, leading to entanglement correlations and other context-dependent measurement effects. This quotient construction has close connections to resource theories derived from symmetry groups, which we also review.

Joint work with Lucas Burns, Lorenzo Catani, Tomas Gonda, and Thomas Galley.

ON THE GEOMETRY OF OPTIMAL MASS TRANSPORT: ADVANCES AND APPLICATIONS IN ENGINEERING AND SCIENCES

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From its early 18th-century beginnings [1] to today [2], the theory of Monge-Kantorovich Optimal Mass Transport (OMT) has provided a natural framework for analysis in a wide range of engineering disciplines, including computer imaging, stochastic and covariance control, uncertainty quantification, fluid mechanics, stochastic thermodynamics, statistics, optimization, and so on. Moreover, and rather recently, it was recognized that OMT provides a Riemannian-like structure on Metric and Probability Measure spaces [3] which opens new possibilities for problems of interpolation, filtering and regression directly in spaces of probability measures. We will discuss recent advances in this direction as well generalizations that allow addressing such problems in the context of matrix-valued and vector-valued distributions [4, 5, 6].

The talk is based on joint works with Yongxin Chen (GaTech), Giovanni Conforti (École Polytechnique), Wilfrid Gangbo (UCLA), Michele Pavon (University of Padova), Luigia Ripani (UCI), Amirhossein Karimi (UCI), and Allen Tannenbaum (Stony Brook).

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ON CONFORMALITY IN FINITE–DIMENSION MODULES OVER HYPERBOLIC NUMBERS

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Let $\mathbb{D}$ be the ring of hyperbolic numbers and set $\mathbb{D}^n := \mathbb{D} \times \cdots \times \mathbb{D}$. We endow the latter with a $\mathbb{D}$–valued norm and the angles between its elements (vectors) are measured with hyperbolic numbers. One of the aims of the talk is to give a detailed description of the geometry and the analysis of $\mathbb{D}^n$; in particular, a number of equivalent characterizations for conformality will be given. A special attention will be dedicated to the case $n = 2$ since $\mathbb{D}^2$ coincides with the ring of bicomplex numbers.
The talk is based on a joint work with M. E. Luna–Elizarrarás.

THE ROLE OF SCHUR FUNCTIONS IN THE STUDY OF QUANTUM WALKS

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We show that scalar as well as operator valued Schur functions play a useful role in the study of recurrence properties of Quantum walks. I will give an ab-initio introduction and discuss a few open questions.

This is joint work with several people, including J. Bourgain, L.Velazquez, A.Werner, R. Werner and J. Wilkening.

VORTEX MOTION AND GEOMETRIC FUNCTION THEORY: THE ROLE OF CONNECTIONS

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We discuss point vortex dynamics on a closed two dimensional Riemannian manifold in the language of affine and other kinds of connections. The speed of a vortex then becomes the difference between an affine connection derived from the coordinate Robin function and the Levi-Civita connection associated to the Riemannian metric.

We also formulate the dynamics in the framework of Hamiltonian systems. The implementation of the Kelvin laws of conservation of circulations then requires of careful analysis of the interaction between periods and poles of the classical Abelian differentials on a compact Riemann surface.

The talk is to expose part of the contents of a recent preprint (arXiv:1811.09430, with author and title as above).

NONCOMMUTATIVE PLURISUBHARMONIC FUNCTIONS ON NC CONVEX SETS

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The talk will describe some progress over the last year on noncommunicative sets and functions. One result completely characterizes the composition r of a convex with an analytic noncommutative rational function, $r(z) = f(q(z))$. These are free analogues of plurisubharmonic functions.

Another result concerns free semialgebraic sets, a.k.a. free spectrahedra, oddities of optimization over them and their free extreme points.

The the talk will choose among these topics.

The work is joint with Meric Augat, Erik Evert, Igor Klep, Scott Mccullough, Jurij Volcic

NEW DEVELOPMENTS IN JOURNAL PUBLISHING AT SPRINGER NATURE

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Technical innovation and changing community needs are driving transformation in academic publishing. Based on examples from *Springer Nature* – the publisher that resulted from the merger of *Springer* and *Nature Publishing Group* – we discuss some recent developments and services that could point the way into the future by catering to scientists needs. Diverse topics, such as e.g. article sharing, reviewer recognition and Open Access, will be discussed.

TRANSFINITE APOLLONIAN METRIC

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It is known that one-point, hyperbolic-type metrics are not Gromov hyperbolic. These metrics are defined as a supremum of one-point metrics (i.e., metrics constructed using one boundary point) and the supremum is taken over all boundary points. In [1] we proved that taking the average of the one-point metrics instead of their supremum yields a Gromov hyperbolic metric. Moreover, the Gromov hyperbolicity constant of the resulting metric does not depend on the number of boundary points used in taking the average. We also provided an example to show that the average of Gromov hyperbolic metrics is not, in general, Gromov hyperbolic. In this talk we extend the idea of averaging metrics to the notion of *transfinite bi-metrics*. In some details we will discuss transfinite Apollonian metric and specific examples of domains where the transfinite Apollonian metric can be computed explicitly.

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REPRESENTATIONS AND MULTI-RESOLUTIONS ASSOCIATED WITH ENDOMORPHISMS

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The notion of multiresolution is from the areas of signal processing, and from wavelet analysis. In the talk, we shall demonstrate that the fundamental notion of scales of resolutions has a much wider reach within pure and applied mathematics; covering dynamics, operator theory, filtration of closed subspaces, identification of scales of resolutions and corresponding detail. Our approach will entail, in turn, a new non-commutative analysis, specifically new representations of the Cuntz algebras. While traditional wavelet analysis, and multi-band filters, takes as starting point a certain harmonic analysis on the disk, and then moves on to an analysis of all wavelet filters, our present approach is much more general. It begins instead with a fixed endomorphism in a measure space. We first point out that this general framework is also a fruitful setting for dynamics, and is of current interest in ergodic theory, and in an analysis of non-invertible dynamical systems. However, so far, a systematic understanding of dynamics for non-invertible maps is only in its infancy. Nonetheless, we shall show that, in the general context of endomorphism in a measure space, one may still construct filtrations of closed subspaces representing scales of resolutions and intermediate detail-subspaces. A key tool in our approach will be a certain infinite-dimensional loop group, and its representations.

This represents joint research with D. Alpay and I. Lewkowicz.

THE MOMENT PROBLEM ON CURVES WITH BUMPS

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The power moments of a positive measure on the real line or the circle are characterised by the non-negativity of an infinite matrix, Hankel, respectively Toeplitz, attached to the given data. Except some fortunate configurations, in higher dimensions there are no non-negativity criteria for the power moments of a measure to be supported by a prescribed closed set. We combine two well studied fortunate situations, specifically a class of curves in two dimensions classified by C Scheiderer and D Plaumann, and compact, basic semi-algebraic sets, with the aim at enlarging the realm of geometric shapes on which the power moment problem is accessible and solvable by non-negativity certificates.

This talk is based on a joint work with Mihai Putinar.

AN OVERVIEW OF COMPLEX FRACTAL DIMENSIONS: FROM FRACTAL STRINGS TO FRACTAL DRUMS AND BACK

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We will give some sample results from the new higher-dimensional theory of complex fractal dimensions developed jointly with Goran Radunovic and Darko Zubrinic in the recently published nearly 700-page research monograph (joint with these same coauthors), Fractal Zeta Functions and Fractal Drums: Higher Dimensional Theory of Complex Dimensions [2], published by Springer in June 2017 in the Springer Monographs in Mathematics series. We will also explain its connections with the earlier one-dimensional theory of complex dimensions developed, in particular, in the research monograph (by the speaker and M. van Frankenhuijsen) entitled "Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings" [1] (Springer Monographs in Mathematics, Springer, New York, 2013; 2nd rev. and enl. edn. of the 2006 edn.).

In particular, to an arbitrary compact subset A of the N-dimensional Euclidean space (or, more generally, to any relative fractal drum), we will associate new distance and tube zeta functions, as well as discuss their basic properties, including their holomorphic and meromorphic extensions, and the nature and distribution of their poles (or 'complex dimensions'). We will also show that the abscissa of convergence of each of these fractal zeta functions coincides with the upper box (or Minkowski) dimension of the underlying compact set A, and that the associated residues are intimately related to the (possibly suitably averaged) Minkowski content of A. Example of classical fractals and their complex dimensions will be provided.

Finally, if time permits, we will discuss and extend to any dimension the general definition of fractality proposed by the author (and M-vF) in their earlier work [1], as the presence of nonreal complex dimensions. We will also provide examples of hyperfractals, for which the critical line $\mathrm{Re}(s) = D$, where D is the Minkowski dimension, is not only a natural boundary for the associated fractal zeta functions, but also consist entirely of singularities of those zeta functions. Fractal tube formulas are obtained which enable us to express the intrinsic oscillations of fractal objects in terms of the underlying complex dimensions and the residues of the associated fractal zeta functions. Intuitively, the real parts of the complex dimensions correspond to the amplitudes of the associated geometric waves, while their imaginary parts correspond to the frequencies of those waves. This is analogous to Riemanns explicit formula in analytic number theory, expressing the counting function of the primes in terms of the underlying zeros of the celebrated Riemann zeta function.

These results are used, in particular, to show the sharpness of an estimate obtained for the abscissa of meromorphic convergence of the spectral zeta functions of fractal drums. Furthermore, we will also briefly discuss recent joint results in which we obtain general fractal tube formulas in this context (that is, for compact subsets of Euclidean space or for relative fractal drums), expressed in terms of the underlying complex dimensions. We may close with a brief discussion of a few of the many open problems stated at the end of the aforementioned book.

PASSIVE LINEAR SYSTEMS
MATRIX-CONVEX INVERTIBLE CONES POINT OF VIEW

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In this talk we show that continuous-time, passive, linear systems, are intimately linked to the structure of maximal matrix-convex, cones, closed under inversion. Moreover, this observation unifies three setups:
(i) differential inclusions,
(ii) matrix-valued rational functions,
(iii) realization arrays associated with rational functions.

SALEM CONDITIONS IN THE NON-PERIODIC CASE

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In the classical sources, Salem's necessary conditions for a trigonometric series to be the Fourier series of an integrable function are given in terms of "some" sums. Realizing that, in fact, they are given in terms of the discrete Hilbert transforms, we generalize these to the non-periodic case, for functions from the Wiener algebra (for details, see [1]).
The obtained result is used to constructing a monotone function with non-integrable cosine Fourier transform in a much easier way than in the classical book [2] by Titchmarsh.
Certain open problems are posed.

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ON THE INTEGRATION OF BICOMPLEX FUNCTIONS AND ITS RELATION WITH HYPERBOLIC CURVES

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In recent years, the development of bicomplex analysis has been strongly influenced by the realizing of the key role that is played by the hyperbolic numbers inside the bicomplex ones and by the corresponding hyperbolic mathematical objects. Among those latter objects there are: hyperbolic–valued norm, hyperbolic angles, hyperbolic curves, hyperbolic–valued probability, etc.
In particular, the question of the proper approach to the integration of bicomplex functions proves to be rather delicate and requiring special attention.
In this talk I will present how we can rethink the definition of the integral of different types of bicomplex functions in such a way that they will be consistent with the hyperbolic objects and with the existing already theory of the bicomplex integrals.

A DATA-DRIVEN FOURIER TRANSFORM LIKE METHOD

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We investigate a signal analysis and decomposition method that is intrinsically adapted to the nature of the signal to be processed.

The basic principle of signal analysis is to decompose the signal into a "*universal*" pre-established family of elementary signals with well-known properties: Fourier basis, Gabor atoms, wavelet basis/frame, Time-Frequency, etc. However, the increasing complexity and the growing heterogeneity of data call for a paradigm shift. The purpose of this talk is to invest a new *model-free* or *ad-hoc-model* based approach, intermediate between the pre-established Fourier-Gabor-Wavelets [1], [2], [3] based model and the model-free of the Empirical Mode Decomposition (EMD) [4]. The idea is to analyze a signal x on an ad-hoc family, built-on and driven-by a given signal y. The interest is to combine the mathematical formalism of the decompositions on bases of functions and the self adaptability of the EMD [5], [6], afforded by the freedom of choice of the decomposing signal y.

A proof of concept is given using the construction of a finite dimensional de Branges space [7].

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ORTHOGONAL POLYNOMIALS ON FRACTALS

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In this talk I will introduce a (non-commutative) theory of orthogonal polynomials on fractals, especially the Sierpinski gasket (SG). In the first part of the talk, we will review the properties of the fractal Laplacian and the theory of polynomials on SG. The second part will be devoted to the theory of Legendre and Sobolev orthogonal polynomials on SG.

THEORY OF HYPERHOLOMORPHIC FUNCTIONS IN BANACH ALGEBRAS

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We present some aspects of the theory of hyperholomorphic functions whose values are taken in a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. Notably, we consider Fueter expansions, Gleasons problem, the theory of hyperholomorphic rational functions, and related problems. Such a framework includes many familiar algebras as particular cases. The quaternions, the split quaternions, the Clifford algebras, the ternary algebra, and the Grassmann algebra are a few examples.

Joint work with Prof. Daniel Alpay and Prof. Daniele Struppa.

TECHNIQUES TO DERIVE ESTIMATES FOR INTEGRAL MEANS AND OTHER GEOMETRIC QUANTITIES RELATED TO CONFORMAL MAPPINGS

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We will describe few methods to derive estimates for integral means and for other asymptotic expressions related to conformal mappings. One method will start from classical inequalities for conformal mappings such as the Goluzin inequalities and the exponential Goluzin inequalities. Then the simple idea of approximating integrals with the aid of their Riemann sums will serve us to obtain such estimates. A second method is to start from a certain elementary identity proved by Hardy in 1915 and use it combined with distortion theorems in S to obtain more integrals estimates. Finally, the main result in the authors Masters thesis which in fact was already known to Bendixon will give us a method to estimate the geometric distance from a point in the image of a conformal mapping to the boundary of this image. The estimate will be in terms of a rather arbitrary sequence in the domain of the definition that converges to the pre-image of the point in the image from which the distance is measured.

CARATHÉODORY FUNCTIONS AND HERGLOTZ THEOREM OVER COMPACT RIEMANN SURFACES AND THE ASSOCIATED DE BRANGES SPACES

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Carathéodory functions, functions with positive real part, play important role in operator theory, $1D$ system theory and the study de Branges-Rovnyak spaces. The Herglotz integral representation theorem associates to each Carathéodory function a positive measure and allows us to further examine these subjects [1]. In this talk, we present these relations when the Riemann sphere is replaced by a real comapct Riemann surface. The generalization of the Herglotz's theorem to the compact real Riemann surface setting will be presented. Furthermore, we examine the de Branges-Rovnyak spaces associated to functions with positive real-part defined on compact Riemann surfaces, that is, reproducing kernel Hilbert spaces containing sections of a certain line bundle.
This is joint work with Daniel Alpay and Victor Vinnikov.

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ON VARIOUS NOTIONS OF SLICE HYPERHOLOMORPHIC FUNCTIONS IN SEVERAL VARIABLES

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In the talk we consider functions in several quaternionic variables according to various possible definitions. We first introduce a notion based on slice functions and we discuss their algebraic analysis, showing the analogy with the functions Cauchy-Fuater regular. We then move to a notion based on converging series of monomials in several noncommuting variables and we discuss some questions that can be tackled in this framework.

PROXIMAL ACTIONS

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An action of a discrete group G on a compact Hausdorff space X is said to be *proximal* if for every two points $x, y \in X$ there is a net $g_\alpha \in G$ such that $\lim g_\alpha x = \lim g_\alpha y$, and *strongly proximal* if the natural action of G on the space $P(X)$ of probability measures on X is proximal. G is said to be *strongly amenable* if all of its proximal actions have a fixed point and *amenable* if all of its strongly proximal actions have a fixed point. In this talk I will present relations between some fundamental operator theoretic concepts to proximal and strongly proximal actions, and hence to strongly amenable and amenable groups. In particular, I will focus on the C^*-algebra of continues functions over the universal minimal proximal G-action and characterize it in the category of G-operator-systems.

This is joint work with Mathew Kennedy and Sven Raum.

ON A FUNCTION OF G.H. HARDY AND J.E. LITTLEWOOD

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In their paper:
Notes on the Theory of Series (XX); On Lambert Series, Proceedings of the London Mathematical Society, Ser. (2), 61 (1936), 257-270
Hardy and Littlewood considered the function

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - e^{\frac{z}{n}}\right), \quad \Re z > 0.$$

They give, without any detail, the expansion, in terms of a Bessel function:

$$f(z) = 2 \log z + 2\gamma - 2 \sum_{n=1}^{\infty} \left\{ K\left(\sqrt{2ni\pi z}\right) + K\left(\sqrt{-2ni\pi z}\right) \right\},$$

γ is the Euler constant. We show that this formula is actually a part of a large construction.

This is joint work with Robert Gay.

ON THE FOUNDATIONS OF THE PENROSE TRANSFORMATION

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In 1966, Penrose discovered that solutions of massless field equations on Minkowski space could be expressed as contour integrals of free holomorphic functions over lines in 3 dimensional complex projective space. This idea of using contour integral formulae to obtain solutions of field equations can in fact be traced back to Whittaker in 1902. The freedom in the function for a fixed solution was later realized to be exactly the freedom of a Cech representative of a sheaf cohomology class, and a formal mechanism was set up by Eastwood, Penrose and Wells to prove isomorphisms between sheaf cohomology groups over a region of the projective space and solutions of massless field equations over a region of spacetime (R. Penrose and W. Rindler, Spinors and Space-Time vol. 1, C U P 1984.) We go back to an epistemological and mathematical basis of this transformation. The Penrose transform was originally ascribed to this particular mechanism for 4 dimensional fiat spacetime. However, it has undergone a considerable amount of generalization and refinement (see thesis by Tsai f.e.).We return to the origins and we develop the basic characteristics of the Penrose transformation by focussing on the construction from the complex framework of $\mathbb{C}^4$ and in the introduction of the complexified projective Minkowski space, which is compactified by its embedding in the $Gr(3,1)$ Grassmanian. We detail some calculations according to Shabat's methods and explain why the transformation allows to describe some solutions of systems of current linear PDE's. of the physics that will be later greatly generalized.

A GENERALIZED FOURIER HYPERFUNCTION THEORY AND ITS APPLICATIONS TO PHYSICS

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Kawai introduced, in the 70s, a space of Fourier Hyperfunctions. Subsequently, Nagamachi and Migibayashi used this space to formulate what they called Fourier Hyperfunction Quantum Theory. In this talk I will present a generalization of Kawais theory that allows a much larger class of spaces to be considered, and we show how such theory allows for a reformulation of quantum field theory. I will then show how to use these ideas to study smooth solutions of the Klein-Gordon equation.

A NEW TYPE OF QUATERNIONIC REGULARITY

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In the general context of hypercomplex analysis, we develop a new type of quaternionic and biquaternionic regularity, based on a primary decomposition of the biquaternionic algebra.

ON TERNARY ANALYSIS: COMMUTATIVE OR NOT

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In the case of ternary commutative case we expand classes of analytic functions defined on to $\mathbb{C}[X]/\langle X^3+1\rangle$, i.e. a commutative algebra given by the linear span of $\{1, e, e^2\}$, where $e \notin \mathbb{C}$ is a generating unit. We define ternary conjugates and we build an analytic theory. This work is joint with Adrian Vajiac, Chapman University.

We also expand the notion of Clifford algebras to include a ternary anti-commutator. Our results are intended to produce an analytic theory related to the cubic factorization of the Laplacian. This work is joint with Paula Cerejeiras, Universidade de Aveiro, Portugal.

AN APPLICATION OF THE NONCOMMUTATIVE METRIC TO A FUNCTIONAL EQUATION ON THE NONCOMMUTATIVE HALF PLANE OVER A FINITE VON NEUMANN ALGEBRA

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Noncommutative functions are functions defined on noncommutative sets (sets closed under direct sums) in a noncommutative space — the disjoint union of square matrices of all sizes over a vector space. Noncommutative function theory is a "quantization" of usual function theory, much like complete positivity is a quantization of positivity and operator space theory is a quantization of functional analysis. In this talk I will review an intrinsically defined (pseudo)metric on noncommutative sets that is analogous to the Kobayashi (pseudo)metric in usual function theory. I will then show an application of this noncommutative metric on the noncommutative upper half plane over a finite von Neumann algebra to solve the functional equation $\omega(b) = b + h(\omega(b))$ for h a noncommutative self map of the noncommutative upper halfplane to itself satisfying a rather mild vanishing condition at infinity. This solves the problem of defining free convolution powers of distributions of unbounded selfadjoint random variables in free probability.

This is joint work with Serban Belinschi.

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POSITIVE EXTENSION PROBLEM FOR MATRICES INDEXED BY HOMOGENEOUS TREES

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The study of isotropic processes indexed by homogeneous trees originated in 1980's in the context of miltiscale signal processing and wavelet transform[1]. A positive extension problem in this setting, while falling outside the scope of the band method[2], admits a generalization of Schur-Levinson algorithm. Another approach to this problem is based on a canonical representation of the input-output operator of a stationary multiscale system[3]. This is joint work with Daniel Alpay.

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THE KACZMARZ ALGORITHM: A CONFLUENCE OF COMPLEX FUNCTION THEORY AND NUMERICAL ANALYSIS

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The Kaczmarz algorithm was introduced in 1937 as an iterative numerical method for solving systems of linear equations. The algorithm is a powerful tool with many applications in signal processing and data science. The algorithm was extended to the infinite dimensional case by Kwapien and Mycielski in 2001 within the context of stationary sequences. Remarkably, in work with John Herr and Palle Jorgensen, we have discovered that there are deep connections between the Kaczmarz algorithm and the model subspaces of the Hardy space of the disc. We will highlight these deep connections, as well as other connections to problems in harmonic analysis that have been solved by utilizing the Kaczmarz algorithm. We will also present ongoing work on the noncommutative extension of the Kaczmarz algorithm, which appears to have connections to vector-valued de Branges (de Branges-Rovnyak) spaces.

This is joint work with John Herr and Palle Jorgensen.

LELONG-POINCARÉ FORMULA AT THE CROSSROAD OF GEOMETRY, ANALYSIS AND ARITHMETICS

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Given a meromorphic section s of an hermitian line bundle $(D, |\ |)$ with Chern form $c_1(D, |\ |)$ over a reduced complex analytic space X, Lelong-Poincaré formula can be formulated as $dd^c(-\log|s|) + [\mathrm{div}(s)] = c_1(D, |\ |)$. It plays a central role in complex analytic geometry as well as in diophantine geometry when the data (X, s, D) are of arithmetic nature. In this lecture, we will look at such an equation from the (classical) complex geometry point of view up to the tropical point for view, let say for example when $s = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha X^\alpha$ is a Laurent polynomial (viewed as a section of a toric line bundle on a complete toric compactification of the complex torus $(\mathbb{C}^*)^n$) with rational coefficients, thus admitting as non-archimedean companions (through the logarithmic map) the affine functions $x \in \mathbb{R}^n \longmapsto \max_{\alpha \in \mathbb{Z}^n}(\log|c_\alpha|_p + \langle \alpha, x\rangle)$, $|\ |_p$ being the p-adic ultrametric absolute value on $\mathbb{Q}$. We will analyze the deep connexion between Lelong-Poincaré equation and Monge-Ampère operators (in the real or complex setting). This will lead us to formulate possible extensions of such equation to some other settings : among them, some commutative ones (such as that of n-bicomplex variables), where its transposition can be carried in a rather straightforward way, and non-commutative ones (as for example the setting of one quaternionic variable, together with the concept of slice-regularity) where we will point out all difficulties that need first to be overcome prior to imagine such a transposition (among them, to clarify the geometric notion of divisor). This talk is dedicated to the memory of my friend Carlos Berenstein. Most the ideas I will discuss here are indeed inspired by our (nearly twenty five years) collaboration from 1985 [1,2]. The work of my students [6,7,8,9] also motivated and inspired this lecture.

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